Validated Constructive Error Estimations for Biharmonic Problems*

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Abstract
This paper presents some constructive error estimates for two-dimensional biharmonic equations by using verified computational techniques. These estimations are expected to provide valuable information for computer-assisted proofs of nonlinear biharmonic problems. Several numerical examples that confirm the effectiveness are reported.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. This paper provides a guaranteed error bound for finite-dimensional approximate solutions for the biharmonic problem

$$\begin{align*}
\Delta^2 u &= f \quad \text{in } \Omega, \\
u &= \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega
\end{align*} \quad (1)
$$

for $f \in L^2(\Omega)$. Here, $\partial u/\partial n$ stands for the outer normal derivative of $u$. The biharmonic problem (1) arises in areas of continuum mechanics, including linear elasticity

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In the one-dimensional case in which the domain is \( J := (a, b) \), several a priori error estimates satisfying

\[
\|u'' - u''_h\|_{L^2(J)} \leq \tilde{C}(h) \|u'''\|_{L^2(J)}
\]

have been presented \([2, 10]\) with numerically determined values for \( \tilde{C}(h) > 0 \). Then, for a rectangular domain such that \( \Omega = J \times J \), by using the estimation \([10]\), the inequality

\[
\|u - u_h\|_{H^2(\Omega)} \leq \tilde{C}(h)|u|_{H^4(\Omega)}
\]
can be derived with the $H^4$ semi-norm:

$$|u|_{H^4(\Omega)} := \left( \|u_{xxxx}\|_{L^2(\Omega)}^2 + 4\|u_{xxyy}\|_{L^2(\Omega)}^2 + 6\|u_{xxyy}\|_{L^2(\Omega)}^2 + 4\|u_{xyyy}\|_{L^2(\Omega)}^2 + \|u_{yyyy}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$  

However, it is not so easy to obtain a numerically determined upper bound $C > 0$ such that

$$|u|_{H^4(\Omega)} \leq C \|\Delta^2 u\|_{L^2(\Omega)}, \quad (12)$$

even if the domain $\Omega$ is a rectangle.

**Remark 1** For example, when $\Omega$ is a unit square, by using the Fourier expansion in which $u = \sum_{m,n=1}^{\infty} a_{mn} \psi_{mn}$ with $\psi_{mn} := \sin(m\pi x)\sin(n\pi y)/2$, it may appear that (12) has been achieved with $C = 1$. It is true if $\hat{a}_{mn} = (\Delta^2 u, \psi_{mn})_{L^2(\Omega)}$ can be restored with $a_{mn} = (u, \psi_{mn})_{L^2(\Omega)}$ by partial integration and with the boundary condition $u = \partial u/\partial n = 0$. It has been reported that if $u \in H^4(\Omega)$ satisfies $u = \Delta u = 0$ on $\partial \Omega$, (12) holds when $C = 1$ [3].

To avoid the need to estimate (12), Nakao et al. [7] proposed a technique that directly determines the constant in the constructive a priori and a posteriori error estimates of (15); they do this by using the finite element approximation. Their procedure is based on verified computational techniques that use the Hermite spline functions for a two-dimensional rectangular domain; several numerical examples have confirmed the effectiveness of this approach.

In this paper, we take another computer-assisted approach that is expected to be applicable to a wide variety of approximation subspaces $S_h \subset H^3_0(\Omega)$.

This paper is organized as follows. Section 2 introduces the notation and several projections with related constants. Section 3 is devoted to some constructive error estimations of biharmonic problems. Several numerical examples are reported in Section 4.

