Optimal Order Constructive a Priori Error Estimates for a Full Discrete Approximation of the Heat Equation*

Takuma Kimura§
Saga University, Saga 840-8502, Japan
tkimura@cc.saga-u.ac.jp

Teruya Minamoto¶
Saga University, Saga 840-8502, Japan
minamoto@is.saga-u.ac.jp

Mitsuhiro T. Nakao‖
Waseda University, Tokyo 169-8555, Japan
mtnakao@aoni.waseda.jp

Abstract

In this paper, we consider constructive a priori error estimates for a full discrete numerical solution of parabolic initial boundary value problems. Our method is based on the finite element Galerkin method with an interpolation in time that uses the fundamental solution for semidiscretization in space. Particularly, we present optimal order error estimates for the linear finite element in both space and time directions. These error estimates are sharper than the existing results in the sense of convergence order to exact solutions. Since the sharply constructive error estimates play an essential role in improving the efficiency of the verification costs, our results are expected to contribute to a new development of the numerical proof for parabolic problems. We also present some numerical examples which confirm that our estimates have the exactly the same order of convergence as the a posteriori errors.

Keywords: Parabolic problem, Galerkin methods, Constructive a priori error estimates

*Submitted: December 1, 2017; Revised: June 26, 2017; Accepted:August 24, 2017.
§T. Kimura was supported by JSPS KAKENHI Grant Number 26870646.
¶T. Minamoto was supported by JSPS KAKENHI Grant Number 26400206.
‖M. T. Nakao was supported by JSPS KAKENHI Grant Number 15K05012.

AMS subject classifications: 35K05, 65M15, 65M60

1 Introduction

We consider constructive a priori error estimates for an approximate solution of the following equations with homogeneous initial and boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u = f(x,t) & \text{in } \Omega \times J, \\ u(x,t) = 0 & \text{on } \partial\Omega \times J, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d (d \in \{1,2,3\})$ is a bounded polygonal or polyhedral domain, $J := (0,T) \subset \mathbb{R}$ (for a fixed $T < \infty$) is an open interval, ν is a positive constant, and $f \in L^2(J; L^2(\Omega))$.

In [3, 4], we defined a full discrete approximation $P_h^k u$ for (1) and derived several error estimates with applications to nonlinear problems, where h and k are mesh size for Ω and J, respectively. Our method is based on the finite element Galerkin method with an interpolation in time that uses the fundamental solution for semidiscretization in space. We will describe in detail the definition of $P_h^k u$ in Section 3.

In [4], the authors studied the optimal order $L^2(J; H_0^1(\Omega))$ and $L^2(J; L^2(\Omega))$ error estimates for $P_h^k u$ with the assumption that $k = h^2$. In this paper, assuming that f is sufficiently smooth, we show the optimal order error estimates can be obtained even if $k \neq h^2$, which is an extension and improvement of the results in [4].

2 Notation

In this section, we introduce some function spaces, operators, and other notation, most of them taken from [4, 5].

Let $L^2(\Omega)$ be the usual Lebesgue spaces on Ω defined by the natural inner product $(u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\,dx$. Let $H^1(\Omega)$ be the usual Sobolev spaces on Ω defined by the inner product $(u,v)_{H^1(\Omega)} := (\nabla u, \nabla v)_{(L^2(\Omega))^d} = \sum_{i=1}^d \int_\Omega \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i}\,dx$. Also, let $H_0^1(\Omega)$ be a subspace of $H^1(\Omega)$ defined by $H_0^1(\Omega) := \{u \in H^1(\Omega)\ ;\ u = 0 \text{ on } \partial\Omega\}$ with inner product $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{(L^2(\Omega))^d}$.

Let $\Delta : L^2(\Omega) \to L^2(\Omega)$ be the Laplace operator defined by $\Delta u(x) = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} u(x)$ that is self-adjoint on the domain $D(\Delta) := \{u \in H_0^1(\Omega)\ ;\ \Delta u \in L^2(\Omega)\}$.

Let $V^1(J)$ be a subspace of $H^1(J)$ defined by $V^1(J) := \{u \in H^1(J)\ ;\ u(0) = 0\}$. Then, $V^1(J)$ is a Hilbert space with inner product $(u,v)_{V^1(J)} := \left(\frac{\partial}{\partial t}u, \frac{\partial}{\partial t}v\right)_{L^2(J)}$.

