A Class of Iterative Methods for Determining p-Solutions of Linear Interval Parametric Systems*

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Abstract

We consider a square linear interval parametric (LIP) system of size n whose elements are affine linear functions of the m-dimensional parameter vector p. Recently, a new type of solution to the LIP system considered (called parameterized or p-solution) has been introduced, which is of a corresponding linear interval (LI) form. An iterative method for determining the linear p-solution has also been suggested.

The objective of the present paper is to generalize the above approach in two directions. First, a new type of p-solution in a corresponding quadratic interval (QI) form is suggested. Second, it is shown that any known iterative method for determining an outer solution to the LIP system given can be modified in a unified manner to produce a corresponding method yielding a linear or quadratic p-solution. Thus, a class of iterative methods for determining p-solutions can be constructed, depending on the iterative scheme chosen and the form, linear or quadratic, of the solution sought. As an illustration, two specific methods for determining a p-solution, based on a simple iterative process and respective LI and QI forms, are suggested.

The proposed p-solutions seem to be useful in solving global optimization problems where the constraint is given as a LIP system. As an example, a parametric linear programming problem is considered.

Keywords: linear parametric systems, linear parameterized solutions, quadratic parameterized solutions
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1 Introduction

Let

$$A(p)x = b(p), \quad p \in \boldsymbol{p} \tag{1a}$$

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denote a square linear interval parametric (LIP) system of size n whose elements $a_{ij}(p)$ and $b_i(p)$ are the affine linear functions

$$a_{ij}(p) = a_{ij} + \sum_{\mu=1}^{m} \alpha_{ij\mu} p_\mu, \quad b_i(p) = \beta_i + \sum_{\mu=1}^{m} \beta_{i\mu} p_\mu. \tag{1b}$$

The united solution set of (1) is the collection of all solutions of (1a), (1b) over $\boldsymbol{p}$, i.e. the set

$$\sum(A(p), b(p), \boldsymbol{p}) = \{x : A(p)x = b(p), p \in \boldsymbol{p}\}.$$

As is well known, the following "interval solutions" to (1) are most often considered (cf., e.g., [20, 4, 16, 7, 8, 17, 24, 15, 9, 23, 18, 25, 11, 21, 22, 19]): (i) interval hull (IH) solution $\boldsymbol{x}^*$: the smallest interval vector containing the united solution set of (1); (ii) outer interval (OI) solution $\boldsymbol{x}$: any interval vector enclosing $\boldsymbol{x}^*$, i.e. $\boldsymbol{x}^* \subseteq \boldsymbol{x}$; (iii) inner estimation of the hull (IEH) solution ζ: an interval vector such that $\zeta \subseteq \boldsymbol{x}^*$.

A new type of solution $\boldsymbol{x}(p)$ to the LIP system (1) (called parameterized or p-solution) has been recently introduced in [12]. It is defined as follows

$$\boldsymbol{x}(p) = \boldsymbol{l}(p) \tag{2a}$$

where

$$\boldsymbol{l}(p) = Lp + \boldsymbol{a}, \quad p \in \boldsymbol{p} \tag{2b}$$

is a corresponding linear interval (LI) form (L is a real $n \times n$ matrix while $\boldsymbol{a}$ is an n-dimensional interval vector). An iterative method for determining $\boldsymbol{x}(p)$ was suggested in [12] which is obtained by modifying each step of a known iterative method for computing $\boldsymbol{x}$ [8]. The solution $\boldsymbol{x}(p)$ (henceforth referred to as a linear parameterized (LP) solution), has a number of useful properties such as: it directly yields an outer interval solution $\boldsymbol{x}$ and an inner approximation ζ of the hull solution $\boldsymbol{x}^*$. However, as underlined in [12], the main advantages of $\boldsymbol{x}(p)$ reside in the fact that it can form the basis of a new paradigm for solving the following class of optimization problems: find the global minimum

$$g_k^* = \min g_k(x, p) \tag{3}$$

subject to the constraint (1) where $g_k(x, p)$ is the kth component of the (in the general case, nonlinear) mapping $g : R^{n+m} \to R^{n'}$, $1 \le n' \le n$. Thus, the lower end $\underline{x}_k^*$ of the kth component $\boldsymbol{x}_k^* = [\underline{x}_k^*, \overline{x}_k^*]$ of $\boldsymbol{x}^*$ is determined as the solution of the following global optimization problem

$$\underline{x}_k^* = \min e_k^T x \tag{4a}$$

(e_k is the kth column of the identity matrix) subject to the constraint (1). In a similar way, $\overline{x}_k^*$ is the solution of

$$\underline{x}_k^* = -\min(-e_k^T x) \tag{4b}$$

and the constraint (1). Combined with a constraint satisfaction technique, such an approach permits determination of the hull $\boldsymbol{x}^*$ as well as the global solution of certain equality-constrained optimization problems [12].

The objective of the present paper is two-fold. First (Section 2.2), we suggest a new type of p-solution in the following quadratic interval (QI) form:

$$\boldsymbol{x}(p) = \boldsymbol{q}(p), \tag{5a}$$

$$\boldsymbol{q}(p) = Q\theta(p) + Lp + \boldsymbol{a}, \quad p \in \boldsymbol{p}, \quad \theta_j(p) = \theta_j(p_j) = p_j^2, \quad j = 1, ..., m \tag{5b}$$

where Q denotes a three-dimensional $n \times m \times m$ array whose ith component Q_i is a $m \times m$ matrix, L and $\boldsymbol{a}$ having the same meaning (but different entries) as in (2b). Second (Section 3), we show that any known iterative method for determining $\boldsymbol{x}$ (e.g. [7], [8], various fixed-point representations such as in [14] or [1], Ch. 12) can be modified in a unified manner to produce a corresponding method for determining $\boldsymbol{x}(p)$. Thus, a whole class of iterative methods yielding $\boldsymbol{x}(p)$ can be constructed. In this context, the method for determining a LI form solution $\boldsymbol{x}(p)$ of [12] is just a representative of this class. As an illustration, two new specific methods for determining $\boldsymbol{x}(p)$, based on a simple iterative process and respective LI and QI forms, are suggested in Subsections 3.1 and 3.2, respectively. It is shown that the latter methods may be better than the former method of [12].

The p-solutions seem promising for solving the global optimization problems (3). The simplest case (4), determining $\boldsymbol{x}^*$, is considered in Subsection 4.1. Two other representatives of (3), linear and quadratic programming, are presented in Subsection 4.2. An example referring to parametric linear programming is solved in Section 5.

2 Parameterized Solutions

2.1 Linear parameterized solution

In this subsection, we recall the definition and basic properties of the linear parameterized (LP) solution of (1) (this information will be useful in the subsequent subsection). Without loss of generality, it is assumed as in [12] that the parameter vector $\boldsymbol{p}$ is a symmetric vector of unit radius, i.e., $\boldsymbol{p}_i = [-1, 1]$ for $i = 1, ..., m$.