## 2 Assumptions and Related Notation

We define the $H^3_0$-projection $P_1 : H^3_0(\Omega) \to S_h$ and the $L^2$-projection $P_0 : L^2(\Omega) \to S_h$ by

$$\left( \nabla (\varphi - P_1 \varphi), \nabla v_h \right)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h, \quad (13)$$

$$\left( \varphi - P_0 \varphi, v_h \right)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h, \quad (14)$$

and we assume that the $H^3_0$-projection $P_1$ has the following approximation property:

$$\|v - P_1 v\|_{L^2(\Omega)} \leq C_0(h) \|\Delta^2 v\|_{L^2(\Omega)}, \quad \forall v \in D(\Delta^2). \quad (15)$$

Here, $C_0(h) > 0$ is a positive constant that is numerically determined such that $C_0(h) \to 0$ as $h \to 0$. Using $C_0(h)$ of (15), we aim to construct $C(h)$ satisfying [9], namely [5].
We assume that the finite-dimensional approximation subspace $S_h$ belongs to $D(\Delta^2)$, and we define the basis function of $S_h$ by $\{\varphi_i\}_{i=1}^K$ for $K := \dim S_h$ and $K \times K$ matrices $A_0$, $A_1$, $A_2$, $A_3$, and $A_4$:

\[
[A_0]_{ij} = (\varphi_j, \varphi_i)_{L^2(\Omega)}, \quad (16)
\]
\[
[A_1]_{ij} = (\Delta \varphi_j, \varphi_i)_{L^2(\Omega)} = -(\nabla \varphi_j, \nabla \varphi_i)_{L^2(\Omega)}, \quad (17)
\]
\[
[A_2]_{ij} = (\Delta \varphi_j, \Delta \varphi_i)_{L^2(\Omega)}, \quad (18)
\]
\[
[A_3]_{ij} = (\Delta^2 \varphi_j, \varphi_i)_{L^2(\Omega)}, \quad (19)
\]
\[
[A_4]_{ij} = (\Delta^2 \varphi_j, \Delta^2 \varphi_i)_{L^2(\Omega)}. \quad (20)
\]

The matrices $A_0$, $A_1$, $A_2$, and $A_4$ are symmetric and nonsingular. Because $A_0$ is positive definite, it can be decomposed as $A_0 = A_0^{1/2} A_0^{T/2}$, where $T$ indicates the transposition, and $A_0^{T/2}$ means $(A_0^{1/2})^T$. Usually, $A_0^{1/2}$ is a lower triangular matrix.

For each $u \in D(\Delta^2)$, by representing the $L^2$-projection $P_0 \Delta^2 u \in S_h$ by (14) and the $H_0^2$-projection $P_2 u \in S_h$ by (7) as

\[
P_0 \Delta^2 u = \sum_{i=1}^K v_i \varphi_i, \quad \mathbf{v} = [v_i] \in \mathbb{R}^K, \quad (21)
\]
\[
P_2 u = \sum_{i=1}^K u_i \varphi_i, \quad \mathbf{u} = [u_i] \in \mathbb{R}^K, \quad (22)
\]

the definition of projections $P_0$ and $P_2$ state that

\[
(P_0 \Delta^2 u, \varphi_i)_{L^2(\Omega)} = (\Delta^2 u, \varphi_i)_{L^2(\Omega)}
\]
\[
= (\Delta u, \Delta \varphi_i)_{L^2(\Omega)}
\]
\[
= (\Delta P_2 u, \Delta \varphi_i)_{L^2(\Omega)}
\]
\[
= (P_0 \Delta^2 P_2 u, \varphi_i)_{L^2(\Omega)}
\]

for all $1 \leq i \leq K$; then, it holds that

\[
\mathbf{u} = A_2^{-1} A_0 \mathbf{v}. \quad (23)
\]

We also assume that an element

\[
\chi_h = \sum_{i=1}^K w_i \varphi_i \in S_h, \quad \mathbf{w} = [w_i] \in \mathbb{R}^K \quad (24)
\]

can be expressed as

\[
\mathbf{w} = F \mathbf{v}, \quad (25)
\]