The time-dependent Lebesgue space $L^2\big(J; L^2(\Omega)\big)$ is defined as a space of square-integrable $L^2(\Omega)$-valued functions on J. Then, $L^2\big(J; L^2(\Omega)\big)$ is a Hilbert space with inner product $(u,v)_{L^2(J;L^2(\Omega))} := \int_J \int_\Omega u(x,t)v(x,t)\,dx\,dt$. We denote the function space $L^2\big(J; L^2(\Omega)\big)$ as L^2L^2, for short. In this paper, abbreviations like L^2L^2 for $L^2\big(J; L^2(\Omega)\big)$ will often be used. Let $L^2\big(J; H_0^1(\Omega)\big)$ be a subspace of L^2L^2 defined by

$$L^2\big(J; H_0^1(\Omega)\big) := \Big\{u \in L^2L^2;\ \nabla u \in (L^2\big(J; L^2(\Omega)\big))^d,\ u(\,\cdot\,,t) = 0 \text{ on } \partial\Omega, \\ \text{for almost all } t \in J\Big\}.$$

Then, $L^2H_0^1 \equiv L^2\big(J; H_0^1(\Omega)\big)$ is a Hilbert space with inner product $(u,v)_{L^2H_0^1} := (\nabla u, \nabla v)_{(L^2L^2)^d}$. Let $V^1\big(J; L^2(\Omega)\big)$ be a subspace of L^2L^2 defined by

$$V^1\big(J;L^2(\Omega)\big) := \left\{ u \in L^2\big(J; L^2(\Omega)\big)\ ;\ \frac{\partial u}{\partial t} \in L^2\big(J; L^2(\Omega)\big),\ u(\,\cdot\,,0) = 0 \text{ in } L^2(\Omega) \right\}.$$

Then, $V^1L^2 \equiv V^1\big(J; L^2(\Omega)\big)$ is a Hilbert space with inner product $(u,v)_{V^1L^2} := \big(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\big)_{L^2L^2}$. We define the Hilbert space $V := V^1L^2 \cap L^2H_0^1$ with inner product $(u,v)_V := (u,v)_{V^1L^2} + (u,v)_{L^2H_0^1} = \big(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\big)_{L^2L^2} + (\nabla u, \nabla v)_{(L^2L^2)^d}$. Last, we define the partial differential operator $\Delta_t : L^2L^2 \to L^2L^2$ by $\Delta_t := \frac{\partial}{\partial t} - \nu\Delta$ on the domain $D(\Delta_t) := V^1L^2 \cap L^2\big(J; D(\Delta)\big)$.

3 Existing Error Estimates

In this section, we define $P_h^k u$ and describe the error estimates derived in [4].

Let $S_h(\Omega)$ be a finite-dimensional subspace of $H_0^1(\Omega)$ dependent on the discretization parameter h. For example, $S_h(\Omega)$ is considered to be a finite element space with mesh size h. Let n be the number of degrees of freedom of $S_h(\Omega)$, and let $\{\varphi_i\}_{i=1}^n \subset H_0^1(\Omega)$ be the basis functions of $S_h(\Omega)$.

In [4], we assume the inverse estimates on $S_h(\Omega)$ as follows:

Assumption 3.1 *There exists a positive constant $C_{inv}(h)$ satisfying*

$$\|u_h\|_{H_0^1(\Omega)} \le C_{inv}(h)\, \|u_h\|_{L^2(\Omega)}, \quad \forall u_h \in S_h(\Omega). \tag{2}$$

For example, if Ω is a bounded open interval in $\mathbb{R}$, and $S_h(\Omega)$ is the P1 finite element space (i.e., spanned by piecewise linear basis functions [1, Section 3]), then Assumption 3.1 is satisfied for $C_{\rm inv}(h) = \frac{\sqrt{12}}{h_{\rm min}}$, where $h_{\rm min}$ is the minimum mesh size in the division of Ω (see e.g., [9, Theorem 1.5]).

Let $P_h^1 : H_0^1(\Omega) \to S_h(\Omega)$ be an H_0^1-projection. Namely, for an arbitrary element $u \in H_0^1(\Omega)$, $P_h^1 u \in S_h(\Omega)$ satisfies the variational equation

$$\big(\nabla(u - P_h^1 u), \nabla v_h\big)_{(L^2(\Omega))^d} = 0, \quad \forall v_h \in S_h(\Omega). \tag{3}$$

We need the following assumptions as the a priori error estimates for P_h^1.

Assumption 3.2 *There exists a positive constant $C_\Omega(h)$ satisfying*

$$\big\|u - P_h^1 u\big\|_{H_0^1(\Omega)} \le C_\Omega(h)\, \|\Delta u\|_{L^2(\Omega)}, \quad \forall u \in D(\Delta), \tag{4}$$

$$\big\|u - P_h^1 u\big\|_{L^2(\Omega)} \le C_\Omega(h)\, \big\|u - P_h^1 u\big\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{5}$$

For example, if Ω is a bounded open interval in $\mathbb{R}$, and $S_h(\Omega)$ is the P1 finite element space, then Assumption 3.2 is satisfied for $C_\Omega(h) = \frac{h}{\pi}$, where h is the mesh size (see e.g., [2, 6]).

