Toward the Optimal Parameterization of Interval-Based Variable-Structure State Estimation Procedures*

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Abstract

Feedback control strategies for continuous-time dynamic systems rely, on the one hand, on a mathematical system model given as a set of (ordinary) differential equations and, on the other hand, on knowledge about the current state variables and system parameters. However, most practical applications are characterized by the fact that not all state variables are directly measurable and that system parameters are either only imprecisely known or may change their values during system operation. This typically leads to the necessity to determine the before-mentioned non-measurable quantities by means of model-based online estimation procedures, for which a guaranteed asymptotically stable convergence to the true, however, unknown values has to be ensured. In this context, variable-structure state estimation procedures represent powerful approaches because candidates for Lyapunov functions are employed in an underlying manner to perform a proof of the required stability properties. In this paper, a novel interval-based variable-structure state and parameter estimation procedure is presented for which a systematic approach toward an optimal parameterization is presented. The parameterization aims at simultaneously attenuating the influence of (stochastic) noise and maximizing the regions in the state and parameter space for which stability can be proven.

Keywords: Variable-structure state estimation, sliding mode observers, Lyapunov stability, interval analysis, stochastic disturbances

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1 Introduction

In previous work, interval-based extensions of sliding mode state observers [3, 4, 5, 7, 22, 23] have been presented by the authors which allow for a guaranteed stabilization of the associated error dynamics [13, 16, 18, 20]. In contrast to well-known Luenberger-type observers, variable-structure approaches have the advantage of an improved robustness against uncertainty and typically show better convergence properties to the true, however, unknown parameters and states (under some preconditions even in finite time).

The parameterization of these observers is so far based on the online evaluation of a suitable candidate for a Lyapunov function and its related time derivative. The variable-structure gain of this type of observer is determined in such a way that the time derivative of the Lyapunov function candidate can be guaranteed to be negative definite despite bounded uncertainty in system parameters as well as in the measured output variables. Besides the treatment of interval uncertainty, extensions were developed which guarantee asymptotic stability in cases in which stochastic measurement noise influences the system dynamics. These stochastic disturbances[1] are taken into account by a replacement of the *classical* time derivative of the Lyapunov function candidate by the so-called Itô differential operator [9, 17, 18].

The limitations of this estimation approach are

- the assumption of negligibly small time discretization errors in a quasi-continuous implementation of the interval-based estimation procedure and
- the use of the stability requirement of the error dynamics as the only design requirement.

While the first requirement can easily be satisfied with a suitable choice of the discretization step size (which has to be smaller by at least one order of magnitude than the smallest time constant in the considered system model), a more detailed investigation of the stability requirement is crucial. So far, it has been assumed that an underlying linear state observer exists with an a-priori fixed gain matrix. This linear observer part aims at stabilizing the system dynamics in domains of the state space where — due to bounded measurement errors — the actual sign of the error in the estimated state variables can no longer be determined. However, a classical choice of this fixed gain matrix by pole assignment or by solving the design conditions for a stationary Kalman Filter may lead to the drawback that stochastic measurement noise is not sufficiently attenuated in the overall variable-structure implementation and that regions in the state space, for which stability of the error dynamics cannot be proven rigorously, become unnecessarily large.

This paper is structured as follows. Sec. 2 gives an overview of the fundamental structure of an interval-based sliding mode state and parameter estimator. A first attempt toward the optimal parameterization of the underlying linear state observer is presented in Sec. 3. Together with a suitable choice of the variable-structure gain in Sec. 2, the rate of convergence of the estimated states toward their true values can be influenced systematically. This allows for a trade-off between the maximum rate of convergence, on the one hand, and the robustness of the estimation as well as the

[1]Throughout the paper, the word *uncertainty* refers to an imprecisely known quantity related to some system parameter, while *disturbances* represent imprecisely known external effects in either the system dynamics or measurement equations. From this point of view, measurement uncertainty, represented by interval variables, refers to the sensor resolution or a sensor bias, while measurement noise represents external random disturbances.

achievable accuracy of the point-valued estimates, on the other hand. Experimental results for prototypical electric drive train applications are summarized in Sec. 4. Finally, Sec. 5 is focused on conclusions and gives an outlook on future work.

2 Variable-Structure Observer Approach

Assume that a dynamic system is given by the ordinary differential equations (ODEs)

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{p}, \mathbf{u}(t)\right) = \mathbf{A} \cdot \mathbf{x}\left(t\right) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{S} \cdot \boldsymbol{\xi}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{1}$$

with the state vector $\mathbf{x}(t)$, the vector of uncertain but bounded parameters $\mathbf{p}(t) \in [\mathbf{p}]$ as well as the input vector $\mathbf{u}(t)$ [18, 20].

The associated vector of measured output variables is denoted by

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}\left(t\right) \quad . \tag{2}$$

The system model above is capable of representing both the nominal linear dynamics in terms of the system, input, and output matrices $\mathbf{A}$, $\mathbf{B}$ as well as $\mathbf{C}$ and all a-priori unknown and nonlinear terms $\mathbf{S} \cdot \xi\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ with $\mathbf{S} \in \mathbb{R}^{n \times q}$ and $\|\xi\left(\mathbf{x}, \mathbf{u}\right)\| \leq \overline{\xi}$. For the term $\boldsymbol{\xi}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$, it is assumed that a fixed upper bound $\overline{\xi}$ exists for a suitable (usually Euclidean) vector norm.

