A Survey of Classes of Matrices Possessing the Interval Property and Related Properties*

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Abstract

This paper considers intervals of real matrices with respect to partial orders and the problem to infer from some exposed matrices lying on the boundary of such an interval that all real matrices taken from the interval possess a certain property. In many cases such a property requires that the chosen matrices have an identically signed inverse. We also briefly survey related problems, e.g., the invariance of matrix properties under entry-wise perturbations.

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1 Introduction

In this paper we consider intervals $[A] = [\underline{A}, \overline{A}]$ of real $n \times n$-matrices with respect to the usual entry-wise partial order and to the checkerboard partial order which is obtained from the entry-wise order by reversing the inequalities between the entries of $\underline{A}$ and $\overline{A}$ in a checkerboard pattern. We call a real matrix A a vertex matrix of $[A]$ if its entries are entries of the matrices $\underline{A}$ and $\overline{A}$. We survey solutions to the problem to infer from some vertex matrices of $[A]$ that all matrices taken from this matrix interval possess a certain property. We do not consider related characterizations which require matrices which may not be vertex matrices, e.g., the midpoint matrix of $[A]$. It turns out that in many cases such a property requires that all minors of fixed order, k say, of the exposed vertex matrices have an identical sign. As a consequence, if $k = n-1$ they have an identically signed inverse. Such matrices are intimately related to bases of functions with optimal shape-preserving properties used in computer aided geometric

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design, see, e.g., [36].

The organization of our paper is as follows: In Section 2 we introduce our notation and matrix intervals. In Section 3 we present matrix properties which can be inferred from two vertex matrices of the matrix interval and in Section 4 properties which require in general more than two vertex matrices. We conclude our paper in Section 5 with a brief survey of some related problems, e.g., the persistence of matrix properties under entry-wise perturbation.

2 Notation and Matrix Intervals

2.1 Notation

We now introduce the notation used in our paper. For κ, n we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. Let A be a real $n \times n$ matrix. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_\kappa) \in Q_{\kappa,n}$, we denote by $A[\alpha|\beta]$ the $\kappa \times \kappa$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\kappa$. We suppress the brackets when we enumerate the indices explicitly. If α and β are formed from consecutive rows and columns we call the submatrix $A[\alpha \mid \beta]$ and $\det A[\alpha \mid \beta]$ *contiguous*. When $\alpha = \beta$, the *principal submatrix* $A[\alpha \mid \alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*. In the special case where $\alpha = (1, 2, \ldots, \kappa)$, we refer to the principal submatrix $A[\alpha]$ as the *leading principal submatrix* (and to $\det A[\alpha]$ as the *leading principal minor*) *of order* κ. We reserve throughout the notation $A^* := JAJ$, where $J := \operatorname{diag}\,(1, -1, \ldots, (-1)^{n+1})$, and $A^\# := SAS$, where $S = (s_{ij})$ is the *anti-diagonal* matrix with $s_{ij} := \delta_{n+1-i,j}$, $i, j = 1, \ldots, n$. The absolute value of vectors and matrices is understood entry-wise.

2.2 Matrix Intervals

Let $\mathbb{R}^{n,n}$ be endowed with a partial order $\preceq$. We consider (matrix) intervals $[A]_\preceq = [\underline{A}, \overline{A}]_\preceq$ with respect to $\preceq$, i.e.,

$$[A]_\preceq = [\underline{A}, \overline{A}]_\preceq = \left\{ A \in \mathbb{R}^{n,n} \mid \underline{A} \preceq A \preceq \overline{A} \right\}, \tag{1}$$

where $\underline{A} \preceq \overline{A}$ with $(\underline{A})_{ij} = \underline{a}_{ij}$, $(\overline{A})_{ij} = \overline{a}_{ij}$, $i, j = 1, \ldots, n$. If the underlying partial order is clear from the context we suppress the explicit reference to it.