Let $V_k^1(J)$ be a finite-dimensional subspace of $V^1(J)$ dependent on the discretization parameter k. For example, $V_k^1(J)$ is considered to be a finite element space with mesh size (time step size) k. Let m be the number of degrees of freedom for $V_k^1(J)$.

We assume that $\Pi_k : V^1(J) \to V_k^1(J)$ is a Lagrange interpolation operator. Namely, if the mesh points on J are taken as $0 = t_0 < t_1 < \cdots < t_m = T$, for any element $u \in V^1(J)$, $\Pi_k u \in V_k^1(J)$ satisfies

$$u(t_i) = \big(\Pi_k u\big)(t_i), \quad \forall i \in \{1, \ldots, m\}. \tag{6}$$

We need the following assumption as the a priori error estimate for Π_k.

Assumption 3.3 *There exists a positive constant $C_J(k)$ satisfying*

$$\|u - \Pi_k u\|_{L^2(J)} \le C_J(k) \|u\|_{V^1(J)}, \quad \forall u \in V^1(J). \tag{7}$$

For example, if $V_k^1(J)$ is the P1 finite element space, then Assumption 3.3 is satisfied by $C_J(k) = \frac{k}{\pi}$ (see e.g., [9, Theorem 2.4]).

Let $V^1\big(J; S_h(\Omega)\big)$ be a subspace of V corresponding to the semidiscretized approximation in the spatial direction, and the space $V_k^1\big(J; S_h(\Omega)\big)$ is defined as the tensor product $V_k^1(J) \otimes S_h(\Omega)$, which corresponds to a full discretization. We now define the semidiscretization operator $P_h : V \to V^1\big(J; S_h(\Omega)\big)$ by the following weak form for any $u \in V$

$$\left(\frac{\partial}{\partial t}\big(u - P_h u\big)(t), v_h\right)_{L^2(\Omega)} + \nu \left(\nabla\big(u - P_h u\big)(t), \nabla v_h\right)_{(L^2(\Omega))^d} = 0,$$
$$\forall v_h \in S_h(\Omega), t \in J. \tag{8}$$

Additionally, corresponding to the homogeneous initial condition, we impose the requirement $P_h u(\cdot, 0) = 0$.

We now define the symmetric and positive definite matrices L_φ and D_φ in $\mathbb{R}^{n\times n}$ by

$$L_{\varphi,i,j} := (\varphi_j, \varphi_i)_{L^2(\Omega)}, \quad D_{\varphi,i,j} := (\nabla\varphi_j, \nabla\varphi_i)_{(L^2(\Omega))^d}, \quad \forall i, j \in \{1, \ldots, n\}.$$

Let $\mathfrak{f} := (\mathfrak{f}_1, \ldots, \mathfrak{f}_n)^T \in L^2(J)^n$ be a vector function defined by $\mathfrak{f}_i := (f, \varphi_i)_{L^2(\Omega)}$. From the fact that $P_h u \in V^1\big(J; S_h(\Omega)\big)$, there exists a coefficient vector $\mathfrak{u} := (\mathfrak{u}_1, \ldots, \mathfrak{u}_n)^T \in V^1(J)^n$ such that

$$P_h u(x, t) = \sum_{j=1}^{n} \varphi_i(x)\mathfrak{u}_j(t) = \varphi(x)^T \mathfrak{u}(t),$$

where $\varphi := (\varphi_1, \ldots, \varphi_n)^T$. Then, the variational equation (8) is equivalent to the following system of linear ODEs with homogeneous initial condition:

$$L_\varphi \mathfrak{u}' + \nu D_\varphi \mathfrak{u} = \mathfrak{f}. \tag{9}$$

Noting that (9) is a system of nonhomogeneous linear ODEs with constant coefficients, by using the fundamental matrix of the system, we obtain

$$\mathfrak{u}(t) = \int_0^t \exp\left((s - t)\nu L_\varphi^{-1} D_\varphi\right) L_\varphi^{-1} \mathfrak{f}(s)\, ds. \tag{10}$$

Here, 'exp' means the exponential of a matrix. By using this representation, we define the full discretization $P_h^k u \in V_k^1\big(J; S_h(\Omega)\big)$ of (1) by the interpolation

$$P_h^k u(x, t_i) = \big(\Pi_k u_h\big)(x, t_i), \quad \forall x \in \Omega, \quad \forall i \in \{1, \ldots, m\}. \tag{11}$$

Thus, the full-discretization operator $P_h^k : V \to V_k^1\big(J; S_h(\Omega)\big)$ is defined as the composition of P_h and Π_k, that is, by $P_h^k := \Pi_k P_h$.