System (1) determines each member of the solution set of (1) as an implicit function of p, i.e. $x(p) = f(p)$, $p \in \boldsymbol{p}$ where $f : p \subset R^m \to R^n$. As in [12] we now temporarily assume that the function $f(p)$ is known (new methods for approximating $f(p)$ will be given in Section 3). It can then be enclosed for $p \in \boldsymbol{p}$ by the linear interval form (2b) [6] so

$$f(p) \in \boldsymbol{l}(p), \quad p \in \boldsymbol{p}. \tag{6}$$

Let $\boldsymbol{l}(\boldsymbol{p})$ denote the range of $\boldsymbol{l}(p)$ over $\boldsymbol{p}$, i.e. $\boldsymbol{l}(\boldsymbol{p}) = \{\boldsymbol{l}(p) : p \in \boldsymbol{p}\}$. On account of (2) and (6), the range $\boldsymbol{x}(\boldsymbol{p})$ of $\boldsymbol{x}(p)$ yields an outer solution $\boldsymbol{x}$ of (1), i.e.

$$\boldsymbol{x} = \boldsymbol{l}(\boldsymbol{p}). \tag{7}$$

Obviously, the range $f_i(\boldsymbol{p})$ over $\boldsymbol{p}$ defines the ith component $\boldsymbol{x}_i^*$ of the IH solution $\boldsymbol{x}^*$ to (1). Since $\boldsymbol{l}(\boldsymbol{p})$ is an outer solution, the component $\boldsymbol{x}_i^*$ of $\boldsymbol{x}^*$ is contained in the component $\boldsymbol{l}_i(\boldsymbol{p})$ of the range $\boldsymbol{l}(\boldsymbol{p})$, i.e.

$$\boldsymbol{x}_i^* \subset \boldsymbol{l}_i(\boldsymbol{p}). \tag{8}$$

The above inclusion can be made more specific [12]. To this end, consider the ith component $\boldsymbol{l}_i(p)$ of $\boldsymbol{l}(p)$. Let L_i denote the ith row of L. Then $\boldsymbol{l}_i(p) = L_i p + \boldsymbol{a}_i$. Let

$$\lambda_i(p) = L_i p, \quad p \in \boldsymbol{p}. \tag{9}$$

and denote

$$\underline{\lambda}_i = \min(L_i p), \quad \overline{\lambda}_i = \max(L_i p), \quad p \in \boldsymbol{p}. \tag{10}$$

Two intervals are now introduced

$$\boldsymbol{e}_i^{(l)} = \underline{\lambda}_i + \boldsymbol{a}_i,\ \boldsymbol{e}_i^{(u)} = \overline{\lambda}_i + \boldsymbol{a}_i,\ \boldsymbol{e}_i^{(l)} = [\underline{e}_i^{(l)}, \overline{e}_i^{(l)}],\ \boldsymbol{e}_i^{(u)} = [\underline{e}_i^{(u)}, \overline{e}_i^{(u)}]. \tag{11}$$

The following result has been recently proved.

Theorem 1 *[12]. Let $\mathbf{e}_i^{(l)}$ and $\mathbf{e}_i^{(u)}$ be the intervals defined by (9) to (11); also let $\underline{x}_i^*$ and $\overline{x}_i^*$ be the endpoints of $\boldsymbol{x}_i^*$. Then*

$$\underline{x}_i^* \in \boldsymbol{e}_i^{(l)}, \overline{x}_i^* \in \boldsymbol{e}_i^{(u)}. \tag{12}$$

Corollary 1 *Introduce the interval*

$$\zeta_i = \begin{cases} [\overline{e}_i^{(l)}, \underline{e}_i^{(u)}], & \text{if} \quad \overline{e}_i^{(l)} \leq \underline{e}_i^{(u)} \\ \text{empty interval}, & \text{otherwise} \end{cases} . \tag{13}$$

Then ζ_i determines the ith component of the IEH solution of (1).

Remark 1. It should be stressed that formula (13) in Corollary 1 of [12] was written incorrectly in the form

$$\zeta_i = \begin{cases} [\overline{e}_i^{(l)}, \underline{e}_i^{(u)}], & \text{if} \quad \overline{e}_i^{(l)} \leq \underline{e}_i^{(u)} \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

It is seen that knowledge of the new solution $\boldsymbol{x}(p)$, $p \in \boldsymbol{p}$, i.e. knowledge of L and $\boldsymbol{a}$, permits determining an outer interval solution $\boldsymbol{x}$, an inner estimation of the hull solution ζ as well intervals containing the endpoints $\underline{x}_i^*$ and $\overline{x}_i^*$ of each component $\boldsymbol{x}_i^*$ of the interval hull solution x^* related to (1).

2.2 Quadratic parameterized solution

The properties of the LP solution (2), (4a) will now be extended to the case of the QP solution (5a), (5b). In this case, $f(p)$ is enclosed for $p \in \boldsymbol{p}$ by the QI form (5b). Thus

$$f(p) \in \boldsymbol{q}(p), \quad p \in \boldsymbol{p}. \tag{14}$$

If $\boldsymbol{q}(\boldsymbol{p})$ denotes the range of $\boldsymbol{q}(p)$ over $\boldsymbol{p}$ then

$$f(p) \in \boldsymbol{q}(\boldsymbol{p}), \quad p \in \boldsymbol{p}. \tag{15}$$

Hence, the range $\boldsymbol{q}(\boldsymbol{p})$ yields an outer solution $\boldsymbol{x}$ of (1), i.e.

$$\boldsymbol{x} = \boldsymbol{q}(\boldsymbol{p}). \tag{16}$$

On account of (16), the ith component $\boldsymbol{x}_i^*$ of the IH solution $\boldsymbol{x}^*$ to (1) is contained in the ith component $\boldsymbol{q}_i(\boldsymbol{p})$ of the range $\boldsymbol{q}(\boldsymbol{p})$, i.e.

$$\boldsymbol{x}_i^* \subset \boldsymbol{q}_i(\boldsymbol{p}). \tag{17}$$

As in the linear case, the above inclusion can be made more specific if we consider closely the ith component $\boldsymbol{q}_i(p)$ of $\boldsymbol{q}(p)$. Let Q_i and L_i denote the ith row of Q and L, respectively. Then from (5b)

$$\boldsymbol{q}_i(p) = Q_i \varphi(p) + L_i p + \boldsymbol{a}_i. \tag{18}$$

Let

$$\lambda_i(p) = Q_i \theta_i(p) + L_i p = \sum_j \varphi_{ij}(p_j), \quad \varphi_{ij}(p_j) = Q_{ij} p_j^2 + L_{ij} p_j \tag{18a}$$

and denote

$$\underline{\lambda}_i = \min(\lambda_i(p)), \quad p \in \boldsymbol{p}, \quad \overline{\lambda}_i = \max(\lambda_i(p)), \quad p \in \boldsymbol{p}. \tag{19}$$

To simplify the presentation, we first consider the case where each $\varphi_{ij}(p_j)$ is monotone within $\boldsymbol{p}_j$. We introduce the following notations:

$$\varphi_{ij}^{-}(p_j) = \varphi_{ij}(-1), \quad \varphi_{ij}^{+}(p_j) = \varphi_{ij}(1), \quad \varphi_{ij}^{(1)} = \min\{\varphi_{ij}^{-}, \varphi_{ij}^{+}\}, \quad \varphi_{ij}^{(2)} = \max\{\varphi_{ij}^{-}, \varphi_{ij}^{+}\}.$$

Since $\lambda_i(p)$ is a separable function

$$\underline{\lambda}_i = \sum_j \varphi_{ij}^{(1)}, \quad \overline{\lambda}_i = \sum_j \varphi_{ij}^{(2)}.$$

Thus, the following two intervals are now introduced for the case of monotone $\varphi_{ij}(p_j)$

$$\boldsymbol{e}_i^{(l)} = \underline{\lambda}_i + \boldsymbol{a}_i, \quad \boldsymbol{e}_i^{(u)} = \overline{\lambda}_i + \boldsymbol{a}_i. \tag{20}$$

In the general case, the functions $\varphi_{ij}(p_j)$ will not be monotone for some indices $j \in J_i$. The set J_i now will be divided into two subsets J_i' and J_i'' if $Q_{ij} > 0$ or $Q_{ij} < 0$, respectively.

Let

$$\underline{\varphi}_{ij} = \min\{\varphi_{ij}(p_j), \quad p_j \in \boldsymbol{p}_j\}, \quad j \in J_i',$$
$$\overline{\varphi}_{ij} = \max\{\varphi_{ij}(p_j), \quad p_j \in \boldsymbol{p}_j\}, \quad j \in J_i''.$$

From geometrical considerations, it is easily seen that if $j \in J_i'$ an additional interval $[\underline{\varphi}_{ij}, \varphi_{ij}^{(1)}]$ arises which contributes to an increase in $\boldsymbol{e}_i^{(l)}$. In a similar way, if $j \in J_i''$ the additional interval increasing $\boldsymbol{e}_i^{(u)}$ is $[\varphi_{ij}^{(2)}, \overline{\varphi}_{ij}]$. To obtain the sum of these intervals, we compute

$$\underline{\varphi}_i = \sum \underline{\varphi}_{ij}^{(1)}, \quad \varphi_i^{(1)} = \sum \underline{\varphi}_{ij}^{(1)}, \quad j \in J_i',$$
$$\overline{\varphi}_i = \sum \underline{\varphi}_{ij}^{(2)}, \quad \varphi_i^{(2)} = \sum \varphi_{ij}^{(1)}, \quad j \in J_i''.$$

Finally

$$\boldsymbol{e}_i^{(l)} = \underline{\lambda}_i + [\underline{\varphi}_i, \varphi_i^{(1)}] + \boldsymbol{a}_i, \quad \boldsymbol{e}_i^{(u)} = \overline{\lambda}_i + [\varphi_i^{(2)}, \overline{\varphi}_i] + \boldsymbol{a}_i. \tag{21}$$

By analogy with the linear case, the following results can be proved.