Here, uncertainty in the linear system components is assumed to be allowed in terms of the interval matrix [6, 12] representations $\mathbf{A} := \mathbf{A}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{A}]$, $\mathbf{B} := \mathbf{B}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{B}]$ and $\mathbf{C} := \mathbf{C}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{C}]$. In such a way, also state dependencies can be included in the before-mentioned matrices in terms of worst-case range enclosures $\mathbf{x}(t) \in [\mathbf{x}]\left(t\right)$. These parameter and state dependencies lead to a convex polytopic uncertainty model for the (quasi-)linear system components. Details on how such models can be constructed can be found, for example, in [15].

As soon as Brownian motions $\mathbf{w}_{\mathrm{p}}$ and $\mathbf{w}_{\mathrm{m}}$ are taken into account for modeling process and measurement noise, the system model turns into the stochastic differential equation

$$\mathrm{d}\mathbf{x} = \Big(\mathbf{A} \cdot \mathbf{x}\left(t\right) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{S} \cdot \boldsymbol{\xi}\left(\mathbf{x}(t), \mathbf{u}(t)\right)\Big)\mathrm{d}t + \mathbf{G}_{\mathrm{p}}\mathrm{d}\mathbf{w}_{\mathrm{p}} \tag{3}$$

with

$$\mathbf{y} = \mathbf{C} \cdot \mathbf{x}\left(t\right) + \mathbf{G}_{\mathrm{m}}\mathbf{w}_{\mathrm{m}} \quad . \tag{4}$$

The interval-based sliding mode observer is then designed by means of the ODEs

$$\begin{aligned} \hat{\bar{\mathbf{f}}} &\in \hat{\mathbf{f}}(\hat{\mathbf{x}}\left(t\right), [\mathbf{p}], \mathbf{u}(t)) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_{s} \cdot \mathrm{sign}(\mathbf{e}_{\mathrm{m}}\left(t\right) + [\Delta \mathbf{y}_{\mathrm{m}}]) \\ &:= [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}\left(t\right) + [\hat{\mathbb{B}}] \cdot \mathbf{u}\left(t\right) + \mathbf{H}_{\mathrm{p}} \cdot [\mathbf{e}_{\mathrm{m}}]\left(t\right) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_{\mathrm{s}}\ \mathrm{sign}(\mathbf{e}_{\mathrm{m}}\left(t\right) + [\Delta \mathbf{y}_{\mathrm{m}}]) \end{aligned} \tag{5}$$

and the output equation

$$\hat{\mathbf{y}}_{\mathrm{m}}\left(t\right) \in [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}\left(t\right) \quad . \tag{6}$$

According to [20], this observer consists of the combination of a *locally valid linear system model* and a *variable-structure part* to stabilize the error dynamics despite uncertainty and nonlinearities. During a real-time implementation, interval arithmetic software libraries such as C-XSC [8] are employed to handle uncertainty in parameters and measurements in such a way that the negative definiteness of the time derivative of a suitable Lyapunov function candidate can be proven.

The use of interval arithmetic becomes necessary if the nominal system, input and output matrices are replaced by interval matrices $[\hat{\mathbb{A}}]$, $[\hat{\mathbb{B}}]$, and $[\hat{\mathbb{C}}]$. These matrices correspond to the interval evaluations of $\hat{\mathbf{A}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{A}}]$, $\hat{\mathbf{B}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{B}}]$, and $\hat{\mathbf{C}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{C}}]$ in the sense of a quasi-linear state-space representation with worst-case bounds for the estimated states $\hat{\mathbf{x}}(t) \in [\hat{\mathbf{x}}](t)$. Moreover, the vector of measurement errors

$$\mathbf{e}_{\mathrm{m}}(t) \in [\mathbf{e}_{\mathrm{m}}](t) = \mathbf{y}_{\mathrm{m}}(t) - \hat{\mathbf{y}}_{\mathrm{m}}(t) + [\Delta \mathbf{y}_{\mathrm{m}}] \tag{7}$$

is extended by the interval uncertainty $[\Delta \mathbf{y}_{\mathrm{m}}]$ to account for bounded measurement tolerances.

According to the structure of the observer ODEs (5), an underlying stabilization of the locally linear error dynamics by the observer gain matrix $\mathbf{H}_{\mathrm{p}}$ is necessary. After computing the matrix $\mathbf{P} = \mathbf{P}^T \succ 0$ as the positive definite solution[2] of the Lyapunov equation

$$\tilde{\mathbf{A}} \cdot \mathbf{P} + \mathbf{P} \cdot \tilde{\mathbf{A}}^T + \mathbf{Q} = \mathbf{0} \quad \text{with} \quad \tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{H}_{\mathrm{p}} \cdot \hat{\mathbf{C}} \quad \text{and} \quad \mathbf{Q} \succ 0 \ , \tag{8}$$

an online evaluation of the switching amplitudes $\mathbf{H}_{\mathrm{s}}$ can be performed in each time step to handle uncertainty and the effects of all nonlinearities that are not captured by the locally valid linear system model described by $\mathbf{A} = \hat{\mathbf{A}}$ and $\mathbf{C} = \hat{\mathbf{C}}$.