Theorem 3.1 (Theorem 5.5 & Theorem 5.6 in [4]) *Under Assumption 3.2, Assumption 3.1, and Assumption 3.3, we have the following constructive a priori error estimates:*

$$\left\|u - P_h^k u\right\|_{L^2 H_0^1} \le C_1(h,k) \|f\|_{L^2 L^2}, \quad \forall u \in V \cap L^2\big(J; D(\Omega)\big), \tag{12}$$

$$\left\|u - P_h^k u\right\|_{L^2 L^2} \le C_0(h,k) \|f\|_{L^2 L^2}, \quad \forall u \in V \cap L^2\big(J; D(\Omega)\big), \tag{13}$$

where

$$C_1(h,k) := \frac{2}{\nu} C_\Omega(h) + C_{inv}(h) C_J(k), \quad C_0(h,k) = \frac{8}{\nu} C_\Omega(h)^2 + C_J(k).$$

If we set $k = h^2$, since $C_\Omega(h) = O(h)$, $C_{\rm inv}(h) = O(h^{-1})$, and $C_J(k) = O(k)$, then $C_1(h,k) = O(h)$ and $C_0(h,k) = O(h^2)$. Therefore, the estimates (12) and (13) are optimal order estimates with $k = h^2$. However, these estimates are not optimal order with $k \neq h^2$, for example, in case of $k = h$.

Example 3.1 *Let $d = 1$ and $V_k^1\big(J; S_h(\Omega)\big)$ be the Q1 finite element space (i.e., spanned by piecewise bilinear basis functions [1, Section 3]), then Theorem 3.1 holds for*

$$C_1(h,k) = \frac{2}{\nu}\,\frac{h}{\pi} + \frac{\sqrt{12}}{h}\,\frac{k}{\pi}, \qquad C_0(h,k) = \frac{8}{\nu}\,\frac{h^2}{\pi^2} + \frac{k}{\pi}.$$

For example, $h = k$ leads to $C_1(h,k) \ge \frac{\sqrt{12}}{\pi}$ and $C_0(h,k) = O(h)$. These estimates are not optimal order with $k = h$.

4 Optimal Order Error Estimates

In the following text, we derive the optimal order error estimates with $k \neq h^2$.

Let $P_0 : L^2(\Omega) \to S_h(\Omega)$ be an L^2-projection satisfying for any $u \in L^2(\Omega)$

$$(u - P_0 u, v_h)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega).$$

Theorem 4.1 ([5, Theorem 4, 5]) *Under Assumption 3.2, the following constructive a priori error estimate holds:*

$$\|u - P_h u\|_{L^2 H_0^1} \le \frac{2}{\nu} C_\Omega(h) \|\Delta_t u\|_{L^2 L^2}, \quad \forall u \in V \cap L^2\big(J; D(\Omega)\big). \tag{14}$$

$$\|u - P_h u\|_{L^2 L^2} \le 4 C_\Omega(h) \|u - P_h u\|_{L^2 H_0^1}, \quad \forall u \in V. \tag{15}$$

Here, we consider the following constructive a priori H_0^1-error estimates.

Theorem 4.2 (H_0^1-error estimates) *Under Assumptions 3.2, 3.3, and that $\frac{\partial}{\partial t} f \in L^2(J; L^2(\Omega))$, the following inequality holds:*

$$\begin{aligned}
&\left\|u - P_h^k u\right\|_{L^2 H_0^1} \\
\le\ & C_\Omega(h) \frac{2}{\nu} \|f\|_{L^2 L^2} + C_J(k) \frac{1}{\sqrt{2\nu}} \sqrt{\|f(\cdot,0)\|_{L^2(\Omega)}^2 + \|f\|_{L^2 L^2}^2 + \left\|\frac{\partial}{\partial t} f\right\|_{L^2 L^2}^2} \\
\le\ & \widehat{C}_1(h,k) \left(\frac{2}{\nu} \|f\|_{L^2 L^2} + \frac{1}{\sqrt{2\nu}} \sqrt{\|f(\cdot,0)\|_{L^2(\Omega)}^2 + \|f\|_{L^2 L^2}^2 + \left\|\frac{\partial}{\partial t} f\right\|_{L^2 L^2}^2} \right),
\end{aligned}$$

where $\widehat{C}_1(h,k) := \max\{C_\Omega(h), C_J(k)\}$.