Theorem 2 *Let $\mathbf{e}_i^{(l)}$ and $\mathbf{e}_i^{(u)}$ be the intervals defined by (20) or (21); also let $\underline{x}_i^*$ and $\overline{x}_i^*$ be the endpoints of $\mathbf{x}_i^*$. Then*

$$\underline{x}_i^* \in \mathbf{e}_i^{(l)}, \quad \overline{x}_i^* \in \mathbf{e}_i^{(u)}. \tag{22}$$

Corollary 2 *Introduce the interval*

$$\zeta_i = \begin{cases} [\overline{e}_i^{(l)}, \underline{e}_i^{(u)}], & \textit{if} \quad \overline{e}_i^{(l)} \le \underline{e}_i^{(u)} \\ \textit{empty interval}, & \textit{otherwise} \end{cases}. \tag{23}$$

Then ζ_i determines the ith component of the IEH solution of (1).

Whenever $\boldsymbol{q}_i(\boldsymbol{p})$ is a narrower interval than $\boldsymbol{l}_i(\boldsymbol{p})$, the QP solution is a better option than the LP solution.

3 Methods for Determining a p-Solution

System (1) is written equivalently in the form

$$A(p)x = b(p), \quad p \in \boldsymbol{p} \tag{24a}$$

$$A(p) = A^0 + \sum_{\mu=1}^{m} A^{(\mu)} p_\mu, \quad b(p) = b^0 + Bp \tag{24b}$$

where A^0, $A^{(\mu)}$ are $n \times n$ real matrices, while B is a $n \times m$ real matrix and b^0 is a real column vector.

A unified iterative scheme for computing $\boldsymbol{x}(p)$ is proposed in this paper. It is based on the following approach comprising two stages: first, (24) is transformed into an equivalent iterative fixed-point representation and second, each iteration is enclosed by a linear interval (LI) or quadratic interval (QI) form. Thus, a whole new class of iterative methods for solving (1) can be constructed, each individual method being determined by the selection of specific ways for addressing the first and second stage of the above unified scheme.

From this general point of view, the method for determining $\boldsymbol{x}(p)$ suggested in [12] is based on the representation

$$\boldsymbol{x}(p) = x^0 + \boldsymbol{v}(p), \quad p \in \boldsymbol{p} \tag{25}$$

where (assuming A^0 nonsingular) x^0 is the solution of the midpoint system $A^0 x = b^0$. In order to determine $\boldsymbol{v}(p)$, the following fixed-point iteration was used

$$\boldsymbol{v}^{(\kappa+1)}(p) = -\left(\sum_{\mu} p_\mu B^{(\mu)}\right) \boldsymbol{v}^{(\kappa)}(p) + C^{(0)} p, \quad \kappa \geq 0, \quad \nu^{(0)} = 0 \tag{26}$$

where $B^{(\mu)}$ and $C^{(0)}$ are computed using $A^{(\mu)}$, B and $(A^0)^{-1}$ [12]; for example

$$B^{(\mu)} = (A^0)^{-1} A^{(\mu)}. \tag{26a}$$

A LI form of the type (4a) was used to approximate each iteration (26).

According to the new general scheme, a new version of the above method could be obtained using the same iteration (26) and applying a corresponding QI form to (26) for each κ.

New methods for determining $\boldsymbol{x}(p)$ will be obtained if alternative to (26) iterative procedures are used. As an illustration, we consider the simplest fixed-point representation of (24)

$$x(p) = (I - A(p))x(p) + b(p), \quad p \in \boldsymbol{p}. \tag{27}$$

On account of (27) and (24b), the iterative process is now

$$x^{(\kappa+1)}(p) = \left(A^{(0)} - \sum_{\mu} p_\mu A^{(\mu)}\right) x^{(\kappa)}(p) + b^0 + Bp, \quad \kappa \geq 0, \quad x^{(0)} = x^0 \tag{28}$$

where $A^{(0)} = I - A^0$ and x^0 is the solution of (24) for $p = 0$ ($\boldsymbol{p}$ is symmetric). As $(A^0)^{-1}$ is now not used, the new iteration (28) is found to be a better alternative than (26) for sparse systems (1) of large enough n. Indeed, as seen from (26a), $B^{(\mu)}$ will be roughly dense matrices even if $A^{(\mu)}$ are fairly sparse.

3.1 Linear iterative method

In this subsection, a linear iterative method for determining the LP solution $\boldsymbol{x}(p)$ (i.e. for computing the associated L and $\boldsymbol{a}$) is suggested which is related to the iteration (28). As in [12], it can be shown that each iteration $x^{(\kappa)}(p)$ in (28) can be enclosed by a corresponding linear interval form

$$\boldsymbol{l}^{(\kappa+1)}(p) = c^{(\kappa+1)} + L^{(\kappa+1)}p + \boldsymbol{s}^{(\kappa+1)}, \quad p \in \boldsymbol{p}, \quad \kappa \geq 0, \tag{29}$$

where $c^{(\kappa+1)}$ and $\boldsymbol{s}^{(\kappa+1)}$ are a real and interval symmetric vectors, respectively.

Indeed, for a fixed p, (28) defines $x^{(\kappa+1)}(p)$ as an explicit function $f(\kappa) : \boldsymbol{p} \in R^m \rightarrow R^n$, i.e.

$$x^{(\kappa+1)}(p) = f^{(\kappa)}(p), \quad \kappa \geq 0, \quad p \in \boldsymbol{p}. \tag{30}$$

Let $S^{(\kappa+1)}$ denote the image of $\boldsymbol{p}$ under $f^{(\kappa)}$. As can be easily seen, $S^{(1)}$ is the image of a linear function $f^{(0)}(p)$. Indeed, for $\kappa = 0$ from (28)

$$x^{(1)}(p) = l^{(1)}(p) = c^{(1)} + L^{(1)}p, \quad p \in \boldsymbol{p} \tag{31a}$$

with

$$c^{(1)} = A^{(0)}c^{(0)} + b^0, \quad c^{(0)} = x^0, \quad L^{(1)} = B^{(1)} + b \tag{31b}$$

where the jth column $B_j^{(1)}$ of $B^{(1)}$ is

$$B_j^{(1)} = -A^{(j)}c^{(0)}. \tag{31c}$$

However, for $\kappa \geq 1$ $S^{(\kappa+1)}$ is not a LI form since $f^{(\kappa)}(p)$ is now a nonlinear function. Thus, for $\kappa = 1$

$$x^{(2)}(p) = \left(A^{(0)} - \sum_{\mu} p_\mu A^{(\mu)}\right) x^{(1)}(p) + b^0 + Bp = T(p)x^{(1)}(p) + b^0 + Bp, \tag{32a}$$

$$T(p) = A^{(0)} - \sum_{\mu} p_\mu A^{(\mu)}. \tag{32b}$$

On account of (31a) and (32a)

$$x^{(2)}(p) = \left(A^{(0)} - \sum_{\mu} p_\mu A^{(\mu)}\right) (c^{(1)} + L^{(1)}p) + b^0 + Bp \tag{32c}$$

so

$$c' = A^{(0)}c^{(1)} + b^0, \quad M^{(2)} = B^{(2)} + B + A^{(0)}L^{(1)}, \quad B_j^{(2)} = -A^{(j)}c^{(1)}. \tag{32d}$$

The product of $\sum_\mu p_\mu A^{(\mu)}$ and $L^{(1)}p$ yields a nonlinear term. As is easily seen, each component $x_i^{(2)}(p)$ of $x^{(2)}(p)$ is a quadratic function of the elements of p

$$x_i^{(2)}(p) = p^T Q_i^{(2)} p + M_i^{(2)} p + c_i^{(2)} = f_i^{(1)}(p), \quad p \in \boldsymbol{p} \tag{33}$$

where Q_i is a $m \times m$ matrix whose elements are $q_{ik\mu}^{(2)} = -\sum_j \alpha_{ijk} l_{j\mu}^{(1)}$. Thus, it has been shown that $S^{(2)}$ is not a LI form.