After the definition of a Lyapunov function candidate

$$V := V(t) = \frac{1}{2}(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^T \mathbf{P} (\mathbf{x}(t) - \hat{\mathbf{x}}(t)) = \frac{1}{2}\mathbf{e}^T(t)\, \mathbf{P} \mathbf{e}(t) \tag{9}$$

with respect to the estimation errors $\mathbf{e} := \mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, the Itô differential operator [9]

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^T \cdot \left(\mathbf{f}([\mathbf{x}], [\mathbf{p}], \mathbf{u}) - \hat{\tilde{\mathbf{f}}}([\hat{\mathbf{x}}], [\mathbf{p}], \mathbf{u})\right) + \frac{1}{2}\mathrm{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G}\right\} \tag{10}$$

is used as a generalization of the total time derivative of V in the stochastic case[3].

In (10), the model for the true dynamic system $\mathbf{f}([\mathbf{x}], [\mathbf{p}], \mathbf{u})$, evaluated with intervals for all not exactly known parameters and states, and the observer parallel model $\hat{\tilde{\mathbf{f}}}([\hat{\mathbf{x}}], [\mathbf{p}], \mathbf{u})$ defined in (5) are included. Moreover, the complete matrix of standard deviations $\mathbf{G} = [\mathbf{G}_{\mathrm{p}} \ \ -\mathbf{H}_{\mathrm{p}}\mathbf{G}_{\mathrm{m}}]$ comprises process and measurement noise and accounts for all stochastically approximated, non-structured uncertainties as well as for all nonlinear phenomena and inaccuracies of sensor measurements that are not analytically included in the system model.

The switching amplitude of the interval-based variable-structure observer is determined by ensuring the negative definiteness of (10). To guarantee an element-wise defined minimum convergence rate $\mathbf{q} \geq \mathbf{0}$ with respect to the components of the estimation error signal, the switching amplitudes are determined from the inequality

$$L(V(t)) \overset{!}{<} -\mathbf{q}^T \left|[\mathbf{e}_{\mathrm{m}}]\right| \ , \tag{11}$$

[2]Note, $\mathbf{M} \succ 0$ denotes positive, $\mathbf{M} \prec 0$ negative definiteness of a square, real-valued, symmetric matrix $\mathbf{M}$.

[3]In the following, the time argument t is omitted for reasons of compactness, whenever the notation is non-ambiguous.

which results in

$$\mathbf{h}_{\mathrm{s}} = \begin{cases} \mathbf{0} & \text{if } [\delta] \cap [\mathbf{e}_{\mathrm{m}}]^T [\mathbf{e}_{\mathrm{m}}] \neq \emptyset \\ \sup\left(|[\mathbf{e}_{\mathrm{m}}]|^{+} \cdot \left([\dot{V}_{\mathrm{a}}] + \frac{1}{2} \cdot \mathrm{trace}\left\{ \mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G} \right\} \right) + \mathbf{q}^T \right) & \text{else} \end{cases} \tag{12}$$

with

$$[\dot{V}_{\mathrm{a}}] = [\mathbf{e}]^T \mathbf{P} \cdot ([\mathbf{f}] - [\hat{\mathbf{f}}] - \mathbf{H}_{\mathrm{p}} \cdot \mathbf{e}_{\mathrm{m}}) \tag{13}$$

and $[\mathbf{f}] := \mathbf{f}([\mathbf{x}], [\mathbf{p}], \mathbf{u})$ as well as $[\hat{\mathbf{f}}] := \hat{\mathbf{f}}([\mathbf{x}], [\mathbf{p}], \mathbf{u})$.

Here, a small interval $[\delta]$ centered around zero is introduced to prevent a division by zero during the gain computation.

The diagonal matrix $\mathbf{H}_{\mathrm{s}} = \mathrm{diag}(\mathbf{h}_{\mathrm{s}}) \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{y}}}$, $\mathbf{y} \in \mathbb{R}^{n_{\mathrm{y}}}$, of the switching amplitudes consists of the vector elements computed in (12).

For the evaluation of the switching amplitudes, the absolute value of the difference between measured and estimated states $|[\mathbf{e}_{\mathrm{m}}]|$ (defined component-wise) according to

$$|[e_{\mathrm{m},i}]| = \begin{cases} \left[-\overline{e}_{\mathrm{m},i} \ ; \ -\underline{e}_{\mathrm{m},i} \right] & \text{for } \overline{e}_{\mathrm{m},i} \leq 0 \\ \left[\underline{e}_{\mathrm{m},i} \ ; \ \overline{e}_{\mathrm{m},i} \right] & \text{for } \underline{e}_{\mathrm{m},i} \geq 0 \\ \left[0 \ ; \ \max\{ |\underline{e}_{\mathrm{m},i}|, |\overline{e}_{\mathrm{m},i}| \} \right] & \text{else} \end{cases} \tag{14}$$

and an interval-valued pseudo inverse

$$|[\mathbf{e}_{\mathrm{m}}]|^{+} = \left(|[\mathbf{e}_{\mathrm{m}}]|^T \, |[\mathbf{e}_{\mathrm{m}}]| \right)^{-1} \cdot |[\mathbf{e}_{\mathrm{m}}]|^T \tag{15}$$

are required together with the interval specifications for control, estimation, and measurement errors according to $[\mathbf{e}] = [\mathbf{x}] - [\hat{\mathbf{x}}]$, $[\mathbf{x}] = \mathbf{x} + [\Delta \mathbf{x}_{\mathrm{c}}]$, $[\hat{\mathbf{x}}] = \hat{\mathbf{x}} + [\Delta \mathbf{x}_{\mathrm{e}}]$. Here, $\mathbf{x}$ is the vector of the true (unknown) states; $\hat{\mathbf{x}}$ are point-valued estimates determined by the variable structure observer; $[\Delta \mathbf{x}_{\mathrm{c}}]$ is chosen such that all true states are contained in the interval $[\mathbf{x}]$ despite control errors; $[\Delta \mathbf{x}_{\mathrm{e}}]$ represents interval bounds for the maximum deviation between the true and estimated states.