Proof: From the triangle inequality, Theorem 4.1, and Assumption 3.3, we have

$$\begin{aligned}\left\|u - P_h^k u\right\|_{L^2 H_0^1} &\leq \|u - P_h u\|_{L^2 H_0^1} + \|P_h u - \Pi_k P_h u\|_{L^2 H_0^1} \\ &\leq \frac{2}{\nu} C_\Omega(h) \|f\|_{L^2 L^2} + C_J(k) \left\|\frac{\partial}{\partial t}\nabla(P_h u)\right\|_{L^2 L^2}. \end{aligned} \tag{16}$$

From (8) we have

$$\left(\frac{\partial}{\partial t} P_h u, v_h\right)_{L^2(\Omega)} + \nu \left(\nabla(P_h u), \nabla v_h\right)_{(L^2(\Omega))^d} = (f, v_h)_{L^2(\Omega)}, \quad \forall v_h \in S_h, \quad t > 0, \tag{17}$$

noting that $P_h u(0) = 0$. Differentiating (17) by t and considering initial condition, we have

$$\left(\frac{\partial^2}{\partial t^2} P_h u, v_h\right)_{L^2(\Omega)} + \nu \left(\frac{\partial}{\partial t}\nabla P_h u, \nabla v_h\right)_{(L^2(\Omega))^d} = \left(\frac{\partial}{\partial t} f, v_h\right)_{L^2(\Omega)}, \quad \forall v_h \in S_h, \quad t > 0, \tag{18}$$

$$\frac{\partial}{\partial t} P_h u(0) = P_0 f(\cdot, 0). \tag{19}$$

By setting $v_h = \frac{\partial}{\partial t} P_h u$ in (18), integrating from 0 to t with condition (19) yields

$$\begin{aligned}&\left\|\frac{\partial}{\partial t} P_h u\right\|_{L^2(\Omega)}^2 + 2\nu \int_0^t \left\|\frac{\partial}{\partial t}\nabla P_h u\right\|_{L^2(\Omega)}^2 dt \\ \leq\ &\|P_0 f(\cdot, 0)\|_{L^2(\Omega)}^2 + \int_0^t \left\|\frac{\partial}{\partial t} f\right\|_{L^2(\Omega)}^2 dt + \int_0^t \left\|\frac{\partial}{\partial t} P_h u\right\|_{L^2(\Omega)}^2 dt\end{aligned} \tag{20}$$

Since

$$\begin{aligned}&\left\|\frac{\partial}{\partial t} P_h u\right\|_{L^2(\Omega)}^2 + \nu \frac{1}{2}\frac{\partial}{\partial t} \|\nabla P_h u\|_{L^2(\Omega)}^2 \\ =\ &\left(\frac{\partial}{\partial t} P_h u, \frac{\partial}{\partial t} P_h u\right) + \nu \left(\nabla P_h u, \nabla \frac{\partial}{\partial t} P_h u\right)_{(L^2(\Omega))^d} = \left(f, \frac{\partial}{\partial t} P_h u\right),\end{aligned}$$

we have

$$\int_0^t \left\|\frac{\partial}{\partial t} P_h u\right\|_{L^2(\Omega)}^2 dt \leq \int_0^t \|f\|_{L^2(\Omega)}^2 dt.$$

Thus taking notice of $\|P_0\|_{\mathcal{L}\left(L^2(\Omega), L^2(\Omega)\right)} \leq 1$, we obtain from (20)

$$\left\|\frac{\partial}{\partial t}\nabla(P_h u)\right\|_{L^2 L^2} \leq \frac{1}{\sqrt{2\nu}} \sqrt{\|f(\cdot, 0)\|_{L^2(\Omega)}^2 + \|f\|_{L^2 L^2}^2 + \left\|\frac{\partial}{\partial t} f\right\|_{L^2 L^2}^2}. \tag{21}$$

Therefore, (16) and (21) prove the desired estimates. □

Next, we consider the constructive a priori L^2-error estimates. Here, we assume that the following inequalities.

Assumption 4.1 *There exists a positive constant $C_J(k)$ satisfying*

$$\|u-\Pi_k u\|_{L^2(J)} \le C_J(k)^2 \left\|\frac{\partial^2}{\partial t^2}u\right\|_{L^2(J)}, \quad \forall u \in H^2(J). \tag{22}$$

$$\left\|\frac{\partial}{\partial t}(u-\Pi_k u)\right\|_{L^2(J)} \le C_J(k) \left\|\frac{\partial^2}{\partial t^2}u\right\|_{L^2(J)}, \quad \forall u \in H^2(J). \tag{23}$$

