An Explicit Enclosure of the Solution Set of Overdetermined Interval Linear Equations*

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Abstract

A new formulation and proof is given for the Hansen-Bliek-Rohn description of the interval hull of the solution set of a system of interval linear equations with unit midpoint. As a consequence we obtain an explicit formula for an enclosure of the solution set of a system of overdetermined interval linear equations.

Keywords: interval linear equations, interval hull, unit midpoint, enclosure
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1 Introduction

For a system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$, where $\boldsymbol{A}$ is an $n \times n$ interval matrix and $\boldsymbol{b}$ is an interval n-vector, the interval hull is defined as

$$\boldsymbol{x}(\boldsymbol{A}, \boldsymbol{b}) = \bigcap_{\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) \subseteq [\underline{y}, \overline{y}]} [\underline{y}, \overline{y}],$$

where

$$\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) = \{ x \mid Ax = b \text{ for some } A \in \boldsymbol{A},\ b \in \boldsymbol{b} \}, \tag{1}$$

i.e., as the narrowest interval vector containing the solution set $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$. Computing the interval hull is NP-hard [12], yet it was shown by Hansen [4], Bliek [2] and Rohn [8] that the hull can be expressed by relatively simple closed-form formulae when the system matrix has unit midpoint, i.e., is of the form $\boldsymbol{A} = [I - \Delta, I + \Delta]$, where I is the unit matrix. However, the proof of this result is by no means straightforward. The formulae not using interval arithmetic were proved in [8], [10], and those formulated in terms of interval arithmetic by Ning and Kearfott [7] (using the result from [8]) and by Neumaier [6].

In this paper we present another proof of the optimality result, based on a new description of the interval hull (Theorem 2.1). This description is expressed in terms of vectors rather than of entries like in [8], and as a direct consequence of the new

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formulation we obtain a formula for an explicit enclosure of the solution set of an overdetermined system of interval linear equations (Theorem 5.1).

Notation used: $\mathrm{diag}(M)$ denotes the diagonal of a matrix M, $M_{k\bullet}$ is the kth row of M, T_z is the diagonal matrix with diagonal vector z, $a \circ b = (a_i b_i)$ stands for the Hadamard (entrywise) product of vectors $a = (a_i)$, $b = (b_i)$ and $a/b = (a_i/b_i)$ for their Hadamard division, minimum/maximum of a finite number of vectors is taken entrywise, I is the identity matrix, e_k is the kth column of I, and e is the vector of all ones.

2 Interval Hull

We shall later make use of the following general characterization of the interval hull. An interval matrix $\boldsymbol{A}$ is called regular if each $A \in \boldsymbol{A}$ is nonsingular.

Theorem 2.1. *Let $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta]$ be regular. Then for each $z \in \{-1, 1\}^n$ the matrix equation*

$$QA_c - |Q|\Delta T_z = I$$

has a unique solution Q_z and for each right-hand side $\boldsymbol{b} = [b_c - \delta, b_c + \delta]$ there holds

$$\boldsymbol{x}(\boldsymbol{A}, \boldsymbol{b}) = \Big[\min_{z \in \{-1,1\}^n} (Q_z b_c - |Q_z|\delta), \max_{z \in \{-1,1\}^n} (Q_z b_c + |Q_z|\delta)\Big]. \qquad (2)$$

Proof: The first part of the theorem is the assertion of [11, Thm. 1], while the second one follows from [9, Thm. 2] if we take $Z = \{-1, 1\}^n$ there. □