By $\mathbb{I}(\mathbb{R}^{n,n})$ we denote all matrix intervals with respect to $\preceq$. A *vertex matrix* of $[A]$ is a matrix $A = (a_{ij})_{i,j=1}^n$ with $a_{ij} \in \left\{\underline{a}_{ij}, \overline{a}_{ij}\right\}$; $\underline{A}$ and $\overline{A}$ are called the *corner matrices.*

Let V be a fixed set of vertex matrices. We say that a set S of matrices *has the interval property* (*with respect to* V) if $[A] \subset S$ whenever $V([A]) \subset S$. Here it is implicitly understood that $S \subset \mathbb{R}^{n,n}$ for an arbitrary, but fixed n. In the sequel we abbreviate "interval property" by "IP" when referring to a specified property. We extend properties of real matrices to matrix intervals by saying that a *matrix interval has* a certain *property* if each real matrix contained in it possesses this property.

3 Matrix Properties Which Can Be Inferred from Two Vertex Matrices

In this section we consider $n \times n$ matrix intervals $[A] = [\underline{A}, \overline{A}]$ with respect to the usual entry-wise partial order and the closely related checkerboard partial order. The interval property refers in both cases to $V([A]) = \left\{\underline{A}, \overline{A}\right\}$.

3.1 Matrix Intervals with Respect to the Usual Entry-wise Partial Order

In this subsection the partial order is the usual entry-wise partial order $\leq$, i.e., the inequality $A \leq B$ between $A, B \in \mathbb{R}^{n,n}$ is understood entry-wise. Likewise the strict inequality $A < B$ is understood entry-wise. Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can also be represented as an *interval matrix*, i.e., as a matrix with entries taken from the set of the compact nonempty real intervals, i.e.,

$$[A] = ([\underline{a}_{ij}, \overline{a}_{ij}])_{i,j=1}^{n}. \tag{2}$$

The first known (nontrivial) interval property concerns inverse nonnegative matrices (also termed *inverse positive matrices*, see, e.g., [32], and *monotone matrices*, see, e.g., [30]).

Definition 3.1. *A matrix $A \in \mathbb{R}^{n,n}$ is called* inverse nonnegative *if A is nonsingular and $0 \leq A^{-1}$; it is an* M-matrix *if it is inverse nonnegative and all its off-diagonal entries are nonpositive.*

IP 3.1.1 [30, Corollary 3.5]: *The inverse nonnegative matrices have the interval property.*

IP 3.1.1 can also be found in [33, Bemerkung 1.2 (v) (a), p.15]. It seems that Metelmann found this result independently of Kuttler ([30] appeared in April 1971, Kurt Metelmann has submitted his dissertation [33] most probably at the end of year 1971 or at the beginning of 1972). In [38, Theorem 4.6] an extension of IP 3.1.1 to more general sign patterns of the inverse matrix is presented. This interval property involves two vertex matrices of type A_{yz} which will be introduced in Subsection 4.2. These sign patterns include the checkerboard like sign pattern, see Subsection 3.2.

We note the following immediate consequence of IP 3.1.1.

IP 3.1.2 [33, pp.27, 32, and 37]: *The following three sets of inverse nonnegative matrices have the interval property:*

a) *The matrices whose leading principal submatrices are all inverse nonnegative (or equivalently,* see [33, Satz 1.8]*, allow an LDU factorization, where L and U are lower and upper triangular matrices with unit diagonal and D is a diagonal matrix, all being inverse nonnegative);*

b) *the matrices whose contiguous principal submatrices are all inverse nonnegative;*

c) *the matrices whose principal submatrices are all inverse nonnegative.*

The matrices considered in IP 3.1.2 c) are just the M-matrices, see [33, Satz 1.16]. So the M-matrices have the interval property; this result can be sharpened in the way that it suffices that the matrix $\overline{A}$ is solely supposed to have only nonpositive off-diagonal entries (without the assumption of being inverse nonnegative), see, e.g., [9, p.119]. Historically, IP 3.1.2 c) has also been found when studying systems of linear interval equations, see, e.g., [9].