The stability proof is successful, if a variable-structure gain exists for which the inequality $L(V(t)) < 0$ (or more generally the inequality (11)) is satisfied. The inequality $L(V(t)) < 0$ corresponds to the stability requirement $\dot{V}(t) < 0$ in the purely set-valued case, that is, if no stochastic disturbances are included in the model. In the close vicinity of the true and estimated system states, where the sign of the estimation errors can no longer be determined unambiguously, the fundamental requirement for the observer is the robust stabilization of the underlying linear dynamics which simultaneously has to prevent the amplification of stochastic measurement noise.

3 Parameterization of the Observer

3.1 Parameterization of the Variable-Structure Part

As long as the sign of the estimation errors can be determined uniquely, an implicit bounding of the variable-structure gain is possible by the choice of the interval $[\delta]$. The rate of convergence in this so-called reaching phase is foremost determined by the choice of the non-negative components of $\mathbf{q}$. Further details are published in [10, 20].

3.2 Parameterization of the Underlying Linear Observer Part

In general, the parameters for the feedback gain of the underlying linear observer part can be determined by the assignment of asymptotically stable eigenvalues located in the open left complex half plane, by the application of a linear quadratic optimization approach (equivalent to the steady-state Kalman Filter solution) or by the solution of linear matrix inequalities (LMIs) [1, 2]. In addition to purely formulating the feasibility problem that is identical to the requirement for asymptotic stability of the linear error dynamics, an optimization approach is introduced in the following that aims at a simultaneous minimization of the domain in the state-space for which asymptotic stability cannot be proven due to the influence of stochastic excitations caused by measurement noise.

3.2.1 LMI-Based Stability Criterion for Linear Time-Invariant Systems

If a linear dynamic system is given by the state-space representation

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y}_{\mathrm{m}} &= \mathbf{C}\mathbf{x} \ , \end{aligned} \tag{16}$$

a suitable linear state observer can be defined as

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{H}_{\mathrm{p}} \cdot (\mathbf{y}_{\mathrm{m}} - \hat{\mathbf{y}}_{\mathrm{m}}) \\ \hat{\mathbf{y}}_{\mathrm{m}} &= \mathbf{C}\hat{\mathbf{x}} \end{aligned} \tag{17}$$

with the estimation errors $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ and the associated error dynamics

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})\,\mathbf{e} \ . \tag{18}$$

The definition of the Lyapunov function candidate

$$V\left(\mathbf{e}(t)\right) = \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} \tag{19}$$

with the positive definite symmetric matrix $\mathbf{P} = \mathbf{P}^T \succ 0$ leads to the time derivative

$$\begin{aligned} \dot{V}\left(\mathbf{e}(t)\right) &= \frac{1}{2}\left(\dot{\mathbf{e}}^T\mathbf{P}\mathbf{e} + \mathbf{e}^T\mathbf{P}\dot{\mathbf{e}}\right) \\ &= \frac{1}{2}\mathbf{e}^T\left((\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})^T\,\mathbf{P} + \mathbf{P}\,(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})\right)\mathbf{e} \ , \end{aligned} \tag{20}$$

in which a bilinear matrix inequality with the yet unknown matrices $\mathbf{H}_{\mathrm{p}}$ and $\mathbf{P}$ needs to be negative definite according to

$$(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})^T\,\mathbf{P} + \mathbf{P}\,(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C}) \prec 0 \tag{21}$$

to ensure asymptotic stability [15]. To recast this inequality in a linear form, the duality between control and observer design is firstly exploited. It makes use of the fact that the eigenvalues of the matrix $(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})$ are identical to those of $(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})^T$. Hence, the matrix inequality

$$(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})\,\mathbf{P} + \mathbf{P}\,(\mathbf{A} - \mathbf{H}_{\mathrm{p}}\mathbf{C})^T \prec 0 \tag{22}$$

has to be satisfied which can be transformed into an LMI by a linearizing change of variables after multiplication with the regular matrix $\mathbf{P}^{-1} \succ 0$ from the left and right with

$$\mathbf{P}^{-1}\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)+\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)^{T}\mathbf{P}^{-1} \prec 0 \ . \tag{23}$$

This leads to the sufficient requirement for asymptotic stability in the LMI form

$$\mathbf{QA}-\mathbf{Y}^{T}\mathbf{C}+\mathbf{A}^{T}\mathbf{Q}-\mathbf{C}^{T}\mathbf{Y} \prec 0 \ , \quad \mathbf{Q}=\mathbf{P}^{-1} \succ 0 \ , \quad \mathbf{H}_{\mathrm{p}}^{T}=\mathbf{YP} \ . \tag{24}$$

If the influence of stochastic noise is considered in both the state and measurement equations according to