3 Matrices Q_z

In this section we show that the matrices Q_z can be expressed explicitly in the case of an interval matrix of the form $\boldsymbol{A} = [I - \Delta, I + \Delta]$. The result, as well as the subsequent ones, is formulated in terms of the matrix

$$M = (I - \Delta)^{-1}.$$

Notice that

$$M\Delta = M - I. \qquad (3)$$

In Theorem 3.1 we shall assume that $M \geq I$. This is equivalent to regularity of $[I - \Delta, I + \Delta]$, see [5]. Define

$$\nu_k = \tfrac{1}{2M_{kk} - 1} \qquad (k = 1, \ldots, n);$$

since $M_{kk} \geq 1$ for each k by assumption, we have $2M_{kk} - 1 \geq 1$, hence all the ν_k's are well defined and are positive. Next, define $\mu(z)$ by

$$\mu_k(z) = \begin{cases} 1 & \text{if } z_k = 1, \\ \nu_k & \text{if } z_k = -1 \end{cases} \qquad (k = 1, \ldots, n). \qquad (4)$$

Vector $\mu(z)$ is again positive.

Theorem 3.1. *Let $M \geq I$. Then for each $z \in \{-1,1\}^n$ the unique solution Q_z of the matrix equation*

$$Q - |Q|\Delta T_z = I \tag{5}$$

is given by

$$Q_z = T_{\mu(z)}(M-I)T_z + I \tag{6}$$

or, alternatively, rowwise by

$$(Q_z)_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = \ \ 1, \\ \nu_k(M_{k1},\dots,-M_{kk},\dots,M_{kn})T_z & \text{if } z_k = -1 \end{cases} \qquad (k=1,\dots,n). \tag{7}$$

Proof: For a given $z \in \{-1,1\}^n$ define a matrix Q rowwise by

$$Q_{k\bullet} = \begin{cases} M_{k\bullet}T_z & \text{if } z_k = \ \ 1, \\ \nu_k(M_{k1},\dots,-M_{kk},\dots,M_{kn})T_z & \text{if } z_k = -1 \end{cases} \qquad (k=1,\dots,n).$$

Then

$$|Q|_{k\bullet} = \begin{cases} M_{k\bullet} & \text{if } z_k = \ \ 1, \\ \nu_k M_{k\bullet} & \text{if } z_k = -1 \end{cases} \qquad (k=1,\dots,n). \tag{8}$$

We shall prove that Q solves (5). Let $k \in \{1,\dots,n\}$. If $z_k = 1$, then

$$\begin{aligned} (|Q|\Delta T_z + I)_{k\bullet} &= M_{k\bullet}\Delta T_z + e_k^T = e_k^T M\Delta T_z + e_k^T = e_k^T(M-I)T_z + e_k^T \\ &= e_k^T MT_z - e_k^T + e_k^T = M_{k\bullet}T_z = Q_{k\bullet} \end{aligned}$$

and if $z_k = -1$, then

$$\begin{aligned} (|Q|\Delta T_z + I)_{k\bullet} &= \nu_k M_{k\bullet}\Delta T_z + e_k^T = \nu_k e_k^T M\Delta T_z + e_k^T = \nu_k e_k^T(M-I)T_z + e_k^T \\ &= \nu_k e_k^T MT_z - \nu_k e_k^T T_z + e_k^T = \nu_k e_k^T MT_z + (\nu_k+1)e_k^T \\ &= \nu_k e_k^T MT_z + 2\nu_k M_{kk}e_k^T = \nu_k e_k^T MT_z - \nu_k(2M_{kk}e_k^T)T_z \\ &= \nu_k(e_k^T M - 2M_{kk}e_k^T)T_z = \nu_k(M_{k1},\dots,-M_{kk},\dots,M_{kn})T_z \\ &= Q_{k\bullet}, \end{aligned}$$

so that in both cases the kth equation of (5) is satisfied, which proves that Q solves (5). In view of Theorem 2.1, under the regularity assumption the equation (5) possesses a unique solution Q_z, hence $Q = Q_z$, so Q_z is given by (7).