3.2 Matrix Intervals with Respect to the Checkerboard Partial Order

In this subsection we employ the checkerboard partial order which is closely related to the partial order considered in Subsection 3.1.

Definition 3.2. *We define the* checkerboard partial order $\leq^*$ *as follows: For* $A, B \in \mathbb{R}^{n,n}$

$$A \leq^* B :\Leftrightarrow \; A^* \leq B^*. \tag{3}$$

Each matrix interval $[A] = [\underline{A}, \overline{A}]_{\leq}$ with respect to the partial order $\leq$ can be represented as a matrix interval $[\downarrow A, \uparrow A]_{\leq^*}$ with respect to the checkerboard partial order and vice versa. The two corner matrices $\downarrow A$, $\uparrow A$ are given by

$$(\downarrow A)_{ij} = \begin{Bmatrix} \underline{a}_{ij} \\ \overline{a}_{ij} \end{Bmatrix}, \quad (\uparrow A)_{ij} = \begin{Bmatrix} \overline{a}_{ij} \\ \underline{a}_{ij} \end{Bmatrix} \quad \text{if} \quad i+j \text{ is } \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}.$$

In this subsection we consider the following matrices. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be a signature sequence, i.e., $\epsilon \in \{1, -1\}^n$. The matrix A is called *strictly sign regular* (abbreviated SSR henceforth) and *sign regular* (abbreviated SR) *with signature* ϵ if $0 < \epsilon_\kappa \det A[\alpha|\beta]$ and $0 \leq \epsilon_\kappa \det A[\alpha|\beta]$, respectively, for all $\alpha, \beta \in Q_{\kappa,n}$, $\kappa = 1, 2, \ldots, n$. If A is SSR (SR) with signature $\epsilon = (1, 1, \ldots, 1)$, then A is called *totally positive* (abbreviated TP) (respectively, *totally nonnegative* (abbreviated TN)). If A is SSR (SR) with signature $\epsilon = (-1, -1, \ldots, -1)$, then A is called *totally negative* (abbreviated *t.n.*) (respectively, *totally nonpositive* (abbreviated *t.n.p.*)). If A is in a certain class of SR matrices and in addition also nonsingular then we affix Ns to the abbreviation of the name of the class.

Following [23], we call a minor *trivial* if it vanishes and its zero value is determined already by the pattern of its zero-nonzero entries. We illustrate this definition by the following example. Let

$$A := \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix},$$

where an asterisk denotes a nonzero entry. Then $\det A[2,3|1,2]$ and $\det A[1,2|1,3]$ are trivial, whereas $\det A$ and $\det A[1,2|2,3]$ are nontrivial minors.

Definition 3.3. [23, Definition 8] *Let* $A \in \mathbb{R}^{n,n}$ *and* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ *be a signature sequence. If for all the nontrivial minors*

$$0 < \epsilon_k \det A[\alpha \mid \beta] \textit{ for all } \alpha, \beta \in Q_{k,n}, \; k = 1, \ldots, n, \tag{4}$$

holds, then A is called almost strictly sign regular *(abbreviated $ASSR$)* with signature ϵ. *If* $\epsilon = (1, \ldots, 1)$*, then A is called* almost totally positive (ATP).

For properties of the $NsASSR$ matrices, in particular, a restriction of the condition (4) to the nontrivial contiguous minors, see [23]. For a new characterization of ATP matrices, see [2, 4].

We present now some classes of SR matrices which possess the interval property. In each case it is implicitly understood that the two corner matrices have the same signature.

We note a consequence of IP 3.1.1, see also [12, Subsection 3.2], [37, Subsection 3.2].

IP 3.2.1 [15, Theorem 1]: *The SSR matrices with a fixed signature ϵ have the interval property; in particular, the sets of the TP and the t.n. matrices have the interval property.*

In [8, Theorem 4.3] we apply IP 3.2.1 to derive a vertex result on the persistence of the number of poles (which are exclusively positive) of the entire family of rational functions, the numerator and denominator of which are both interval polynomials.