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \left(\mathbf{Ax}+\mathbf{Bu}\right)\mathrm{d}t+\mathbf{G}_{\mathrm{p}}\mathrm{d}\mathbf{w}_{\mathrm{p}} \\ \mathbf{y}_{\mathrm{m}} &= \mathbf{Cx}+\mathbf{G}_{\mathrm{m}}\mathbf{w}_{\mathrm{m}} \ , \end{aligned} \tag{25}$$

the time derivative in (20) can be replaced by the Itô differential operator [9]

$$L(V)=\frac{1}{2}\left(\tilde{\mathbf{e}}^{T}\left(\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)\mathbf{P}+\mathbf{P}\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)^{T}\right)\tilde{\mathbf{e}}\right)+\frac{1}{2}\mathrm{trace}\left\{\mathbf{G}^{T}\mathbf{PG}\right\}<0 \tag{26}$$

with $\mathbf{G}=[\mathbf{G}_{\mathrm{p}} \quad -\mathbf{H}_{\mathrm{p}}\mathbf{G}_{\mathrm{m}}]$. However, the stochastic excitation due to process and measurement noise leads to a domain around the point of vanishing estimation errors $\mathbf{e}=\mathbf{0}$ (respectively, $\tilde{\mathbf{e}}=\mathbf{0}$, $\dot{\tilde{\mathbf{e}}}=\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)^{T}\tilde{\mathbf{e}}$), where the inequality above cannot be satisfied. Therefore, the stability boundary is represented by

$$L(V)=0 \ . \tag{27}$$

The corresponding non-provable stability domain is then given by the interior of an ellipsoid

$$\tilde{\mathbf{e}}^{T}\mathbf{M}^{-1}\tilde{\mathbf{e}}-1=0 \tag{28}$$

with

$$\mathbf{M}^{-1}=\left(\frac{-\bar{\mathbf{A}}}{\mathrm{trace}\left\{\mathbf{G}^{T}\mathbf{PG}\right\}}\right) \tag{29}$$

and

$$\bar{\mathbf{A}}:=\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)\mathbf{P}+\mathbf{P}\left(\mathbf{A}-\mathbf{H}_{\mathrm{p}}\mathbf{C}\right)^{T} \prec 0 \ . \tag{30}$$

3.2.2 Optimality of the Observer Gain

During the observer parameterization, the ellipsoid for which stability cannot be proven, is minimized. The volume of the ellipsoid (28) to be minimized is proportional to

$$\sqrt{\det\left\{\mathbf{M}\right\}} \overset{!}{=} \min \tag{31}$$

as well as proportional to

$$\sqrt{\frac{\mathrm{trace}\left\{\mathbf{G}^{T}\mathbf{PG}\right\}}{\det\left(-\bar{\mathbf{A}}\right)}} \overset{!}{=} \min \ . \tag{32}$$

Non-vanishing observer gains $\mathbf{H}_{\mathrm{p}}$ are achieved by an equivalent maximization according to

$$\frac{\sqrt{\det\left\{\mathbf{Q}\right\}}}{\sqrt{\frac{\mathrm{trace}\left\{\mathbf{G}^{T}\mathbf{PG}\right\}}{\det\left(-\bar{\mathbf{A}}\right)}}} \overset{!}{=} \max \ . \tag{33}$$

In (33), the numerator term maximizes the error domain with a bounded linear feedback in the observer ODEs. The overall optimization task is hence equivalent to

$$\sqrt{\frac{\mathrm{trace}\left\{\mathbf{G}^T\mathbf{P}\mathbf{G}\right\}}{\det\left(-\bar{\mathbf{A}}\right)}} \cdot \frac{1}{\sqrt{\det\left\{\mathbf{Q}\right\}}} \stackrel{!}{=} \min \ , \tag{34}$$

which has the same minimum as

$$\frac{\mathrm{trace}\left\{\mathbf{G}^T\mathbf{P}\mathbf{G}\right\}}{\det\left(-\bar{\mathbf{A}}\right)\cdot\det\left\{\mathbf{Q}\right\}} \stackrel{!}{=} \min \tag{35}$$

and

$$\ln\left(\mathrm{trace}\left\{\mathbf{G}^T\mathbf{P}\mathbf{G}\right\}\right) - \ln\left(\det\left(-\bar{\mathbf{A}}\right)\right) - \ln\left(\det\left\{\mathbf{Q}\right\}\right) \stackrel{!}{=} \min \ , \tag{36}$$

where the latter variant replaces nonlinear couplings in the cost function by suitable differences.

Due to further multiplicative couplings between $\mathbf{P}$ and $\mathbf{H}_\mathrm{p}$ in the first term of (36), the before-mentioned optimization problem is not yet solvable by standard LMI techniques. Hence, the term $\mathbf{G}^T\mathbf{P}\mathbf{G}$ is replaced by the matrix inequality

$$\mathbf{N} \succ \mathbf{G}^T\mathbf{P}\mathbf{G} \quad \Longrightarrow \quad \mathbf{N} - \mathbf{G}^T\mathbf{Q}^{-1}\mathbf{G} \succ 0 \tag{37}$$

that can be re-written by applying the Schur complement formula according to

$$\begin{bmatrix} \mathbf{N} & \mathbf{G}^T \\ \mathbf{G} & \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \begin{bmatrix}\mathbf{G}_\mathrm{p} & -\mathbf{H}_\mathrm{p}\mathbf{G}_\mathrm{m}\end{bmatrix}^T \\ \begin{bmatrix}\mathbf{G}_\mathrm{p} & -\mathbf{H}_\mathrm{p}\mathbf{G}_\mathrm{m}\end{bmatrix} & \mathbf{Q} \end{bmatrix} \succ 0 \ , \quad \mathbf{N} = \mathbf{N}^T \succ 0 \ . \tag{38}$$