Now, from (8) and (4) we can see that $|Q_z|_{k\bullet} = \mu_k(z)M_{k\bullet}$ for each k which can be written as

$$|Q_z| = T_{\mu(z)}M.$$

Then, from the fact that Q_z solves (5), it follows that

$$Q_z = |Q_z|\Delta T_z + I = T_{\mu(z)}M\Delta T_z + I = T_{\mu(z)}(M-I)T_z + I$$

(see (3)), which is (6). $\square$

4 HBR Optimality Result

As is well known, arithmetic operations with one-dimensional intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$, $\boldsymbol{b} = [\underline{b}, \overline{b}]$ are defined by the general rule

$$\boldsymbol{a} \diamondsuit \boldsymbol{b} = \{\, a \diamondsuit b \mid a \in \boldsymbol{a},\ b \in \boldsymbol{b} \,\},$$

where $\diamondsuit \in \{+, -, *, \div\}$. We shall later use only division of intervals for which a simple continuity argument shows that

$$\frac{[\underline{a}, \overline{a}]}{[\underline{b}, \overline{b}]} = [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}] \tag{9}$$

holds, under assumption that $0 \notin [\underline{b}, \overline{b}]$.

The Hansen-Bliek-Rohn (abbreviated as HBR) optimality result gives an explicit formula for the interval hull of an interval linear system of the form

$$\boldsymbol{I}x = \boldsymbol{b},$$

where $\boldsymbol{I} = \langle I, \Delta \rangle = [I - \Delta, I + \Delta]$.

Theorem 4.1. *Let $M := (I - \Delta)^{-1} \geq I$. Then we have*

$$\boldsymbol{x}(\langle I, \Delta \rangle, \langle b_c, \delta \rangle) = \frac{\langle x_*,\, x^* - |x_*| \rangle}{\langle d,\, d - e \rangle}, \tag{10}$$

where

$$\begin{aligned} d &= \operatorname{diag}(M), \\ x_* &= d \circ b_c, \\ x^* &= M(|b_c| + \delta). \end{aligned}$$

Comment. In (10) we use the Hadamard (entrywise) division of interval vectors in their midpoint-radius representation, i.e., $\langle a, b \rangle = [a - b, a + b]$. To be perfectly clear, (10) means that

$$\boldsymbol{x}(\boldsymbol{I}, \boldsymbol{b}) = \left(\frac{\langle x_*,\, x^* - |x_*| \rangle_i}{\langle d,\, d - e \rangle_i} \right)_{i=1}^{n}.$$

Evidently, the shortened version (10) is less cumbersome.

Proof: Denote $[\underline{x}, \overline{x}] = \boldsymbol{x}(\boldsymbol{I}, \boldsymbol{b})$. Let $k \in \{1, \ldots, n\}$. We shall first derive a formula for $\overline{x}_k$. From (2), we have

$$\overline{x}_k = \max_{z \in \{-1,1\}^n} (Q_z b_c + |Q_z|\delta)_k = \max_{z \in \{-1,1\}^n} ((Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet}\delta),$$

so, according to (7), for each $z \in \{-1, 1\}^n$ we must consider two cases: $z_k = 1$ and $z_k = -1$.

If $z_k = 1$, then by Theorem 3.1

$$\begin{aligned} (Q_z)_{k\bullet} b_c + |Q_z|_{k\bullet}\delta &= M_{k\bullet} T_z b_c + M_{k\bullet}\delta \\ &= \sum_{j \neq k} M_{kj} z_j (b_c)_j + M_{kk}(b_c)_k + M_{k\bullet}\delta \\ &\leq \sum_{j \neq k} M_{kj} |(b_c)_j| + M_{kk}(b_c)_k + M_{k\bullet}\delta. \end{aligned}$$