where $\mathbf{v}$ is defined in (21) and $F \in \mathbb{R}^{K \times K}$. The element $\chi_h \in S_h$ is introduced by Lemma 3.1 in the next section, and the relation (25) between $\mathbf{w}$ for $\chi_h$ and $\mathbf{v}$ for $P_0 \Delta^2 u$ will be presented in connection with Lemmas 3.2 and 3.3 in the next section.
Finally, we define matrices
\[ Q_1 := A_0^{-1/2} A_1 F A_0^{-T/2}, \]
\[ Q_2 := -A_0^{T/2} A_2^{-1} A_1^T F A_0^{-T/2}, \]
\[ Q_3 := A_0^{T/2} A_3^{-1} A_4 A_2^{-1} A_0^{T/2}, \]
\[ Q_4 := A_0^{1/2} F^T A_4 F A_0^{-T/2}, \]
\[ B_1 := Q_2 + Q_T^2 + Q_3 + Q_4, \]
\[ B_2 := Q_1 + Q_T^1 + Q_2 + Q_T^2 + Q_3 + Q_4 - I, \]
where \( I \) stands for the identity matrix.

3 Constructive Error Estimations of Biharmonic Problems

For the error estimation of the \( P_2 \)-projection \( \tilde{u} \) with \( C_0(h) \), we begin by showing the following lemma.

Lemma 3.1 For each \( u \in D(\Delta^2) \) and \( \chi_h \in S_h \), it is true that
\[ \| u - P_2u \|_{H^2(\Omega)} \leq C_0(h) \| \Delta^2(u - P_2u) + \Delta \chi_h \|_{L^2(\Omega)}. \] (32)

Proof: Set \( u_\perp = u - P_2u \in D(\Delta^2) \). Using \( [7] \), two partial integrations, \( [13] \), the Cauchy-Schwarz inequality, and \( [15] \), we have
\[ \| \Delta u_\perp \|_{L^2(\Omega)}^2 = (\Delta u_\perp, \Delta u_\perp)_{L^2(\Omega)} = (\Delta u_\perp, \Delta(u_\perp - P_1 u_\perp))_{L^2(\Omega)} = -(\nabla \Delta u_\perp, \nabla(u_\perp - P_1 u_\perp))_{L^2(\Omega)} = -(\nabla (\Delta u_\perp + \chi_h), \nabla(u_\perp - P_1 u_\perp))_{L^2(\Omega)} = (\Delta^2 u_\perp + \Delta \chi_h, u_\perp - P_1 u_\perp)_{L^2(\Omega)} \leq \| \Delta^2 u_\perp + \Delta \chi_h \|_{L^2(\Omega)} \| u_\perp - P_1 u_\perp \|_{L^2(\Omega)} \leq \| \Delta^2 u_\perp + \Delta \chi_h \|_{L^2(\Omega)} C_0(h) \| \Delta u_\perp \|_{L^2(\Omega)}, \]
which implies (32). \( \Box \)

Note that (32) holds for any \( \chi_h \in S_h \) and there are some choice of \( \chi_h \) depending on the finite-dimensional subspace \( S_h \). We show several concrete examples of \( \chi_h \) in the last section.

Now, we consider the estimation of \( C_1(h) > 0 \) satisfying
\[ \| \Delta^2(u - P_2u) + \Delta \chi_h \|_{L^2(\Omega)} \leq C_1(h) \| \Delta^2 u \|_{L^2(\Omega)}. \] (33)

We show two approaches for \( C_1(h) \) satisfying (33). The choice will depend on \( S_h \) and the computational cost. The following lemma is one of the approaches.