A further reformulation is necessary because (38) is still not in the form of an LMI. Rearranging the blocks in the rows and columns leads to

$$\begin{bmatrix} \mathbf{Q} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{N} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \begin{bmatrix}\mathbf{G}_\mathrm{p} & -\mathbf{H}_\mathrm{p}\mathbf{G}_\mathrm{m}\end{bmatrix} \\ \begin{bmatrix}\mathbf{G}_\mathrm{p} & -\mathbf{H}_\mathrm{p}\mathbf{G}_\mathrm{m}\end{bmatrix}^T & \mathbf{N} \end{bmatrix} \succ 0 \ , \quad \mathbf{N} = \mathbf{N}^T \succ 0 \ . \tag{39}$$

A left and right multiplication with the block diagonal matrix blkdiag$\{\mathbf{Q},\mathbf{I}\}$ results in the expression

$$\begin{bmatrix} \check{\mathbf{Q}}^3 & \begin{bmatrix}\mathbf{Q}\mathbf{G}_\mathrm{p} & -\mathbf{Y}^T\mathbf{G}_\mathrm{m}\end{bmatrix} \\ \begin{bmatrix}\mathbf{Q}\mathbf{G}_\mathrm{p} & -\mathbf{Y}^T\mathbf{G}_\mathrm{m}\end{bmatrix}^T & \mathbf{N} \end{bmatrix} \succ 0 \ , \quad \mathbf{N} = \mathbf{N}^T \succ 0 \tag{40}$$

which is solved iteratively due to the nonlinearity $\mathbf{Q}^3$. For that reason, the matrix $\check{\mathbf{Q}}^3$ in (40) — that has been obtained in the previous iteration stage — serves as a substitute for the nonlinear term $\mathbf{Q}^3$. The solution of the matrix inequality (40) is constrained by the stability requirement

$$\mathbf{Q}\mathbf{A} - \mathbf{Y}^T\mathbf{C} + \mathbf{A}^T\mathbf{Q} - \mathbf{C}^T\mathbf{Y} \prec 0 \ , \quad \mathbf{Q} = \mathbf{P}^{-1} \succ 0 \ , \quad \mathbf{H}_\mathrm{p}^T = \mathbf{Y}\mathbf{P} \tag{41}$$

according to (24) as well as by the optimality criterion

$$\ln\left(\mathrm{trace}\left\{\mathbf{N}\right\}\right) - \ln\left(\det\left(-\bar{\mathbf{A}}\right)\right) - \ln\left(\det\left\{\mathbf{Q}\right\}\right) \stackrel{!}{=} \min \ . \tag{42}$$

In (42), the logarithms of determinants and matrix traces are coupled in such a way that the structure is not accepted by the LMI solver SeDuMi [11, 21]. Hence, the criterion (35) is directly included in the iteration scheme in the form

$$\frac{\mathrm{trace}\left\{\mathbf{N}\right\}}{\det\left(-\check{\bar{\mathbf{A}}}\right)\cdot\det\left\{\check{\mathbf{Q}}\right\}} \stackrel{!}{=} \min \tag{43}$$

with

$$\breve{\tilde{\mathbf{A}}} := \left(\mathbf{A} - \breve{\mathbf{H}}_{\mathrm{p}}\mathbf{C}\right)\breve{\mathbf{Q}}^{-1} + \breve{\mathbf{Q}}^{-1}\left(\mathbf{A} - \breve{\mathbf{H}}_{\mathrm{p}}\mathbf{C}\right)^{T} \tag{44}$$

and the observer gain

$$\breve{\mathbf{H}}_{\mathrm{p}}^{T} = \breve{\mathbf{Y}}\breve{\mathbf{Q}}^{-1} \tag{45}$$

computed by the results of the previous iteration. The iteration is continued until the matrix $\mathbf{Q}$ and the optimality criterion remain practically identical in two subsequent iteration steps.

3.2.3 Generalization to a Polytopic Uncertainty Representation of the Underlying Linear Observer

In case of polytopic uncertainty in the system model, a joint gain $\mathbf{H}_{\mathrm{p}}$ is determined for all vertex matrices $(\mathbf{A}_\nu, \mathbf{C}_\nu)$. In this case, (41) has to be satisfied for each of the vertex matrices, where the performance criterion is evaluated as the average ellipsoid volume for all ν.

4 Experimental Validation of the Variable-Structure Observer Approach

To validate the observer for a practical application scenario, the drive train test rig [19] depicted in Fig. 1 is considered. It consists of an electric drive attached rigidly to the drive-side shaft. The brake on the load-side shaft (connected to the drive-side shaft via a non-elastic toothed belt) provides an unknown velocity-proportional load torque. The goal of the following observer is the estimation of the before-mentioned load torque (in terms of the corresponding parameter α in the following model) and the overall mass moment of inertia (β^{-1}) from pure angle measurements.