Introducing the vector $\overline{z}(k) \in \{-1, 1\}^n$ by

$$\overline{z}(k)_j = \begin{cases} 1 & \text{if } j = k, \\ 1 & \text{if } j \neq k \text{ and } (b_c)_j \geq 0, \\ -1 & \text{if } j \neq k \text{ and } (b_c)_j < 0 \end{cases} \qquad (j = 1, \ldots, n),$$

we can write

$$\sum_{j\neq k} M_{kj}|(b_c)_j| + M_{kk}(b_c)_k + M_{k\bullet}\delta = M_{k\bullet}T_{\overline{z}(k)}b_c + M_{k\bullet}\delta = (Q_{\overline{z}(k)})_{k\bullet}b_c + |Q_{\overline{z}(k)}|_{k\bullet}\delta.$$

Hence, for each $z \in \{-1,1\}^n$ with $z_k = 1$ we have

$$(Q_z)_{k\bullet}b_c + |Q_z|_{k\bullet}\delta \le (Q_{\overline{z}(k)})_{k\bullet}b_c + |Q_{\overline{z}(k)}|_{k\bullet}\delta,$$

and the upper bound is obviously attained.

If $z_k = -1$, then, again by Theorem 3.1,

$$\begin{aligned}
(Q_z)_{k\bullet}b_c + |Q_z|_{k\bullet}\delta &= \nu_k(M_{k1},\dots,-M_{kk},\dots,M_{kn})T_z b_c + \nu_k M_{k\bullet}\delta \\
&= \nu_k \sum_{j\neq k} M_{kj}z_j(b_c)_j + \nu_k M_{kk}(b_c)_k + \nu_k M_{k\bullet}\delta \\
&\le \nu_k \sum_{j\neq k} M_{kj}|(b_c)_j| + \nu_k M_{kk}(b_c)_k + \nu_k M_{k\bullet}\delta \\
&= \nu_k(M_{k1},\dots,-M_{kk},\dots,M_{kn})T_{\underline{z}(k)} b_c + \nu_k M_{k\bullet}\delta \\
&= (Q_{\underline{z}(k)})_{k\bullet}b_c + |Q_{\underline{z}(k)}|_{k\bullet}\delta
\end{aligned}$$

where we have employed the vector $\underline{z}(k)$ given by

$$\underline{z}(k)_j = \begin{cases} -1 & \text{if } j = k, \\ 1 & \text{if } j \neq k \text{ and } (b_c)_j \ge 0, \\ -1 & \text{if } j \neq k \text{ and } (b_c)_j < 0 \end{cases} \qquad (j = 1,\dots,n).$$

Hence, for each $z \in \{-1,1\}^n$ with $z_k = -1$ we have

$$(Q_z)_{k\bullet}b_c + |Q_z|_{k\bullet}\delta \le (Q_{\underline{z}(k)})_{k\bullet}b_c + |Q_{\underline{z}(k)}|_{k\bullet}\delta,$$

and the upper bound is again obviously attained. In this way we have proved the formula

$$\overline{x}_k = \max\{Q_{\overline{z}(k)})_{k\bullet}b_c + |Q_{\overline{z}(k)}|_{k\bullet}\delta,\ (Q_{\underline{z}(k)})_{k\bullet}b_c + |Q_{\underline{z}(k)}|_{k\bullet}\delta\}.$$

Now,

$$\begin{aligned}
(Q_{\overline{z}(k)})_{k\bullet}b_c + |Q_{\overline{z}(k)}|_{k\bullet}\delta &= \sum_{j\neq k} M_{kj}|(b_c)_j| + M_{kk}(b_c)_k + M_{k\bullet}\delta \\
&= M_{k\bullet}(|b_c| + \delta) + M_{kk}((b_c)_k - |b_c|_k) \\
&= (x_* + x^* - |x_*|)_k \\
&= \tilde{x}_k,
\end{aligned}$$