The quadratic form $f_i^{(1)}$ is written as (omitting the superscript (2))

$$f_i^{(1)}(p) = \sum_k (q_{ikk}p_k^2 + m_{ik}p_k) + \sum_{k<\mu} (q_{ik\mu} + q_{i\mu k})p_k p_\mu + c_i, \quad p \in \boldsymbol{p}. \tag{34}$$

We now look for an enclosure of (34). The second term in (34) gives rise to a symmetric interval

$$\boldsymbol{s}'_i = s'_i[-1,1], \quad s'_i = \sum_{k<\mu} |q_{ik\mu} + q_{i\mu k}|. \tag{35a}$$

Each component $\varphi_{ik}(p_k) = q_{ikk}p_k^2 + m_{ik}p_k$ is approximated outwardly in $\boldsymbol{p}_k$ by a LI form $\boldsymbol{l}_{ik}(p_k)$ (in a similar manner as in [12]) so $\varphi_{ik}(p_k) \in \boldsymbol{l}_{ik}(p_k) = c''_{ik} + L''_{ik}p_k + \boldsymbol{s}''_{ik}$. Thus, new components

$$c''_i = \sum c'_{ik}, \quad L^{(2)}_{ik} = L''_{ik}, \quad \boldsymbol{s}''_i = \sum \boldsymbol{s}'_{ik}, \tag{35b}$$

are produced. On account of (32d) and (35)

$$\boldsymbol{l}^{(2)}_i = c^{(2)}_i + L^{(2)}_i p + \boldsymbol{s}^{(2)}_i, \quad \boldsymbol{s}^{(2)}_i = [-s^{(2)}_i, s^{(2)}_i] \tag{36}$$

where $c^{(2)}_i$ and $\boldsymbol{s}^{(2)}_i$ are the sums of the corresponding components c'_i, c''_i and $\boldsymbol{s}'_i$, $\boldsymbol{s}''_i$. Using $\boldsymbol{l}^{(2)}_i(p)$, $i = 1, ..., n$ we form the LI form

$$\boldsymbol{l}^{(2)}(p) = c^{(2)} + L^{(2)}p + \boldsymbol{s}^{(2)} \tag{37}$$

that encloses $x^{(2)}(p)$ (with components given in (33)) in $\boldsymbol{p}$, i.e.

$$x^{(2)}(p) = f^{(1)}(p) \in \boldsymbol{l}^{(2)}(p). \tag{38}$$

Next, we processed to constructing an approximation for $x^{(3)}(p)$. With this in mind, we first replace the relationship

$$x^{(3)}(p) = T(p)x^{(2)}(p) + b^0 + Bp, \quad T(p) = A^{(0)} - \sum_{\mu} p_\mu A^{(\mu)} \tag{39a}$$

with

$$\begin{aligned} x^{(3)}(p) &= T(p)\boldsymbol{l}^{(2)}(p) + b^0 + Bp \\ &= T(p)(c^{(2)} + L^{(2)}p) + b^0 + Bp + T(p)\boldsymbol{s}^{(2)} = g^{(2)}(p) + T(p)\boldsymbol{s}^{(2)}. \end{aligned} \tag{39b}$$

Obviously, by analogy with the previous case of $f^{(1)}_i$, each component $g^{(2)}_i(p)$ of $g^{(2)}$ is a quadratic function

$$g^{(2)}_i(p) = p^T Q^{(3)}_i p + L^{(3)}_i p + c^{(3)}_i \tag{40}$$

which can be linearized to give a corresponding LI form

$$\boldsymbol{l}^{(3)}_i(p) = c^{(3)}_i + L^{(3)}_i p + \boldsymbol{t}'_i. \tag{41a}$$

However, unlike the case of $f^{(1)}_i$, the product $T(p)\boldsymbol{s}^{(2)}$, $p \in \boldsymbol{p}$ gives rise to an additional symmetric interval

$$\boldsymbol{t}''_i = \left(\sum_j \left(|A^{(0)}_{ij}| + \sum_\mu |A^{(\mu)}_{ij}| \right) s^{(2)}_j \right) [-1,1]. \tag{41b}$$

Hence

$$\boldsymbol{l}^{(3)}(p) = c^{(3)} + L^{(3)}p + \boldsymbol{s}^{(3)} \tag{41c}$$

where the components of $\boldsymbol{s}^{(3)}$ are $\boldsymbol{s}^{(3)}_i = \boldsymbol{t}'_i + \boldsymbol{t}''_i$. It is seen that $\boldsymbol{l}^{(3)}(p)$ in (41c) has the same form as $\boldsymbol{l}^{(2)}(p)$ in (37). Evidently, this process of successively generating new quadratic functions and their subsequent interval linearization can continue for $\kappa \geq 2$.

Consider the range $\boldsymbol{l}^{(\kappa)}(\boldsymbol{p})$ of the corresponding LI form $\boldsymbol{l}^{(\kappa)}(p)$. Clearly, $\boldsymbol{l}^{(\kappa)}(\boldsymbol{p})$ is an interval vector having the properties

$$S^{(k)} \subset \boldsymbol{l}^{(\kappa)}(\boldsymbol{p}), \tag{42}$$

$$\boldsymbol{l}^{(\kappa)}(\boldsymbol{p}) \subset \boldsymbol{l}^{(\kappa+1)}(\boldsymbol{p}). \tag{43}$$

The distance between two such interval vectors (43) will be assessed using the formula

$$d(\boldsymbol{a}, \boldsymbol{b}) = \max\{\max_i |\underline{a}_i - \underline{b}_i|, \max_i |\overline{a}_i - \overline{b}_i|\}. \tag{44}$$

Now we can formulate the main result of the subsection.

Theorem 3 *Assume that the matrix $A^{(0)}$ is non-singular and the sequence $\{\boldsymbol{l}^{(\kappa)}(\boldsymbol{p})\}$, $\kappa \leq 1$ is convergent in the sense of (44) to a limit $\boldsymbol{l}^{(\infty)}(\boldsymbol{p})$. Then:*

(i) the linear interval form

$$\boldsymbol{x}(p) = \boldsymbol{l}^{(\infty)}(p) = c^{(\infty)} + L^{(\infty)}p + \boldsymbol{s}^{(\infty)}, \quad p \in \boldsymbol{p} \tag{45}$$

determines a LP solution to (1),

(ii) the interval vector

$$\boldsymbol{x} = \boldsymbol{l}^{(\infty)}(\boldsymbol{p}) \tag{46}$$

(46) is an OI solution to (1),

(iii) the matrix $A(p)$ is non-singular for each $p \in \boldsymbol{p}$.

The above method for determining will be referred to as linear interval form (LIF) method.

3.2 Quadratic iterative method

In this subsection, an iterative method for determining will be suggested which is based on the QI form (5b) and iteration (28). The new method (referred to as QIF method) is expected to outperform the LIF method for relatively narrower $\boldsymbol{p}$ intervals.