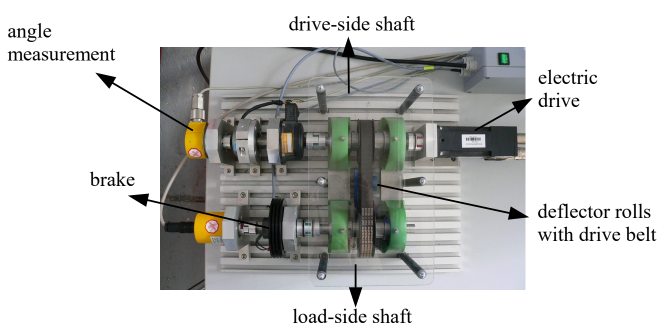


Figure 1: Test rig for the validation of the variable-structure observer approach.

To eliminate the influence of static friction, a combined feedforward and feedback control strategy is implemented in the real-time control system that turns the state equations of the test rig into

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u}) = \mathbf{A}\cdot\mathbf{x} + \mathbf{b}\cdot u = \begin{bmatrix}0 & 1\\ 0 & \alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \beta\end{bmatrix} u\ . \tag{46}$$

Here, x_1 is the angle of rotation of the drive-side shaft and x_2 its corresponding angular velocity. For the observer implementation, the state variable x_1 corresponds to the only measured system output with

$$y = \mathbf{c}^T \cdot \mathbf{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \mathbf{x} = x_1 \ . \tag{47}$$

Besides the angular velocity $x_2 = \dot{x}_1$, the observer has to estimate the not a-priori known parameters $\alpha = -\frac{d}{J}$ and $\beta = \frac{1}{J}$ in real time. In these parameter definitions, d is a velocity-proportional friction coefficient resulting from a suitable actuation of the magnetic powder brake on the load-side shaft. As mentioned above, both shafts are connected with each other by a toothed belt with negligible elasticity, so that the only remaining parameter is the overall mass moment of inertia J of all rotating elements.

According to [10, 20], the observer consists of two subsystems. The first subsystem (implemented as a variable-structure observer) is employed to estimate the first three derivatives of the measured angle. These estimates serve as virtual measurements for the second variable-structure observer that is employed to estimate the parameters α and β that are appended to the state equations (46) via the integrator disturbance models $\dot{\alpha} = 0$ and $\dot{\beta} = 0$.

4.1 Parameterization by Designing a Stationary Kalman Filter

As a reference solution for the variable-structure observer, experimental estimation results are shown in Fig. 2 for the initial parameter intervals $\alpha \in [-1.5\ ;\ 4.5]$ and $\beta \in [30\ ;\ 90]$.

Here, the observer gain matrices $\mathbf{H}_\mathrm{p}$ were determined for both linear subsystems by solving the algebraic Riccati equation

$$\mathbf{P}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{P} - \mathbf{A}\mathbf{P} - \mathbf{P}\mathbf{A}^T - \mathbf{Q} = \mathbf{0} \ , \tag{48}$$

where the final gain is given by

$$\mathbf{H} = \mathbf{P}\mathbf{C}^T\mathbf{R}^{-1} \ . \tag{49}$$

A corresponding cost function (dual to the linear quadratic regulator design) is then given by

$$J = \frac{1}{2}\int_0^\infty \left(\Delta\mathbf{x}^T\mathbf{Q}\Delta\mathbf{x} + \Delta\mathbf{y}^T\mathbf{R}\Delta\mathbf{y}\right)\mathrm{d}t \tag{50}$$

with the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ as well as the state and output errors $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$. Note that (48) coincides with the design criterion for a stationary Kalman Filter [14].

For the first subsystem, described in detail in [20], the matrices for the standard deviations of process and measurement noise were set to

$$\mathbf{G}_\mathrm{p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}^T \quad \text{and} \quad G_\mathrm{m} = 0.005 \tag{51}$$

with the weighting matrices

$$\mathbf{Q} = \mathrm{diag}\left\{\begin{bmatrix} 1^2 & 50^2 & 50^2 & 50^2 & 70^2 \end{bmatrix}\right\} \quad \text{and} \quad R = 0.001^2 \ . \tag{52}$$

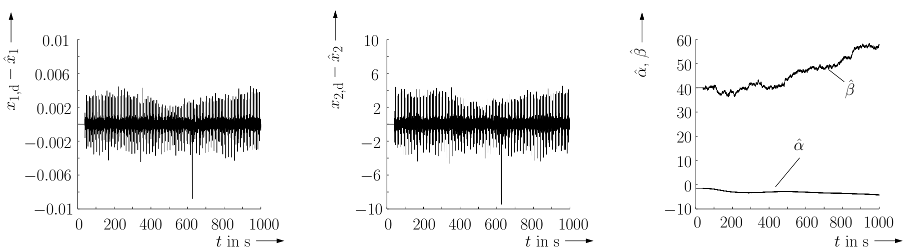
For the second subsystem, the parameters

$$\mathbf{G}_{\mathrm{p}} = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \quad \text{and} \quad \mathbf{G}_{\mathrm{m}} = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \tag{53}$$

as well as

$$\mathbf{Q} = \mathrm{diag}\left\{\begin{bmatrix} 10^2 & 10^2 & 0.01^2 & 0.01^2 & 0.1^2 \end{bmatrix}\right\} \quad \text{and} \quad \mathbf{R} = \mathrm{diag}\left\{\begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}\right\} \tag{54}$$

were chosen. Note that especially the entries in each of the matrices $\mathbf{Q}$ and $\mathbf{R}$ need to be chosen heuristically to achieve sufficient robustness and satisfactory convergence rates of the underlying linear observer. Unfortunately, taking directly matrices that coincide with $\mathbf{G}_{\mathrm{p}}$ and $\mathbf{G}_{\mathrm{m}}$ (as suggested by the theory of the Kalman Filter design) leads to unsatisfying convergence rates of the observer due to the fact that information concerning provable stability domains (as it is done in the following subsection) cannot be taken into account directly.