where we have denoted $\tilde{x} = x_* + x^* - |x_*|$, and similarly

$$\begin{aligned}
(Q_{\underline{z}(k)})_{k\bullet}b_c + |Q_{\underline{z}(k)}|_{k\bullet}\delta &= \nu_k \sum_{j\neq k} M_{kj}|(b_c)_j| + \nu_k M_{kk}(b_c)_k + \nu_k M_{k\bullet}\delta \\
&= \nu_k(M_{k\bullet}(|b_c| + \delta) + M_{kk}((b_c)_k - |b_c|_k)) \\
&= \nu_k(x_* + x^* - |x_*|)_k \\
&= \nu_k\tilde{x}_k
\end{aligned}$$

which together gives

$$\overline{x}_k = \max\{\tilde{x}_k, \nu_k \tilde{x}_k\}.$$

Since

$$\nu_k \tilde{x}_k = \tilde{x}_k/(2M_{kk} - 1),$$

we finally obtain

$$\overline{x} = \max\{\tilde{x}, \tilde{x}/(2d - e)\},$$

where we have used the Hadamard (entrywise) division of vectors.

To prove the formula for $\underline{x}$, consider the system $\boldsymbol{I}x = -\boldsymbol{b}$, where $\boldsymbol{I} = [I - \Delta, I + \Delta]$ as before and $-\boldsymbol{b} = \{\, -b \mid b \in \boldsymbol{b} \,\} = [-b_c - \delta, -b_c + \delta]$. Then $\boldsymbol{X}(\boldsymbol{I}, -\boldsymbol{b}) = -\boldsymbol{X}(\boldsymbol{I}, \boldsymbol{b})$, so $\boldsymbol{x}(\boldsymbol{I}, -\boldsymbol{b}) = [-\overline{x}, -\underline{x}]$. Now we can apply the previously derived formula for the upper bound of the interval hull:

$$\begin{aligned} -\underline{x} \quad = \quad & \max\{-d \circ b_c + M(|b_c| + \delta) - |d \circ b_c|, \\ & \quad (-d \circ b_c + M(|b_c| + \delta) - |d \circ b_c|)/(2d - e)\}, \end{aligned}$$

hence

$$\begin{aligned} \underline{x} &= \min\{d \circ b_c - M(|b_c| + \delta) + |d \circ b_c|,\ (d \circ b_c - M(|b_c| + \delta) + |d \circ b_c|)/(2d - e)\} \\ &= \min\{x_* - x^* + |x_*|,\ (x_* - x^* + |x_*|)/(2d - e)\} \\ &= \min\{\underset{\sim}{x},\ \underset{\sim}{x}/(2d - e)\}, \end{aligned}$$

where $\underset{\sim}{x} = x_* - x^* + |x_*|$. This proves that

$$\boldsymbol{x}(\boldsymbol{I}, \boldsymbol{b}) = [\min\{\underset{\sim}{x},\ \underset{\sim}{x}/(2d - e)\},\ \max\{\tilde{x},\ \tilde{x}/(2d - e)\}]. \tag{11}$$

Because $\underset{\sim}{x} \le \tilde{x}$ and $\nu > 0$, we can write (11) as

$$\boldsymbol{x}(\boldsymbol{I}, \boldsymbol{b}) = [\min\{\underset{\sim}{x}/e,\ \underset{\sim}{x}/(2d-e), \tilde{x}/e,\ \tilde{x}/(2d-e)\},\ \max\{\underset{\sim}{x}/e,\ \underset{\sim}{x}/(2d-e), \tilde{x}/e,\ \tilde{x}/(2d-e)\}],$$

which is the Hadamard division performed in interval arithmetic (see (9)):

$$\boldsymbol{x}(\boldsymbol{I}, \boldsymbol{b}) = \frac{[\underset{\sim}{x},\ \tilde{x}]}{[e,\ 2d - e]}. \tag{12}$$