Initially, the QIF method repeats the first steps $\kappa = 0$ and $\kappa = 1$ of the LIF method until the quadratic representation (33) is formed. At that point, the quadratic part is, however, not linearized and the next step $\kappa = 2$ is started with

$$x_i^{(2)}(p) = p^T Q_i^{(2)} p + L_i^{(2)} p + c_i^{(2)}, \quad i = 1, ..., n. \tag{47}$$

We now use the relationship

$$x^{(3)}(p) = T(p)x^{(2)}(p) + b^0 + Bp = f^{(2)}(p). \tag{48}$$

As can be easily seen, each component $f_i^{(2)}(p)$ of $f^{(2)}(p)$ is a cubic expression which is written in the form

$$f_i^{(2)}(p) = \sum_j C_{ij}(p_j) + R_{ij}, \quad C_{ij}(p_j) = a_0^{ij} + a_1^{ij} p_j^1 + a_2^{ij} p_j^2 + a_3^{ij} p_j^3, \quad j = 1, ..., m \tag{49}$$

where R_{ij} regroups the remaining (quadratic, linear and mixed) terms. Now each jth cubic term $C_{ij}(p_j)$ is approximated outwardly by a corresponding quadratic interval form so we get

$$\boldsymbol{q}_i^{(3)}(p) = Q_i^{(3)} \theta_i(p) + L_i^{(3)} p + c_i^{(3)} + \boldsymbol{s}_i^{(3)}. \tag{50}$$

At the next step for $k = 3$, the relationship

$$x^{(4)}(p) = T(p)x^{(3)}(p) + b^0 + Bp \tag{51}$$

is replaced with $x^{(4)}(p) = T(p)\boldsymbol{q}^{(3)}(p) + b^0 + Bp = f^{(3)}(p)$.

Now $f^{(3)}(p)$ again contains a cubic function which is anew approximated in a quadratic manner. Evidently, this process of successively generating new cubic functions and their subsequent quadratic approximation can continue for $\kappa \geq 3$ so on account of (50)

$$\boldsymbol{q}_i^{(\kappa)}(p) = Q_i^{(\kappa)}\theta_i(p) + L_i^{(\kappa)}p + c_i^{(\kappa)} + \boldsymbol{s}_i^{(\kappa)}, \quad i = 1, ..., n \tag{52}$$

Consider the range $\boldsymbol{q}^{(\kappa)}(\boldsymbol{p})$ of the corresponding QI form $\boldsymbol{q}^{(\kappa)}(p)$. Clearly, $\boldsymbol{q}^{(\kappa)}(\boldsymbol{p})$ is an interval vector having the properties

$$S^{(k)} \subset \boldsymbol{q}^{(\kappa)}(\boldsymbol{p}), \tag{53}$$

$$\boldsymbol{q}^{(\kappa)}(\boldsymbol{p}) \subset \boldsymbol{q}^{(\kappa+1)}(\boldsymbol{p}). \tag{54}$$

The distance between two such interval vectors (54) will be again assessed using (54). Now we can formulate the main result of the subsection.

Theorem 4 *Assume that the matrix $A^{(0)}$ is non-singular and the sequence $\{\boldsymbol{q}^{(\kappa)}(\boldsymbol{p})\}$, $\kappa \geq 1$ is convergent in the sense of (54) to a limit $\{\boldsymbol{q}^{(\infty)}(\boldsymbol{p})\}$. Then:*

(i) the quadratic interval form

$$\boldsymbol{x}(p) = \boldsymbol{q}^{(\infty)}(p) = c^{(\infty)} + L^{(\infty)}p + Q^{(\infty)}\varphi(p) + \boldsymbol{s}^{(\infty)}, \quad p \in \boldsymbol{p} \tag{55}$$

determines a QP solution to (1),

(ii) the interval vector

$$\boldsymbol{x} = \boldsymbol{q}^{(\infty)}(\boldsymbol{p}) \tag{56}$$

is an OI solution to (1),

(iii) the matrix $A(p)$ is non-singular for each $p \in \boldsymbol{p}$.

3.3 An algorithm

A method pertaining to the new class suggested in the present paper can be categorized according to the following features:

A – iterative scheme employed:

A1 – fixed-point iterations on the $\boldsymbol{p}$-solution $x(p)$ of the non-preconditioned system (27) according to (28) or

A2 – fixed-point iterations on the centered solution $\boldsymbol{v}(p) = \boldsymbol{x}(p) - x^{(0)}$ (25) of a pre-conditioned system resulting in the iterative process (26) [12];

B – type of $\boldsymbol{p}$-solution used:

B1 – LI form solution or

B2 – QI form solution. Each of the above methods can be implemented using an appropriate algorithm. An algorithm for the method employing the characteristics A2 and B1 has been presented in [12].

We now sketch an algorithm for the method A1.B2 from Section 3.2.

Algorithm A1.B2

The algorithm starts with a preliminary stage where the components $f_i^{(1)}(p)$ of $f^{(1)}(p)$

are obtained in the form the quadratic expressions (34). First, we find the affine form (31), i.e. $c^{(1)}$ and $L^{(1)}$ by (31b), (31c). Next, $c^{(2)}$ and $L^{(2)}$ are computed by (32c), (32d). Using these data, we compute the matrices $Q_i^{(2)}$ and vectors $L_i^{(2)}$ and $c_i^{(2)}$ in the quadratic form (33) and (34). The basic cycle of the algorithm is now initiated.

Step 1. Using (32b), (48) and (47), we find the coefficients a_0, a_1, a_2 and a_3 in the cubic expressions $C_{ij}(p_j)$ for every $f_i^{(2)}(p)$ in (49).

Step 2. For $i = 1, ..., n$ each cubic term is now enclosed by a corresponding QI polynomial (using Procedure 1 given below)

$$\boldsymbol{q}_{ij}(p_j) = q_{0ij} + q_{1ij}p_j + q_{2ij}p_j^2 + \boldsymbol{s}_{ij}, \quad p \in [-1, 1]. \tag{57}$$

Step 3. Replacing in (49) each $C_{ij}(p_j)$ by (57), we obtain the corresponding QI form $\boldsymbol{q}_i^{(3)}(p)$ in (50).

Step 4. Using (52) and (32b), find the corresponding cubic function $C_{ij}(p_j)$ in $f_i^{(3)}(p)$.

The iteration process is resumed from Step 2 after renaming the index (k) of the current iteration as $(k-1)$. The algorithm is terminated whenever the distance between two successive iterations in (54) computed by (44) becomes smaller than a threshold ϵ or the number of iterations reaches a limit value.

The outward approximation of the cubic function by a quadratic interval function (57) can be done in the following way.

Procedure 1. We write $C_{ij}(p_j)$ and $\boldsymbol{q}_{ij}(p_j)$ as

$$x(t) = a_0 + a_1t + a_2t^2 + a_3t^3, \quad t \in [-1, 1], \tag{58}$$

$$\boldsymbol{q}(t) = q_0 + q_1t + q_2t + \boldsymbol{s}, \quad t \in [-1, 1], \quad \boldsymbol{s} = [-s, s] \tag{59a}$$

and approximate (in Chebyshev sense) only the part $y(t) = a_1t + a_3t^3, \quad t \in [-1, 1]$ of $x(t)$. Applying Procedure 1 from [5], Section 2.1, we get the LI form $\boldsymbol{y}(t) = y_1t + \ \boldsymbol{y}$, $t \in [-1, 1]$, $y_1 = a_1 + a_3$, $t_1 = \sqrt{\frac{q_1 - a_3}{3a_3}}$, $s_y = |q_1t_1 - y(t_1)|$ ($\boldsymbol{y}$ is a symmetric interval of radius s_y). Finally, the QI approximation polynomial sought is given by (59a) if

$$q_0 = a_0, \ q_1 = y_1, \ q_2 = a_2, \ s = s_y. \tag{59b}$$

Remark 2. As is seen from the foregoing, obtaining a linear or quadratic $\boldsymbol{p}$-solution requires much more computational effort (especially for the quadratic form) than the original interval method. Therefore, if the final objective consists in computing an outer solution $\boldsymbol{x}$ of (1), the $\boldsymbol{p}$-solutions should be computed and used only if $\boldsymbol{x}$ is not narrow enough and an improved enclosure is needed. On the other hand, whenever a global optimization problem (3), (1) is to be solved, the use of the $\boldsymbol{p}$-solutions may turn out to be a better choice (see Section 5).

4 Applications

To illustrate the potential of the new approach in globally solving optimization problems of type (3), we first consider the problem of determining the IH solution $\boldsymbol{x}^*$ of the LIP system (1). Other applications are given in Subsection 4.2 and Section 5.