In relaxing the strict sign condition, we obtain the following two classes of SR matrices possessing the interval property.

IP 3.2.2: *The following two sets have the interval property:*

a) The $NsASSR$ matrices with a fixed signature ϵ [7, Theorem 5.5] [17, Theorem 1 for $\epsilon = (1, \ldots, 1)$]*;*

b) the tridiagonal $NsSR$ matrices with a fixed signature ϵ [7, Theorem 5.11] [15, Theorem 4 for $\epsilon = (1, \ldots, 1)$].

Each SR matrix can be arbitrarily closely approximated by SSR matrices, see, e.g., [14, Satz 17, p.311]. Furthermore, this approximation can be accomplished in a two-sided way with respect to $\le$ [15, Lemma 2]. Therefore, the nonsingularity assumption can be dropped.

Theorem 3.1. [15, Theorem 2] *Let $[\downarrow A, \uparrow A] \in \mathbb{I}(\mathbb{R}^{n,n})$ be such that*

either

$$\forall\, i, j \in \{1, \ldots, n\} \quad \underline{a}_{ij} = \overline{a}_{ij} \;\Rightarrow\; i + j \text{ is even},$$

or

$$\forall\, i, j \in \{1, \ldots, n\} \quad \underline{a}_{ij} = \overline{a}_{ij} \;\Rightarrow\; i + j \text{ is odd}.$$

Then the following two statements are equivalent:

(i) $[\downarrow A, \uparrow A]$ is SR (respectively, $NsSR$) with the same signature.

(ii) $\downarrow A$, $\uparrow A$ are SR (respectively, $NsSR$) with the same signature.

The rather obscure condition on the parity of the sum of indices means that entries with no variation have either an even or an odd index sum. This condition stems from the construction of a sequence of approximating intervals with respect to the checkerboard partial order. If this condition is removed the interval property does not hold. In [15] it was conjectured that the interval property holds in the TN case if the assumption of the nonsingularity of the matrices $\downarrow A$ and $\uparrow A$ is added (then by IP 3.1.1 the matrix interval $[\downarrow A, \uparrow A]$ is nonsingular). Subsequently, the interval property has been established for some subclasses of the $NsTN$ matrices. The conjecture was finally settled in [3] by making use of the so-called Cauchon algorithm [20, 31]; for a compressed form and further properties of this algorithm see [2, 4].

IP 3.2.3: *The following sets of matrices have the interval property:*

a) The $NsTN$ matrices [3, Theorem 3.6]*;*

b) the $NsTN$ matrices with a fixed pattern of their zero-nonzero minors [3, Theorem 3.4]*;*

c) special $NsTN$ band matrices arising, e.g., in the discretization of certain boundary value problems [33, 34].

In [8, Theorem 3.6] we apply IP 3.2.3 a) to derive a new sufficient condition for the Hurwitz stability of an interval family of polynomials.

In some instances, the assumption of nonsingularity in IP 3.2.3 a) can be relaxed.

Theorem 3.2. [3, Corollary 3.7] *Let $[\downarrow A, \uparrow A] \in \mathbb{I}(\mathbb{R}^{n,n})$ and $Z \in [\downarrow A, \uparrow A]$. If $\downarrow A$ and $\uparrow A$ are TN and $\downarrow A[2, \ldots, n]$ or $\downarrow A[1, \ldots, n-1]$ is nonsingular, then Z is TN.*

IP 3.2.4 [3, Corollary 3.8]: *The tridiagonal TN matrices have the interval property.*

Now we present related results for the *t.n.p.* matrices.

IP 3.2.5 [7, Theorem 5.7]: *The $Ns.t.n.p.$ matrices A with $\overline{a}_{nn} < 0$ have the interval property.*

In passing over to $A^{\#}$ and back, IP 3.2.5 remains in force if we replace the condition $\overline{a}_{nn} < 0$ by $\overline{a}_{11} < 0$. By [7, Remark 1] the assumption of the negativity of $\overline{a}_{nn}$ (and $\overline{a}_{11}$) is not necessary. The following theorem shows that the nonsingularity assumption in IP 3.2.5 can be relaxed.