Since

$$[\underset{\sim}{x},\ \tilde{x}] = [x_* - (x^* - |x_*|),\ x_* + (x^* - |x_*|)] = \langle x_*,\ x^* - |x_*| \rangle$$

and

$$[e,\ 2d - e] = \langle d,\ d - e \rangle,$$

(12) implies (10). $\square$

5 The Overdetermined Case

Now we shall turn to the general case of an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with an $m \times n$ interval matrix $\boldsymbol{A} = \langle A_c, \Delta \rangle$ and an interval m-vector $\boldsymbol{b} = \langle b_c, \delta \rangle$. We shall assume below that A_c has full column rank, which already implies that $m \ge n$, i.e., that the system is overdetermined. In this case we are no longer able to describe the interval hull by simple formulae (the problem is NP-hard, see [3, Thm. 2.38]);

instead, we construct an enclosure (i.e., an interval vector containing the solution set) computable in polynomial time. As it will be seen in the following theorem, all that is needed is evaluation of one pseudoinverse, one inverse and several matrix-vector multiplications. In the proof, we shall essentially use the Oettli-Prager description [3, Thm. 2.9] of the solution set (1) by

$$\boldsymbol{X}(\langle A_c, \Delta\rangle, \langle b_c, \delta\rangle) = \{\, x \mid |A_c x - b_c| \le \Delta|x| + \delta \,\}.$$

In the description below we shall employ the pseudoinverse of the midpoint matrix. As is well known, for each matrix $A \in \mathbb{R}^{m\times n}$ there exists exactly one matrix $A^\dagger \in \mathbb{R}^{n\times m}$ satisfying

$$\begin{aligned} AA^\dagger A &= A, \\ A^\dagger AA^\dagger &= A^\dagger, \\ (A^\dagger A)^T &= A^\dagger A, \\ (AA^\dagger)^T &= AA^\dagger. \end{aligned}$$

This matrix is called the pseudoinverse (or Moore-Penrose inverse) of A. If A has full column rank, then $A^\dagger$ is given explicitly by $A^\dagger = (A^TA)^{-1}A^T$, so that $A^\dagger A = I$ in this case. If A is square nonsingular, then $A^\dagger = A^{-1}$.

Theorem 5.1. *Let A_c have full column rank and let $M := (I - |A_c^\dagger|\Delta)^{-1} \ge I$. Then we have*

$$\boldsymbol{X}(\langle A_c, \Delta\rangle, \langle b_c, \delta\rangle) \subseteq \frac{\langle x_*,\, x^* - |x_*|\rangle}{\langle d,\, d - e\rangle}, \tag{13}$$

where

$$\begin{aligned} d &= \operatorname{diag}(M), && (14)\\ x_* &= d \circ (A_c^\dagger b_c), && (15)\\ x^* &= M(|A_c^\dagger b_c| + |A_c^\dagger|\delta). && (16) \end{aligned}$$

Proof: Because A_c has full column rank, $A_c^\dagger A_c = I$. Thus, if $x \in \boldsymbol{X}(\langle A_c, \Delta\rangle, \langle b_c, \delta\rangle)$, the Oettli-Prager theorem implies

$$|A_c x - b_c| \le \Delta|x| + \delta$$

and hence also

$$|x - A_c^\dagger b_c| = |A_c^\dagger(A_c x - b_c)| \le |A_c^\dagger| \cdot |A_c x - b_c| \le |A_c^\dagger|\Delta|x| + |A_c^\dagger|\delta$$

which, again by the Oettli-Prager theorem, means that

$$x \in \boldsymbol{X}(\langle I, |A_c^\dagger|\Delta\rangle, \langle A_c^\dagger b_c, |A_c^\dagger|\delta\rangle).$$

In this way we have proved that

$$\boldsymbol{X}(\langle A_c, \Delta\rangle, \langle b_c, \delta\rangle) \subseteq \boldsymbol{X}(\langle I, |A_c^\dagger|\Delta\rangle, \langle A_c^\dagger b_c, |A_c^\dagger|\delta\rangle).$$