4.1 Solving the IH problem

We consider the problem of determining the component $\boldsymbol{x}_k^* = [\underline{x}_k^*, \overline{x}_k^*]$ of the IH solution $\boldsymbol{x}^*$ of the LIP system (1), confining ourselves to using the QIF method of Subsection 3.2. The value of $\underline{x}_k^*$ is found as the global solution of the following optimization problem:

$$\underline{x}_k^* = \min e_k^T x \tag{60a}$$

subject to the constraint

$$A(p)x = b(p), \quad p \in \boldsymbol{p}. \tag{60b}$$

The computational scheme for solving (60) involves two basic phases at each iteration: (i) find in $\boldsymbol{p}$ an upper bound x_k^u on $\underline{x}_k^*$, (ii) using x_k^u and a related constraint equation, try to reduce the current domain $\boldsymbol{p}$ to a narrower domain $\boldsymbol{p}'$ applying some constraint satisfaction technique. The iterative process continues until the width of the current domain becomes smaller than a given threshold ε_p.

In [12], the above two phases were both implemented using the iteration (26) and the corresponding LP solution which results in a linear constraint equation. We now show that the efficiency of the method based on the associated iteration (28) can be improved by resorting to the related QP solution. Indeed, as can be easily seen, the constraint equation is now nonlinear. Indeed, on account of (52b) the constraint is (dropping the superscript (κ))

$$c_k + L_k p + Q_k \varphi(p) + \boldsymbol{s}_k = x_k^u - \overline{e}_k^{(l)} \tag{61}$$

($\overline{e}_k^{(l)}$ is the upper end of $\boldsymbol{e}_k^{(l)}$) which is rewritten as

$$Q_{kj_0} p_{j_0}^2 + L_{kj_0} p_{j_0} + \boldsymbol{r}_{kj_0} = 0, \quad p_{j_0} \in \boldsymbol{p}_{j_0} \tag{62}$$

where j_0 is a chosen index, $\boldsymbol{r}_{kj_0}$ is an interval combining the interval extensions of the remaining terms $j \neq j_0$. Equation (62) is put in the form $p_{j_0}^2 + 2bp_{j_0} + \boldsymbol{c} = 0$ so

$$\boldsymbol{p}_{j_0}^{(1)} = -b - \sqrt{b^2 - \boldsymbol{c}}, \quad \boldsymbol{p}_{j_0}^{(2)} = -b + \sqrt{b^2 - \boldsymbol{c}} \tag{63}$$

if

$$\boldsymbol{c} < b^2. \tag{64}$$

We now intersect $\boldsymbol{p}_{j_0}^{(1)}$ and $\boldsymbol{p}_{j_0}^{(2)}$ with $\boldsymbol{p}_{j_0}$ to obtain (hopefully) a reduction of $\boldsymbol{p}_{j_0}$. This approach seems to offer better possibilities to contract the current $\boldsymbol{p}$ as compared to the known linear constraint technique in [12] (according to [2], Subsection 10.3 to Subsection 10.6, nonlinear constraints are more effective than linear ones). An additional contracting effect appears whenever (64) is violated since $\boldsymbol{c}$ involves a sum of quadratic expressions $\alpha_j \varphi_j(p_j)$ for $j \neq j_0$.

Remark 3. The above nonlinear constraint satisfaction technique is also possible, to a lesser effect, in the framework of the LIF method from Subsection 3.1. Indeed, at the last iteration for computing $\boldsymbol{x}(p)$, we can save the quadratic form (34) and use it in exactly the same way as above.

4.2 Other applications

We next consider two problems of type (3) where the mapping g_k is linear or quadratic.

Parametric linear programming (PLP)

The PLP problem is formulated as follows [12]: given the linear parametric objective function

$$l(x,p) = c^T(p)x \tag{65}$$

(where $c_i(p)$ are, in general, nonlinear functions of p and the constraint

$$A(p)x = b(p), \quad p \in \boldsymbol{p} \tag{66}$$

determine the range

$$\boldsymbol{l}^*(A(p), b(p), c(p), \boldsymbol{p}) = \{l = c^T(p)x : A(p)x = b(p),\ p \in \boldsymbol{p}\}. \tag{67}$$

The PLP (65), (66) is a parametric generalization of the known interval linear programming problem (e.g. [3]) where interval matrix $\boldsymbol{A}$ and interval vectors $\boldsymbol{b}$, $\boldsymbol{c}$ are involved.

Obviously, the end-points $\underline{l}^*$ and $\overline{l}^*$ of the range $\boldsymbol{l}^*$ can be determined as the global solutions of the following two optimization problems

$$\underline{l}^* = \min\{l = c^T(p)x : A(p)x = b(p), \quad p \in \boldsymbol{p}\}, \tag{68}$$

$$\overline{l}^* = \max\{l = c^T(p)x : A(p)x = b(p), \quad p \in \boldsymbol{p}\}. \tag{69}$$

The p-solution $\boldsymbol{x}(p)$ of (66) can be used for solving (68) in a similar way as this was done for the case of the problem in Subsection 4.1. Thus, an iterative method can be developed where $l(x,p)$ is put in the form

$$l(x,p) = \sum_i c_i(p)\boldsymbol{x}_i(p), \quad p \in \boldsymbol{p} \tag{70}$$

and exploited as a constraint at each iteration. Another possibility is to check monotonicity conditions

$$\frac{\partial l(x,p)}{\partial p_i} = \frac{\partial c_i(p)}{\partial p_i}\boldsymbol{x}_i(p) + c_i(p)\frac{\partial \boldsymbol{x}_i}{\partial p_i}, \quad p \in \boldsymbol{p}. \tag{71}$$

It should be stressed that unlike [12] where the LP solution (26) was used, now better efficiency of (70) and (71) could be obtained if the QP versions from Subsection 2.2 related to the iterative scheme (26) or (28) are used to obtain $\boldsymbol{x}_i(p)$ or $\boldsymbol{d}_i(p) = \frac{\partial \boldsymbol{x}_i(p)}{\partial p_i}$.

It is worth mentioning the following special case

$$l(p) = \sum_i c_i x_i(p), \quad p \in \boldsymbol{p}, \tag{72}$$

where c_i are constraints. Obviously, the range $\boldsymbol{l}^*(\boldsymbol{p})$ provides an "interval hull" bound on the projection of the united solution set of (66) along the line defined by the coefficients c_i.

We illustrate the new approach to solving the class of optimization problems (3) using the example of the simple parametric linear programming (PLP) problem (72), (66). More specifically, we seek an outer bound $\boldsymbol{l}$ on the range $\boldsymbol{l}^*(\boldsymbol{p})$.

Two approaches to computing $\boldsymbol{l}$ will be considered. The first one, referred to as the standard approach, consists in using an outer solution $\boldsymbol{x}$ to (66) which is obtained by a method not employing p-solutions. In this case the outer bound $\boldsymbol{l}$ is given as

$$\boldsymbol{l}_1 = \sum_i c_i \boldsymbol{x}_i. \tag{73}$$

The second one is the new approach based on the use of the LP or QP solution of (66). Now another outer bound $\boldsymbol{l}_2$ is computed as the range of

$$\boldsymbol{l}(p) = \sum_i c_i \boldsymbol{x}_i(p), \quad p \in \boldsymbol{p}, \tag{74a}$$

that is

$$\boldsymbol{l}_2 = \boldsymbol{l}(\boldsymbol{p}). \tag{74b}$$

To simplify the presentation, we confine ourselves to the LP solution (45) so

$$\boldsymbol{x}_i(p) = x_i^c + \sum_j L_{ij} p_j + [-s_i, s_i], \quad p \in \boldsymbol{p}, \quad i = 1, ..., n. \tag{75a}$$

(x_i^c is the centre of $\boldsymbol{x}_i(p)$). From (74a)

$$\boldsymbol{l}(p) = l_0 + \sum_j L_j^0 p_j + s^0[-1, 1], \quad p_j \in \ \boldsymbol{p}_j, \tag{75b}$$

$$l_0 = \sum_i c_i x_i^c, \quad L_j^0 = \sum_i L_{ij}, \quad s^0 = \sum_i |c_i| s_i. \tag{75c}$$