Theorem 3.3. [7, Corollary 5.8] *Let $[\downarrow A, \uparrow A] \in \mathbb{I}(\mathbb{R}^{n,n})$, $Z \in [\downarrow A, \uparrow A]$, $\downarrow A$ and $\uparrow A$ be t.n.p. with $\overline{a}_{nn} < 0$, and*

(i) $\downarrow A[2, \ldots, n]$ nonsingular and $\overline{a}_{11} < 0$,

or

(ii) $\downarrow A[1, \ldots, n-1]$ nonsingular.

Then Z is t.n.p.

If A is a $NsSR$ matrix with signature $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, then SA and $-A$ have signatures $((-1)^{\frac{i(i-1)}{2}}\epsilon_i)$ and $((-1)^i\epsilon_i)$, respectively. This fact can be used to identify further sets of the $NsSR$ matrices exhibiting the interval property.

IP 3.2.6 [2, Theorem 4.10]: *The $NsSR$ matrices with each of the following signatures $\epsilon = (\epsilon_i)_{i=1}^n$ have the interval property:*

(i) $\epsilon_i = (-1)^i$,

(ii) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}}$,

(iii) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}}$,

(iv) $\epsilon_i = (-1)^{i+1}$,

(v) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}+1}$,

(vi) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}+1}$.

Based on the variety of subclasses of the $NsSR$ matrices which possess the interval property we were led to the following conjecture. For a partial result in favor of this conjecture see IP 4.3.

Conjecture 3.1. *The set of the NsSR matrices with a fixed signature has the interval property.*

We conclude this section with two classes of matrices which are considered in [1] and called SDB and $SSDB$ matrices. Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$ be a signature sequence and let $K :=$ diag $(k_1, k_2, \ldots, k_n)$ be the diagonal matrix with

$$k_1 := 1, \quad k_j := \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_{j-1}, \quad j = 2, \ldots, n.$$

Barreras and Peña showed via the matrix K that the SBD and $SSDB$ matrices are signature similar to the $NsTN$ and TP matrices, respectively [1, Theorem 1]. From this property they obtained directly by using IP 3.2.1 and IP 3.2.3 a) the following theorem.

Theorem 3.4. [1, Theorems 3 and 11] *Let $A, B, Z \in \mathbb{R}^{n,n}$ and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$ be a signature sequence. If $KAK \leq^* KZK \leq^* KBK$ and A and B are (S)SBD matrices with the signature sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$, then Z is a (S)SBD matrix with the same signature sequence.*

4 Matrix Properties Which Require in General More than Two Vertex Matrices

In this section we consider instances in which the interval property requires in general more than two vertex matrices. The underlying partial ordering is the usual entry-wise partial order.

4.1 Properties Requiring at Most 2^{n^2-n} or $2^{(n^2-n)/2}$ Vertex Matrices

a) **Inverse M-Matrices:**

Definition 4.1. *A matrix A is an* inverse M-matrix *if it is nonsingular and A^{-1} is an M-matrix.*

For properties and examples of these matrices the reader is referred to [24, 26].

IP 4.1 [25, Theorem, p.241], see also [26, Theorem 9.7]: *The set of the inverse M-matrices has the interval property with respect to all vertex matrices.*

In [25] examples of matrix intervals are presented which show that we cannot expect that IP 4.1 is true with respect to a smaller set of vertex matrices. However, the set $V([A])$ can slightly be restricted to the subset containing all vertex matrices $A = (a_{ij})_{i,j=1,\dots,n}$ with $a_{ii} = \underline{a}_{ii}$, $i = 1,\dots,n$, since for each inverse M-matrix A and each nonnegative diagonal matrix D the matrix $A + D$ is an inverse M-matrix, too, [26, Theorem 1.7].