Theorem 4.3 (L^2-error estimates) *Under assumptions 3.2, 3.3, 4.1 and that $\frac{\partial}{\partial t}f \in L^2(J;L^2(\Omega))$ and $f(\cdot,0)\in H_0^1(\Omega)$, the following inequality holds:*

$$\begin{aligned}
&\left\|u-P_h^k u\right\|_{L^2L^2} \\
\le\ & \frac{8}{\nu}C_\Omega(h)^2\|f\|_{L^2L^2} + C_J(k)^2\sqrt{\nu\|\nabla P_0 f(\cdot,0)\|^2_{L^2(\Omega)} + \left\|\frac{\partial}{\partial t}f\right\|^2_{L^2L^2}} \\
\le\ & \widehat{C}_0(h,k)\left(\|f\|_{L^2L^2} + \left\|\frac{\partial}{\partial t}f\right\|_{L^2L^2} + \|\nabla P_0 f(\cdot,0)\|_{L^2(\Omega)}\right),
\end{aligned}$$

where $\widehat{C}_0(h,k) := \max\left\{\frac{8}{\nu}C_\Omega(h)^2, \sqrt{\nu}C_J(k)^2, C_J(k)^2\right\}$.

Proof: From the triangle inequality with Theorem 4.1 and Assumption 4.1, we have

$$\begin{aligned}
\left\|u-P_h^k u\right\|_{L^2L^2} &\le \|u-P_h u\|_{L^2L^2} + \|P_h u - \Pi_k P_h u\|_{L^2L^2} \\
&\le \frac{8}{\nu}C_\Omega(h)^2\|f\|_{L^2L^2} + C_J(k)^2\left\|\frac{\partial^2}{\partial t^2}P_h u\right\|_{L^2L^2}.
\end{aligned} \tag{24}$$

By setting $v_h = \frac{\partial^2}{\partial t^2}P_h u$ in (18) and integrating by t, we have

$$\begin{aligned}
&\int_0^t \left\|\frac{\partial^2}{\partial t^2}P_h u\right\|^2_{L^2(\Omega)} dt + \nu\left\|\nabla\frac{\partial}{\partial t}P_h u(t)\right\|^2_{L^2(\Omega)} dt \\
\le\ & \nu\left\|\nabla\frac{\partial}{\partial t}P_h u(0)\right\|^2_{L^2(\Omega)} + \int_0^t \left\|\frac{\partial}{\partial t}f\right\|^2_{L^2(\Omega)} dt \\
=\ & \nu\|\nabla P_0 f(\cdot,0)\|^2_{L^2(\Omega)} + \int_0^t \left\|\frac{\partial}{\partial t}f\right\|^2_{L^2(\Omega)} dt.
\end{aligned} \tag{25}$$

Here, we have used the fact that

$$\nabla\frac{\partial}{\partial t}P_h u(0) \equiv \lim_{t\to 0}\nabla\frac{\partial}{\partial t}P_h u(t) = \nabla P_0 f(\cdot,0) \in (L^2(\Omega))^d.$$

Therefore, combining (25) with (24), we have the desired result. □

Now, to get the optimal-order V^1-estimates, we need the following lemma.

Lemma 4.1 *Let u be a solution of (1). Assuming that $\frac{\partial}{\partial t}f \in L^2(J;L^2(\Omega))$ and $f(\cdot,0)\in H_0^1(\Omega)$, the following inequalities hold.*

$$\left\|\Delta\frac{\partial}{\partial t}u\right\|^2_{L^2L^2} \le \frac{2}{\nu^2}\left(2\left\|\frac{\partial}{\partial t}f\right\|^2_{L^2L^2} + \nu\|\nabla f(\cdot,0)\|^2_{L^2(\Omega)}\right) \tag{26}$$

Since the proof of this lemma follows by some standard arguments for the solution of the equation (1) and its differentiated form in t, we omit it.

We now present the following optimal order V^1-error estimates.

Theorem 4.4 (V^1-error estimates) *Under the same assumptions in Theorem 4.3, we have the estimates*

$$\left\| u - P_h^k u \right\|_{V^1 L^2} \le 4C_\Omega(h)^2 \frac{\sqrt{2}}{\nu} \left\{ 2 \left\| \frac{\partial}{\partial t} f \right\|_{L^2 L^2}^2 + \nu \left\| \nabla f(\cdot, 0) \right\|_{L^2(\Omega)}^2 \right\}^{\frac{1}{2}} + C_J(k) \left\{ \left\| \frac{\partial}{\partial t} f \right\|_{L^2 L^2}^2 + \nu \left\| \nabla P_0 f(\cdot, 0) \right\|_{L^2(\Omega)}^2 \right\}^{\frac{1}{2}}. \quad (27)$$

Proof: We only describe a sketch of the proof. As before, we use the following triangle inequality:

$$\begin{aligned} \left\| u - P_h^k u \right\|_{V^1 L^2} &= \left\| \frac{\partial}{\partial t}(u - P_h u + P_h u - \Pi_k P_h u) \right\|_{L^2 L^2} \\ &\le \left\| \frac{\partial}{\partial t}(u - P_h u) \right\|_{L^2 L^2} + \left\| \frac{\partial}{\partial t}(P_h u - \Pi_k P_h u) \right\|_{L^2 L^2}. \end{aligned} \quad (28)$$

First, we estimate the first term in the right hand side of the above, which is done by using techniques similar to that in the proof of Theorem 5 in [5].