Lemma 3.2 The constant \( C_1(h) > 0 \) of (33) can be taken as
\[ C_1(h) = 1 + \sqrt{\| B_1 \|_2}. \] (34)
Proof: Because
\[
\|\Delta^2(u - P_2u) + \Delta \chi_h \|_{L^2(\Omega)} \leq \|\Delta^2 u\|_{L^2(\Omega)} + \|\Delta^2 P_2 u - \Delta \chi_h\|_{L^2(\Omega)},
\]
using (20), (19), (18), (22), (24), (25), (23), (28), (27), (29), and (30) we obtain
\[
\|\Delta^2 P_2 u - \Delta \chi_h\|_{L^2(\Omega)}^2 = (\Delta^2 P_2 u - \Delta \chi_h, \Delta^2 P_2 u - \Delta \chi_h)_{L^2(\Omega)}
\]
which, by (36) and Hölder’s inequality, yields
\[
\|\Delta^2 P_2 u - \Delta \chi_h\|_{L^2(\Omega)}^2 \leq u^T A_4 u - w^T A_3 u - u^T A_1 w + w^T A_2 w
\]
\[
= v^T A_0 A_2^{-1} A_4 A_2^{-1} A_0 v - v^T F^T A_3 A_2^{-1} A_0 v - v^T A_0 A_2^{-1} A_3^T F v + v^T F^T A_2 F v
\]
\[
= (A_0^{T/2} v)^T \left( A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2} - A_0^{1/2} F^T A_3 A_2^{-1} A_0^{1/2}
\right)
\]
\[
- A_0^{T/2} A_2^{-1} A_3^T F A_0^{-T/2} + A_0^{-1/2} F^T A_2 F A_0^{-T/2} \right) A_0^{T/2} v
\]
\[
= (A_0^{T/2} v)^T \left( Q_2 + Q_4^T \right) A_0^{T/2} v
\]
\[
= (A_0^{T/2} v)^T B_1 A_0^{T/2} v
\]
\[
\leq B_1 \|z(A_0^{T/2} v)^T A_0^{T/2} v\|
\]
\[
= B_1 \|z v^T A_0 v\|
\]
\[
\leq \|z\| \|\Delta^2 u\|_{L^2(\Omega)}^2
\]
\[
\leq \|z\| \|\Delta^2 u\|_{L^2(\Omega)}^2.
\]
then the conclusion. \hfill \Box

Remark 2 In the case of \(\chi_h = 0\), we can take
\[
C_1(h) = 1 + \sqrt{\|A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2}\|_2^2},
\]
which based on Lemma 3.2 and then \(\|A_0^{T/2} A_2^{-1} A_4 A_2^{-1} A_0^{1/2}\|_2\) coincides with the maximum eigenvalue of the matrix \(A_2^{-1} A_4 A_2^{-1} A_0\). For the verified bounds for the 2-norm (spectral norm) of a matrix, see [5].

Now we show an alternative to Lemma 3.2

Lemma 3.3 The constant \(C_2(h) > 0\) of \[[33]\] can be taken as
\[
C_1(h) = \sqrt{1 + \|B_2\|_2}.
\]

Proof: When there exists \(K_h > 0\) satisfying
\[
\|P_0 \Delta^2 u - \Delta^2 P_2 u + \Delta \chi_h\|_{L^2(\Omega)} \leq K_h \|P_0 \Delta^2 u\|_{L^2(\Omega)},
\]
using (36) and Hölder’s inequality, we obtain
\[
\|\Delta^2(u - P_2 u) + \Delta \chi_h\|_{L^2(\Omega)} = \|(I - P_0) \Delta^2 u + P_0 \Delta^2 u - \Delta^2 P_2 u + \Delta \chi_h\|_{L^2(\Omega)}
\]
\[
\leq \|(I - P_0) \Delta^2 u\|_{L^2(\Omega)} + K_h \|P_0 \Delta^2 u\|_{L^2(\Omega)}
\]
\[
\leq \sqrt{1 + K_h^2} \sqrt{\|(I - P_0) \Delta^2 u\|_{L^2(\Omega)}^2 + \|P_0 \Delta^2 u\|_{L^2(\Omega)}^2}
\]
\[
= \sqrt{1 + K_h^2} \|\Delta^2 u\|_{L^2(\Omega)}.
\]
For $K_h$ satisfying (36), using partial integration and (16), (18), (19), and (20), we have

\[
\|P_0\Delta^2 u - \Delta^2 P_0 u + \Delta \chi_h\|^2_{L^2(\Omega)}
= (P_0\Delta^2 u - \Delta^2 P_0 u + \Delta \chi_h, P_0\Delta^2 u - \Delta^2 P_0 u + \Delta \chi_h)_{L^2(\Omega)}
= (P_0\Delta^2 u, P_0\Delta^2 u)_{L^2(\Omega)} - (P_0\Delta^2 u, \Delta \chi_h)_{L^2(\Omega)} + (\Delta \chi_h, P_0\Delta^2 u)_{L^2(\Omega)}
\]