But $p_j \in [-1, 1]$ so

$$\boldsymbol{l}_2 = l_0 + \left(\sum_j |L_j^0| \right) [-1, 1] + s^0[-1, 1]. \tag{76}$$

It is logical to expect that $\boldsymbol{l}_2$ provides, in general, a narrower bound on $\boldsymbol{l}^*(\boldsymbol{p})$ than $\boldsymbol{l}_1$. Indeed, (75) and (76) show that the interdependencies between p_j in (74a) are accounted for (in a linear manner) whereas they are completely ignored in (73). To show quantitatively that (76) is superior to (73), we assume that $\boldsymbol{x}_i$ in (73) are found as the range of $\boldsymbol{x}_i(p)$ in (75a). Then $\boldsymbol{x}_i = x_i^c + \left(\sum_j |L_{ij}| \right) [-1, 1] + s_i[-1, 1]$, hence, from (73)

$$\boldsymbol{l}_1 = l_0 + \left(\sum_{ij} |c_i||L_{ij}^0| \right) [-1, 1] + s^0[-1, 1]. \tag{77}$$

Let $r_1 = \mathrm{rad}(\boldsymbol{l}_1)$ and $r_2 = \mathrm{rad}(\boldsymbol{l}_2)$ (rad stands for radius) and consider

$$dr = r_1 - r_2. \tag{78a}$$

As is seen from (76) and (77)

$$dr = \sum_{ij} |c_j||L_{ij}^0| - \sum_{ji} |c_i L_{ij}^0| \geq 0. \tag{78b}$$

Therefore, it is advantageous to use the new approach.

Parametric quadratic programming (PQP)

In this case, the objective function is quadratic

$$f(x, p) = \frac{1}{2} x^T Q(p) x + c^T(p) x, \tag{79}$$

the constraint is the LIP system (66) and the problem is to determine the range $\boldsymbol{f}^*$ of $f(x,p)$ over $\boldsymbol{p}$. Obviously, the end-points $\underline{f}^*$ and $\overline{f}^*$ of $\boldsymbol{f}^*$ can be found as the global solutions of

$$\underline{f}^* = \min\{f(x,p) : A(p)x = b(p), \quad p \in \boldsymbol{p}\}, \tag{80a}$$

$$\overline{f}^* = \max\{f(x,p) : A(p)x = b(p), \quad p \in \boldsymbol{p}\}. \tag{80b}$$

The $\boldsymbol{p}$-solutions of (66) can be used for solving (80) in a similar way as in the PLP problem. Now (79) is written as

$$\boldsymbol{f}(x,p) = \frac{1}{2}\boldsymbol{x}^T(p)Q(p)\boldsymbol{x}(p) + c^T(p)\boldsymbol{x}(p), \quad p \in \boldsymbol{p}$$

and is exploited as a constraint propagation equation or modified monotonicity condition.

It is worth mentioning the following two special cases of (79):

$$f(x,p) = \sum_{k=1}^{n} p_k x_k^2(p), \quad m = n, \tag{81a}$$

and

$$f(x,p) = \sum_{k=1}^{n} x_k^2(p). \tag{81b}$$

Problem (81a) arises in determining the dissipated active power in electric circuits [10]. The latter problem (given in [12]) consists in determining the range of the length squared of the vector $x(p)$ whose components are $x_k(p)$.

5 An Example

To illustrate the new approach based on using $\boldsymbol{p}$-solutions, we consider the problem of determining an outer interval solution of: (i) a given LIP system and (ii) a simple parametric linear programming (PLP) problem associated with the given LIP system.

5.1 Outer solution of a LIP system

The LIP system is [19], [12]

$$A(p)x = b(p), \tag{82a}$$

$$A(p) = \begin{bmatrix} p_1 & p_2+1 & -p_3 \\ -p_2+1 & -3 & p_1 \\ 2-p_3 & 4p_2+1 & 1 \end{bmatrix}, \quad b(p) = \begin{bmatrix} 2p_1 \\ p_3-1 \\ -1 \end{bmatrix} \tag{82b}$$

which can be written in the form (24b)

$$A(p) = A^{(0)} + \sum_{\mu=1}^{3} A^{(\mu)} p_\mu, \quad b(p) = b^{(0)} + Bp. \tag{82c}$$

The outer solution $\boldsymbol{x}$ sought will be determined on the basis of the LP solution $\boldsymbol{x}(p)$ obtained by the A2.B1 method from [12] (i.e. using the LI version of the iteration (26)). The results of this method (referred to as method M2) will be compared with those of a known method [24] (referred to as method M1) for parametric intervals $\boldsymbol{p}$ of variable width. Thus, we shall need parameter vectors (boxes) of the form

$$\boldsymbol{p}(\rho) = p^0 + \rho[-r^0, r^0], \tag{83a}$$

where ρ is variable while

$$p^0 = (0.5 \;\; 0.5 \;\; 0.5), \quad r^0 = (0.5 \;\; 0.5 \;\; 0.5). \tag{83b}$$

To apply method M2, system (82) is first rewritten in an equivalent form to get the interval parameters involved to be symmetric:

$$, A^{(0))} := A^{(0)} + A^{(1)}p_1^0 + A^{(2)}p_2^0 + A^{(3)}p_3^0 = \begin{bmatrix} 0.5 & 1.5 & -0.5 \\ 1.5 & -3 & 0.5 \\ 1.5 & 3 & 1 \end{bmatrix} \tag{83c}$$

$$b^{(0)} := b^{(0)} + Bp^{(0)} = b^{(0)} + b^{(1)}p_1^0 + b^{(2)}p_2^0 + b^{(3)}p_3^0 = (1 \;\; -0.5 \;\; -1)^T \tag{83d}$$

where $b^{(\mu)}$ is the μth column of B. Thus, (83a) becomes

$$\boldsymbol{p}(\rho) = \rho r^0[-1, 1] \tag{84}$$

so

$$\boldsymbol{p}' = \boldsymbol{p}(\rho) = [-r', r'], \quad r' = \gamma[1, \; 1, \; 1], \quad \gamma = 0.5\rho. \tag{84a}$$

Now each $A^{(\mu)}$ and $b^{(\mu)}$ is multiplied by γ, i.e.

$$A^{(\mu)} := \gamma A^{(\mu)}, \quad b^{(\mu)} := \gamma b^{(\mu)}, \quad \mu = 1, ...m \tag{84b}$$

to obtain an equivalent form LIP system (82c), (83c), (83d) and (84b) where $\boldsymbol{p}_\mu = [-1, 1]$.

We first choose $\rho = 0.3$. Using method M2, we determine the corresponding LP solution

$$\boldsymbol{x}(p) = x_c + Lp + [-s, s], \quad p \in \boldsymbol{p}, \tag{85a}$$

$$x_c = \begin{bmatrix} 0.2964 \\ 0.0430 \\ -1.5802 \end{bmatrix}, \quad L = \begin{bmatrix} 0.2363 & -0.0231 & -0.0741 \\ -0.0133 & 0.0032 & -0.0407 \\ -0.3146 & -0.0007 & 0.2778 \end{bmatrix}, \quad s = \begin{bmatrix} 0.1143 \\ 0.0403 \\ 0.1767 \end{bmatrix}. \tag{85b}$$

The outer solution of the (transformed) system (82) obtained by the LP solution (85) will be denoted $\boldsymbol{x}^{(2)}$ and is given as

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}(\boldsymbol{p}). \tag{86a}$$

From (85)

$$\boldsymbol{x}^{(2)} = ([-0.1514, \; 0.7442], [-0.0545, \; 0.1406], [-2.3501, \; -0.8104])^T. \tag{86b}$$

Next, we compute the outer interval solution $x^{(1)}$ of (82) using method M1:

$$\boldsymbol{x}^{(1)} = ([-0.2906, \; 0.8620], [-0.0894, \; 0.1846], [-2.5012, \; -0.6417])^T \tag{87}$$

It is seen that

$$\boldsymbol{x}^{(2)} \subset \boldsymbol{x}^{(1)} \tag{88}$$

(the inclusion is meant component-wise).