We consider the following dual problem for the original equation (1)

$$\begin{cases} \dfrac{\partial w}{\partial t} + \nu \Delta w = e_t, & \text{in } \Omega \times J, \\ w(x,t) = 0, & \text{on } \partial\Omega \times J, \\ w(x,T) = 0, & \text{in } \Omega, \end{cases} \quad (29)$$

where $e_t := \frac{\partial}{\partial t}(u - P_h u)$. By arguments analogous to those in [5], we have the following estimates for the time derivative of error $u - P_h u$:

$$\| e_t \|_{L^2 L^2} \le 4C_\Omega(h)^2 \left\| \Delta \frac{\partial}{\partial t} u \right\|_{L^2 L^2}. \quad (30)$$

Therefore, by using the estimate (26) in Lemma 4.1, we have the bound

$$\left\| \frac{\partial}{\partial t}(u - P_h u) \right\|_{L^2 L^2} \le 4C_\Omega(h)^2 \frac{\sqrt{2}}{\nu} \left\{ 2 \left\| \frac{\partial}{\partial t} f \right\|_{L^2 L^2}^2 + \nu \left\| \nabla f(\cdot, 0) \right\|_{L^2(\Omega)}^2 \right\}^{\frac{1}{2}}. \quad (31)$$

On the other hand, the second term of the right hand side of (28) is estimated as follows: By using the estimates (23) in the Assumption 4.1 and (25) for $\frac{\partial^2}{\partial t^2} P_h u$, we have

$$\begin{aligned} \left\| \frac{\partial}{\partial t}(P_h u - \Pi_k P_h u) \right\|_{L^2 L^2} &\le C_J(k) \left\| \frac{\partial^2}{\partial t^2} P_h u \right\|_{L^2 L^2} \\ &\le C_J(k) \left\{ \nu \left\| \nabla f(\cdot, 0) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial t} f \right\|_{L^2 L^2}^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (32)$$

Combining the estimates (31) and (32) with (28), we have the desired result. □

5 Numerical Examples

In this section, we show several numerical results by three kinds of proposed estimates and two existing estimates. We set f to be the exact solution such that $u(x,t) = \sin(\pi x)\sin(\pi t)$ and parameter $\nu = 1$. In these examples, we used the finite element subspaces $S_h(\Omega)$ and $V_k^1(J)$ spanned by piecewise linear basis functions with uniform mesh size h and k, respectively. Since the exact solutions are known, the upper bounds of the exact errors for approximate solutions can be validated in the a posteriori sense.

All computations are carried out on MATLAB R12a by using INTLAB 9 [8] to take care of rounding errors. INTLAB is a MATLAB toolbox for interval arithmetic.

Tables 1 - 2 illustrate the results of $L^2(J;H_0^1(\Omega))$ error estimates, namely, Theorem 5.5 in [4], Theorem 4.2, and $L^2(J;H_0^1(\Omega))$ norm of exact error. These tables show the estimates presented in Theorem 4.2 give the optimal order $O(h)$, even if the mesh size $k \neq h^2$, but $k = h$.

Table 1: $L^2(J;H_0^1(\Omega))$ error estimates for $k = h^2$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 5.5 in [4]		Theorem 4.2		exact error
h	$C_1(h,k)$	error bound	$\hat{C}_1(h,k)$	error bound	
1/2	0.870	4.50	0.1592	2.62	0.690
1/4	0.435	2.25	0.0796	1.07	0.353
1/8	0.217	1.13	0.0398	0.48	0.178
1/16	0.109	0.57	0.0199	0.22	0.089

Table 2: $L^2(J;H_0^1(\Omega))$ error estimates for $k = h$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 5.5 in [4]		Theorem 4.2		exact error
h	$C_1(h,k)$	error bound	$\hat{C}_1(h,k)$	error bound	
1/2	1.42	7.36	0.1591	3.59	0.752
1/4	1.26	6.53	0.0796	1.79	0.362
1/8	1.18	6.12	0.0398	0.90	0.179
1/16	1.14	5.92	0.0199	0.45	0.090

Next, Tables 3 - 4 illustrate the results of $L^2(J;L^2(\Omega))$ error estimates, namely, Theorem 5.6 in [4], Theorem 4.3, and $L^2(J;L^2(\Omega))$ norm of exact error. These tables show the L^2-error estimates in Theorem 4.3 also present the optimal order $O(h^2)$ independent of mesh size for space and time directions. This optimality comes from the well known interpolation theory on the approximation by piecewise linear functions [7].