Therefore, we can take

\[
\|A_0^{1/2} A_0^{T/2} v\|_{L^2(\Omega)} \leq \|B_2\| \|A_0^{T/2} A_0^{1/2} v\|_{L^2(\Omega)}.
\]

Then, noting that $A_0 = A_0^{1/2} A_0^{T/2}$, (22) and (20) can be used to derive

\[
\|P_0\Delta^2 u - \Delta^2 P_0 u + \Delta \chi_h\|^2_{L^2(\Omega)}
= v^T A_0 v - v^T A_0 A_0^{-1} A_0 v + v^T F^T A_1 v
- v^T A_2 A_2^{-1} A_2 v + v^T A_0 A_0^{-1} A_4 A_2^{-1} A_0 v
- v^T A_1 A_1 v - v^T A_0 A_0^{-1} A_3 A_2^{-1} A_0 v
\]

\[
\leq \|A_0^{1/2} v\|^2_{L^2(\Omega)} \|B_2\| \|A_0^{T/2} A_0^{1/2} v\|_{L^2(\Omega)}.
\]

Therefore, we can take $K_h^2 = \|B_2\|_2$.

\[\square\]

**Remark 3** In the case of $\chi_h = 0$ in Lemma 3.3, we can take

\[
C_1(h) = \sqrt{1 + \|A_0^{1/2} A_0^{-1} A_4 A_2^{-1} A_0^{1/2} - I\|_2}.
\]

Lemma 3.1, Lemma 3.2, and Lemma 3.3 imply our main result.

**Theorem 3.1** For the solution $u \in D(\Delta^2)$ of the biharmonic equation (1) and the approximate solution $u_h \in S_h$ satisfying (6), it is true that

\[
\|u - u_h\|_{H^2_0(\Omega)} \leq C(h) \|f\|_{L^2(\Omega)},
\]  
with

\[
C(h) := C_0(h) C_1(h),
\]

where $C_1(h)$ is given constructively by (34) or (35).
4 Numerical Examples

In this section, we report several numerical examples of a finite-dimensional approximation of $H^2_0(\Omega)$ by Legendre polynomials [2] on the unit square domain $\Omega = (0, 1) \times (0, 1)$. For $N > 0$, define

$$\psi_n(x) := \frac{(-1)^{n+1}\sqrt{2n+3}}{(n+1)!} \left( \frac{d}{dx} \right)^{n-1} (x-x^2)^{n+1}, \quad 1 \leq n \leq N, \quad (41)$$

and

$$\varphi_k(x, y) := \psi_m(x) \times \psi_n(y), \quad (42)$$

with some change of indices $(m, n) \rightarrow k$. Then, we can assure that $K = N^2$, $h = 1/N$, and $S_h = \text{span} \{ \varphi_k \}_{k=1}^K$ is a finite-dimensional subspace of $H^2_0(\Omega)$ satisfying $S_h \subset D(\Delta^2)$. Moreover, $C_0(h) > 0$ of (15) can be taken as

$$C_0(h) = \begin{cases} \sqrt{c_2(N+3)/4} & \text{if } 1 \leq N \leq 16, \\ \sqrt{c_3(N+3)/4} & \text{if } N \geq 17, \end{cases} \quad (43)$$

where

$$c_2(L) := \frac{2}{\sqrt{2L-5(2L-3)^2\sqrt{2L-1}}(2L+1)} + \frac{4}{(2L-3)\sqrt{2L-1}(2L+1)\sqrt{2L+3(2L+5)}} + \frac{1}{\sqrt{2L-1}(2L+1)(2L+3)(2L+5)\sqrt{2L+7}} + \frac{10L-3}{(2L-3)^2(2L-1)(2L+1)(2L+3)}, \quad (44)$$

and

$$c_3(L) := \frac{1}{\sqrt{2L-5(2L-3)(2L-1)(2L+1)\sqrt{2L+3}}} + \frac{4}{(2L-3)\sqrt{2L-1}(2L+1)\sqrt{2L+3(2L+5)}} + \frac{6}{(2L-1)(2L+1)(2L+5)(2L+7)} + \frac{4}{(2L+1)\sqrt{2L+3(2L+5)\sqrt{2L+7(2L+9)}}} + \frac{1}{\sqrt{2L+3(2L+5)(2L+7)(2L+9)\sqrt{2L+11}}}, \quad (45)$$