Table 1: Outer solution of the LIP system (82) to (84): comparison of the enclosure tightness η_{12} and the computational efficiency τ_{21} of method M1 and method M2 in function of ρ.

ρ	0.1	0.2	0.3	0.4	0.5	0.6
$\eta_{12} = r_1/r_2$	1.057	1.125	1.208	1.311	1.443	1.617
$\tau_{21} = t_2/t_1$	1.319	1.864	2.200	2.896	3.080	3.759

To assess quantitatively the improvement of the new approach over the standard one, we compare the largest components $\boldsymbol{x}_3^{(2)}$ and $\boldsymbol{x}_3^{(1)}$ of the respective outer interval solutions using the following merit figure

$$\eta_{12} = \frac{r_1}{r_2} \tag{89}$$

where r_1 and r_2 are the radii of the corresponding components intervals. It is seen from (86b) and (87) that for $\rho = 0.3$ the quotient $r_1/r_2 = 1.208$.

The dependence of η_{12} on ρ for $\rho = 0.1$ up to $\rho = 0.6$ is given in the second row of Table 1. In the third row, data on the relative computational efficiency of the two methods involved are provided, measured by

$$\tau_{21} = \frac{t_2}{t_1} \tag{90}$$

where t_1 and t_2 are the computer times needed by the respective method to compute $\boldsymbol{x}$. As expected, method M2 has better enclosure efficiency ($r_2 < r_1$) than method M1 at the cost, however, of larger computational expenses.

We also compare the applicability radii $r_a(\mathrm{M2})$ and $r_a(\mathrm{M1})$ of the respective methods M2 and M1. In accordance with [11], we approximately determine each r_a by letting ρ increase with an increment $\Delta\rho$ until inapplicability of the method is reached. Choosing $\Delta\rho = 0.001$, we have obtained $r_a(\mathrm{M2}) = 0.738$ (M2 becomes inapplicable for $\rho = r_a(\mathrm{M2}) + \Delta\rho = 0.739$). In a similar way, we have $r_a(\mathrm{M2}) = 0.744$. It is seen that the two methods have comparable applicability radii.

5.2 Outer solution of a PLP problem

We consider the PLP problem (65), (82) for the special case where

$$c^T = (1, 1, 1), \tag{91}$$

seeking an outer bound $\boldsymbol{l}$ on the range $\boldsymbol{l}^*(\boldsymbol{p})$. As in Subsection 5.1, we apply methods M1 and M2 to obtain the respective bounds $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. On account of (73), (74), (82) and (91), the results for $\rho = 0.3$ are

$$\boldsymbol{l}_1 = [-2.8811,\ 0.4049], \tag{92a}$$

$$\boldsymbol{l}_2 = [-1.8473,\ -0.6343]. \tag{92b}$$

It is seen that

$$\boldsymbol{l}_2 \subset \boldsymbol{l}_1. \tag{93}$$

We show that the above inclusion remains valid for variable ρ using an analogous figure merit (89) where now r_1 and r_2 are the radii of the respective intervals $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. The related data are given in the second row of Table 2. It is also shown that the better enclosure efficiency of method M2 is achieved at the cost of higher computational effort (third row of the table).

Table 2: Outer solution of the PLP problem (72), (82) and (91): comparison of the enclosure tightness η_{12} and the computational efficiency τ_{21} of methods M1 and M2 in function of ρ.

ρ	0.1	0.2	0.3	0.4	0.5	0.6
$\eta_{12} = r_1/r_2$	3.222	2.943	2.709	2.510	2.339	2.189
$\tau_{21} = t_2/t_1$	1.401	1.978	2.350	2.903	3.160	3.819

A better but more costly outer bound $\boldsymbol{l}_1^*$ on the range $\boldsymbol{l}^*(\boldsymbol{p})$ can be obtained if $\boldsymbol{l}_1$ defined by (73) is replaced with

$$\boldsymbol{l}_1^* = \sum_i c_i \boldsymbol{x}_i^* \tag{94}$$

where $\boldsymbol{x}_i^*$ are the components of the interval hull solution $\boldsymbol{x}^*$ to (66). It turns out that, up to a certain width of $\boldsymbol{p}$, $\boldsymbol{l}_2$ is narrower even than $\boldsymbol{l}_1^*$. To show this, we fix $\rho = 0.3$ and find an approximate interval hull solution $\boldsymbol{x}^a$ of (82) (in fact, an inner hull estimation) using the Monte-Carlo method for $N = 10^6$ trials

$$\boldsymbol{x}^a = ([0.0218,\ 0.6929], [-0.0174,\ 0.1043], [-2.2533,\ -1.0510])^T. \tag{95a}$$

From (91) and (94)

$$\boldsymbol{l}_1^a = [-2.2487,\ -0.2508]. \tag{95b}$$

Now (89) is replaced by

$$\eta^a = \frac{\mathrm{rad}(\boldsymbol{l}_2^a)}{\mathrm{rad}(\boldsymbol{l}_1^a)} \tag{95c}$$

so

$$\eta^a = 1.64. \tag{95d}$$

It is seen that the new approach leads to a better result even in the case where the narrowest possible interval $\boldsymbol{l}_1^*$ has been used to compute η. It is important to underline that the narrower outer solution $\boldsymbol{l}_2$ of the PLP problem considered is achieved in spite of the fact that $\boldsymbol{x}^{(2)}$ is broader than $\boldsymbol{x}^*$.

The datum related to $\rho = 0.3$ is reported in the fourth column of Table 3.

Next, we show the dependence of η^a on ρ. As seen from the data in Table 3, the improvement in the enclosure tightness is significant for lower values of ρ and decreases as ρ grows. Starting from roughly $\rho = 0.45$, the use of the corresponding hull solutions $\boldsymbol{x}^*$ is preferable since $\mathrm{rad}(\boldsymbol{l}_1^*) < \mathrm{rad}(\boldsymbol{l}_2)$.

It should be noted that the enclosing efficiency of the new approach as compared to the standard approach will be enhanced for the general form LPL problem (65) where the coefficients $c_i(p)$ in the cost function depend on p.

Table 3: Outer solution of the PLP problem (72), (82) and (91): comparison of the enclosure tightness η^a of the outer solution $\boldsymbol{l}_2$ and the solution $\boldsymbol{l}_1^a$ in function of ρ.

ρ	0.1	0.2	0.3	0.4	0.5	0.6
η_a	2.78	2.17	1.64	1.21	0.82	–

6 Conclusion

A class of iterative methods for determining a p-solution $\boldsymbol{x}(p)$ of the LIP systems (1) has been suggested in Sections 2 and 3. The methods pertaining to this class differ in the chosen iterative scheme (26) or (28), on the one hand, and the employed form, linear parametric (2) or quadratic parametric (5), on the other. As an illustration, two new specific methods for determining $\boldsymbol{x}(p)$, based on the simple iterative process (28) and linear or quadratic forms, have been suggested in Subsection 3.1 and 3.2, respectively. The latter methods circumvent the need for inverting the real midpoint matrix A^0 and, therefore, may be better than the previous method of [12] for large-size sparse LIP systems (1).

Two applications of the new approach are mentioned in Subsections 4.1 and 4.2. A numerical example (Section 5) referring to the simple linear parametric programming problem (72), (82) illustrates the potential of the new approach to solving parametric global optimization problems of the type (3), (1).

Presently, the methods for computing p-solutions require much more computational effort (especially for the quadratic form (5)) than their interval parametric but non-parameterized counterparts. Therefore, future research should focus on the development of new computationally more efficient methods yielding $\boldsymbol{x}(p)$. One such possibility is to take into consideration some special features of system (1). Thus, an iterative method for LP or QP solutions might be constructed in the case where the matrix $A(p)$ of the LIP system at hand has the specific structure of [15], modifying the original method of [15]. Also, a direct method for computing a LP solution can be suggested, which is based on the known direct method of [24] proposed for computing a standard outer interval solution $\boldsymbol{x}$ of (1).

The basic unifying property of the iterative methods suggested in the present paper is the fact that the successive iterations of the p-solution are expanding. An interesting alternative is to construct iterative schemes where the iterations are contractive. The p-solution form version [13] of the direct method [24] can be used as an initial iteration.

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