Now, the right-hand side solution set is by definition contained in its interval hull:

$$\boldsymbol{X}(\langle I, |A_c^\dagger|\Delta\rangle, \langle A_c^\dagger b_c, |A_c^\dagger|\delta\rangle) \subseteq \boldsymbol{x}(\langle I, |A_c^\dagger|\Delta\rangle, \langle A_c^\dagger b_c, |A_c^\dagger|\delta\rangle),$$

which by Theorem 4.1 implies

$$\boldsymbol{X}(\langle A_c, \Delta\rangle, \langle b_c, \delta\rangle) \subseteq \boldsymbol{x}(\langle I, |A_c^\dagger|\Delta\rangle, \langle A_c^\dagger b_c, |A_c^\dagger|\delta\rangle) = \frac{\langle x_*,\, x^* - |x_*|\rangle}{\langle d,\, d - e\rangle},$$

where in the formulae for M, d, x_* and x^* in Theorem 4.1 the values of Δ, b_c and δ were replaced by $|A_c^\dagger|\Delta$, $A_c^\dagger b_c$ and $|A_c^\dagger|\delta$, respectively, which gives (14)-(16). □

The enclosure (13) can be naturally also applied to the square case simply by replacing $A_c^\dagger$ by A_c^{-1}.

6 Example

Consider the example by Bentbib [1]

$$\boldsymbol{A}x = \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{pmatrix} [0.1, 0.3] & [0.9, 1.1] \\ [8.9, 9.1] & [0.4, 0.6] \\ [0.9, 1.1] & [6.9, 7.1] \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [0.8, 1.2] \\ [-0.2, 0.2] \\ [1.8, 2.2] \end{pmatrix}.$$

When we attempt to visualize the solution set with the help of the file `EqnWeak2D.m` by Sharaya [13], we get the message

```
>> EqnWeak2D(Ac-Delta,Ac+Delta,bc-delta,bc+delta)
Solution set is empty (it does not have boundary intervals)
```

which shows that no solution exists. If we, however, replace the right-hand side $\boldsymbol{b}$ by

$$\boldsymbol{b}' = \begin{pmatrix} [0.8, 1.2] \\ [0.3, 0.7] \\ [6.8, 7.2] \end{pmatrix},$$

then the same file `EqnWeak2D.m` depicts the solution set of

$$\boldsymbol{A}x = \boldsymbol{b}' \tag{17}$$

as nonempty (Fig. 1), an exponential orthant-by-orthant algorithm determines the interval hull as

$$\begin{pmatrix} [-0.0370, 0.0359] \\ [0.9522, 1.0494] \end{pmatrix},$$

and our method (13) yields the enclosure

$$\begin{pmatrix} [-0.0372, 0.0372] \\ [0.9471, 1.0548] \end{pmatrix}$$

whose overestimation can be considered acceptable.

Nevertheless, these results show that overdetermined systems require additional care when compared with square systems, where regularity of the interval matrix already guarantees nonemptiness of the solution set.

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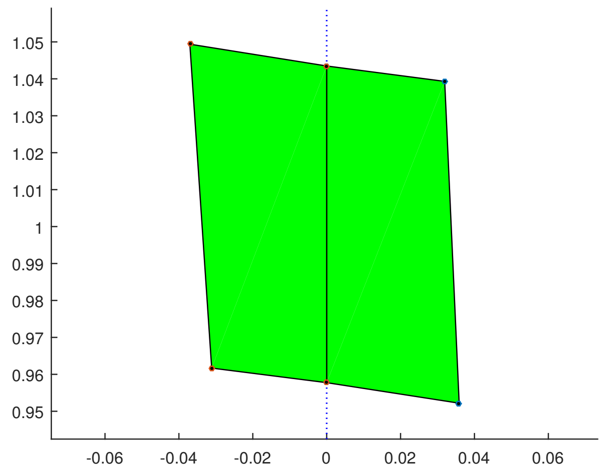

Figure 1: Plot of the solution set of (17).

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