Note that by using Theorem 3.7 in [2], it would be possible to further improve $C_0(h)$. Table 1 shows the bounds of $C_1(h)$ obtained by Wolfram Mathematica 10.0.2.0 with 100-digit multiple precision. To avoid rounding-error effects, this should be confirmed analytically, which can be accomplished by interval arithmetic software (e.g., [4, 9]). In Table 1, we consider three types of the matrix $F$. The notation "0" indicates $\chi_h = 0$, "$A_1^{-1}A_3^{-1}A_0^{-1}$" indicates that $u$ in (24) satisfies

$$(\Delta \chi_h - \Delta^2 P_2u, \Delta \varphi_i)_{L^2(\Omega)} = 0, \quad 1 \leq i \leq K,$$
which ensures that $Q_2 + Q_4 = 0$, and $A_2^{-1}(A_3 A_2^{-1} A_0 - A_1)$ indicates that $w$ is taken such that $$ (\Delta \chi_h - \Delta^2 P_2 u + P_0 \Delta^2 u, \Delta \varphi_i)_{L^2(\Omega)} = 0, \quad 1 \leq i \leq K. $$

The simplest case, $F = 0$, is very unstable; in other cases, there is some improvement in $C_1(h)$.

Table 1: Constructive constants of $C_1(h)$ in Lemma 3.2 and Lemma 3.3.

<table>
<thead>
<tr>
<th>$F$</th>
<th>$A_2^{-1} A_3 A_2^{-1} A_0$</th>
<th>$A_2^{-1}(A_3 A_2^{-1} A_0 - A_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>Lemma 2</td>
<td>Lemma 3</td>
</tr>
<tr>
<td>5</td>
<td>3.3305</td>
<td>2.3305</td>
</tr>
<tr>
<td>10</td>
<td>5.7256</td>
<td>4.7256</td>
</tr>
<tr>
<td>15</td>
<td>8.6612</td>
<td>7.6612</td>
</tr>
<tr>
<td>20</td>
<td>12.0622</td>
<td>11.0622</td>
</tr>
</tbody>
</table>

Table 2 shows the bounds of each constant by using Lemma 3 with $F = A_2^{-1}(A_3 A_2^{-1} A_0 - A_1)$.

$C(h)$ seems to be approximately $O(h)$, which means it should provide a “good” verification of nonlinear biharmonic problems.

Table 2: Constructive error estimates for the biharmonic equation.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$C(h)$</th>
<th>$C_0(h)$</th>
<th>$C_1(h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$3.7742 \times 10^{-3}$</td>
<td>$1.3117 \times 10^{-4}$</td>
<td>2.8774</td>
</tr>
<tr>
<td>20</td>
<td>$2.2329 \times 10^{-3}$</td>
<td>$4.2161 \times 10^{-4}$</td>
<td>5.2962</td>
</tr>
<tr>
<td>30</td>
<td>$1.6453 \times 10^{-3}$</td>
<td>$2.1133 \times 10^{-4}$</td>
<td>7.7851</td>
</tr>
<tr>
<td>40</td>
<td>$1.3051 \times 10^{-3}$</td>
<td>$1.2672 \times 10^{-4}$</td>
<td>10.2997</td>
</tr>
<tr>
<td>50</td>
<td>$1.0823 \times 10^{-3}$</td>
<td>$8.4375 \times 10^{-5}$</td>
<td>12.8265</td>
</tr>
</tbody>
</table>

It is not clear why $C_1(h)$ shows a tendency to become large as $h \to 0$. As an area of future work, we intend to investigate much finer spacing of $F$ for $C(h)$ and to use another finite-dimensional basis, e.g., finite element functions; we also will try to verify these solutions of nonlinear biharmonic equations, especially the two-dimensional Navier-Stokes equations.

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