Finally, Tables 5 - 6 show the results of $V^1(J;L^2(\Omega))$ error estimates namely, Theorem 4.4 and $V^1(J;L^2(\Omega))$ norm of exact error. These tables confirm that we can also attain the optimal order error estimates for V^1-error, which is actually $O(h^2)$ if we take mesh size as $k = h^2$. We can say this fact exceeds our usual expectation.

Table 3: $L^2(J; L^2(\Omega))$ error estimates for $k = h^2$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 5.6 in [4]		Theorem 4.3		exact
h	$C_0(h,k)$	error bound	$\hat{C}_0(h,k)$	error bound	error
1/2	0.282	1.460	0.2026	1.0800	1.23E-01
1/4	0.071	0.365	0.0507	0.2640	2.83E-02
1/8	0.018	0.092	0.0127	0.0657	6.86E-03
1/16	0.004	0.023	0.0032	0.0164	1.70E-03

Table 4: $L^2(J; L^2(\Omega))$ error estimates for $k = h$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 5.6 in [4]		Theorem 4.3		exact
h	$C_0(h,k)$	error bound	$\hat{C}_0(h,k)$	error bound	error
1/2	0.362	1.870	0.2026	1.1700	1.82E-01
1/4	0.130	0.674	0.0507	0.2931	5.06E-02
1/8	0.053	0.272	0.0127	0.0732	1.30E-02
1/16	0.024	0.119	0.0032	0.0183	3.27E-03

Table 5: $V^1(J; L^2(\Omega))$ error estimates for $k = h^2$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 4.4	exact
h	error bound	error
1/2	4.88E+00	4.56E-01
1/4	1.21E+00	1.16E-01
1/8	3.03E-01	2.91E-02
1/16	7.58E-02	7.28E-03

Table 6: $V^1(J; L^2(\Omega))$ error estimates for $k = h$. $u(x,t) = \sin(\pi x)\sin(\pi t)$

	Theorem 4.4	exact
h	error bound	error
1/2	6.31E+00	7.34E-01
1/4	2.28E+00	3.60E-01
1/8	9.20E-01	1.79E-01
1/16	4.06E-01	8.91E-02

6 Conclusion

The error estimates presented here are sharper than the existing estimates in [4]. Particularly, the optimal order V^1 estimates are considered as a special advantage from the fact that we used the approximation scheme in the time direction by using the fundamental matrix for ODE corresponding to semidiscretization in space.

Acknowledgements

The authors would like to thank two anonymous reviewers for their very useful comments to improve this article.

References

[1] Claes Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Dover Publications, New York, 2009.

[2] Seiji Kimura and Nobito Yamamoto. On explicit bounds in the error for the $\{H_0^1\}$-projection into piecewise polynomial spaces. *Bulletin of Informatics and Cybernetics*, 31(2):109–115, 1999.

[3] Takehiko Kinoshita, Takuma Kimura, and Mitsuhiro T. Nakao. On the a posteriori estimates for inverse operators of linear parabolic equations with applications to the numerical enclosure of solutions for nonlinear problems. *Numerische Mathematik*, 126(4):679–701, 2014.

[4] Mitsuhiro T Nakao, Takuma Kimura, and Takehiko Kinoshita. Constructive a priori error estimates for a full discrete approximation of the heat equation. *SIAM Journal on Numerical Analysis*, 51(3):1525–1541, 2013.

[5] Mitsuhiro T Nakao, Takehiko Kinoshita, and Takuma Kimura. On a posteriori estimates of inverse operators for linear parabolic initial-boundary value problems. *Computing*, 94(2-4):151–162, 2012.

[6] Mitsuhiro T Nakao, Nobito Yamamoto, and Seiji Kimura. On the best constant in the error bound for the h_0^1-projection into piecewise polynomial spaces. *Journal of Approximation Theory*, 93(3):491–500, 1998.

[7] John T. Oden and Junuthula N. Reddy. *An Introduction to the Mathematical Theory of Finite Elements*. John Wiley & Sons, New York, 1976.

[8] Siegfried M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. `http://www.ti3.tuhh.de/rump/`.

[9] Martin H Schultz. *Spline Analysis*. Prentice-Hall, Englewood Cliffs, N.J., 1973.