(a) Estimation error x_1. (b) Estimation error x_2. (c) Parameter estimates $\hat{\alpha}$, $\hat{\beta}$.

Figure 2: Experimental results.

The accuracy of the parameter estimates for α and β is checked by the simulation results in Tab. 1.

Table 1: Quantification of the remaining modeling errors by means of simulation: Comparison of least-squares identification (LS) and parameterization by the state-of-the-art approach for the interval sliding mode observer design (ISMO).

	LS	ISMO	Improvement
x_1	$\Delta_{x_1,\mathrm{LS}} = 2730$	$\Delta_{x_1,\mathrm{ISMO}} = 268.61$	90.16%
x_2	$\Delta_{x_2,\mathrm{LS}} = 4.79$	$\Delta_{x_2,\mathrm{ISMO}} = 3.01$	37.10%

There, the estimated time series of the parameters were substituted into the state equations which are then simulated in terms of an initial value problem with the experimental motor torque $u(t)$ as input signal. The reference solution for this simulation was determined by a least-squares estimate that minimizes the square of the deviation between simulated and measured angles x_1 by determining the system parameters as constant values for time windows of eight seconds each (a full driving cycle consisting of acceleration, constant velocity, and deceleration phases). It can be seen that the interval-based sliding mode observer (ISMO) significantly outperforms the results of the least-squares identification (LS).

4.2 Parameterization by the Proposed Optimization Procedure

For the novel optimization procedure proposed in this paper, only information about the standard deviations of process and measurement noise is necessary. There, directly the technologically motivated values

$$\mathbf{G}_\mathrm{p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 10 \end{bmatrix}^T \quad \text{and} \quad G_\mathrm{m} = 0.001 \tag{55}$$

can be used for the first subsystem. Moreover, the corresponding values for the second subsystem are given by

$$\mathbf{G}_\mathrm{p} = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.003 & 0 & 0 \\ 0 & 0 & 0 & 0.003 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \quad \text{and} \quad \mathbf{G}_\mathrm{m} = \begin{bmatrix} 0.06 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} . \tag{56}$$

Obviously, this significantly reduces the parameterization effort because the gain matrices $\mathbf{H}_\mathrm{p}$ now directly result from solving the LMI-constrained optimization problem according to (40), (41), (43), (44), and (45). This straightforward optimization approach removes the necessity for intelligent guessing of weighting factors and leads to the estimation results in Tab. 2 which were determined in analogy to Tab. 1.

Table 2: Quantification of the remaining modeling errors by means of simulation: Comparison of least-squares identification and novel optimization-based observer parameterization.

	LS	ISMO	Improvement
x_1	$\Delta_{x_1,\mathrm{LS}} = 2730$	$\Delta_{x_1,\mathrm{ISMO}} = 215.48$	92.11%
x_2	$\Delta_{x_2,\mathrm{LS}} = 4.79$	$\Delta_{x_2,\mathrm{ISMO}} = 3.09$	35.41%

It can be seen that the estimation results for the first state variable were further improved by the novel parameterization procedure, while the accuracy of the estimate for the second state variable remains practically unchanged.

5 Conclusions and Future Work

In this paper, an optimization procedure was presented for the parameterization of the underlying linear observer gain matrix of an interval-based variable-structure observer. This observer is capable of simultaneously estimating system states and parameters with improved accuracy as compare to linear gain-scheduled approaches or (if parameter estimation is concerned) with respect to offline least-squares identification routines applied over finite time intervals.

The optimization of the gain matrix is performed by solving an LMI-constrained optimization problem which is characterized by the fact that asymptotic stability of the linear system part is guaranteed despite bounded uncertainty and that the domain in the state-space is minimized for which stability cannot be proven rigorously. The stability proof is based on the online evaluation of a Lyapunov function candidate including stochastic noise processes for both the state and measurement equations. For that purpose, the Itô differential operator is used to determine the time derivative of the Lyapunov function candidate including the before-mentioned noise processes. In contrast to classical parameterizations of the underling linear observer, the proposed approach can be applied in a straightforward manner to system models with a polytopic uncertainty representation.

Due to the duality of the design of linear feedback controllers and state estimators, the proposed optimization procedure can be easily applied to the design of linear feedback control components in interval-based sliding mode controllers. A further point for future work is to rigorously account for time discretization errors in cases in which the assumptions on the sampling time — that were made in the introduction of this paper — are not satisfied. Moreover, future work will also aim at the development of a procedure that simultaneously performs the guaranteed stabilizing design of controllers and state estimators. Currently, the controller and observer design is made independently with a subsequent careful check of the overall system stability. However, the joint design of both components is necessary for nonlinear systems if no additional stability proof is desired. This is due to the fact that the separation principle — that is well-known from linear dynamics — no longer holds in the case of nonlinear observer-based control structures.

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