b) **Diagonal stability:**

Definition 4.2. *A matrix $A \in \mathbb{R}^{n,n}$ is called* positive semidefinite *if $0 \leq x^T Ax$ for each $x \in \mathbb{R}^n$ and* positive definite *if $0 < x^T Ax$ for each $x \in \mathbb{R}^n \setminus \{0\}$.*

Definition 4.3. *A matrix A is called* diagonal stable *if a positive definite diagonal matrix D exists such that $AD + DA^T$ is positive definite.*

Examples of diagonal stable matrices are the M-matrices and the inverse M-matrices [24, Theorem 2]. For properties and many applications of these matrices see the monograph [27]. We choose $V([A])$ as the set of all vertex matrices $A = (a_{ij})_{i,j=1,\dots,n}$ with $a_{ii} = \underline{a}_{ii}$, $i = 1,\dots,n$, and the property that if $a_{ij} = \underline{a}_{ij}$ (respectively, $\overline{a}_{ij}$) then $a_{ji} = \underline{a}_{ji}$ (respectively, $\overline{a}_{ji}$), $j = i+1,\dots,n$. The cardinality of this vertex set is at most $2^{n(n-1)/2}$ and we have the following interval property.

IP 4.2 [11, Theorem 1 (ii)]: *The set of the diagonally stable matrices has the interval property.*

4.2 Properties Requiring at Most 2^{2n-1} Vertex Matrices

Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can be represented as $\{A \in \mathbb{R}^{n,n} \mid | A - A_c | \leq \Delta\}$, where $A_c := \frac{1}{2}(\overline{A} + \underline{A})$ is the *midpoint matrix* and $\Delta := \frac{1}{2}(\overline{A} - \underline{A})$ is the *radius matrix*, in particular, $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$.

With $Y_n := \{y \in \mathbb{R}^n \mid | y_i |= 1,\ i = 1,\dots,n\}$ and $T_y := \mathrm{diag}\,(y_1, y_2,\dots,y_n)$ we define matrices $A_{yz} := A_c - T_y \Delta T_z$ for all $y, z \in Y_n$. The definition implies that for all $i, j = 1,\dots,n$

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i(\Delta)_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if} \quad y_i z_j = -1, \\ \underline{a}_{ij} & \text{if} \quad y_i z_j = 1, \end{cases}$$

so that all matrices A_{yz} are vertex matrices. In this subsection we choose $V([A])$ as the matrices A_{yz} for $y, z \in Y_n$. Since $A_{yz} = A_{-y,-z}$ for all $y, z \in Y_n$, the cardinality of $V([A])$ is at most 2^{2n-1}.

The following properties of $[A]$ can be inferred from the set $V([A])$:

a) **Nonsingularity:** Forty necessary and sufficient conditions for a matrix interval to be nonsingular are presented in [41]; some of them involve the set $V([A])$.

Theorem 4.1. [41, Theorem 4.1 (xxxii), (xxxiii)] *Let $[A] \in \mathbb{I}(\mathbb{R}^{n,n})$. The following three statements are equivalent:*

(i) $[A]$ *is nonsingular.*

(ii) $0 < \det A_{yz} \cdot \det A_{y'z'}$ *for each* $y, z, y', z' \in Y_n$.

(iii) $0 < \det A_{yz} \cdot \det A_{y'z}$ *for each* $y, y', z \in Y_n$ *such that* y *and* y' *differ in exactly one entry.*

The equivalence of (i) and (ii) in Theorem 4.1 was already proven in [10]. In [29, Theorem 2.2] it was shown that in statement (ii) the set $V([A])$ cannot be replaced by a nonempty proper subset.

b) **Nonsingular sign regular matrices**, see Subsection 3.2: Inspection of the proof of [16, Theorem 4] shows that the proof does not depend on the special choice of the sign of the minors of fixed order (in [16] all signs are taken as 1, i.e., the TN case is considered) and we obtain therefore the following interval property, cf. Conjecture 3.1.

IP 4.3: *The set of the* $NsSR$ *matrices with a fixed signature has the interval property.*

c) **Inverse stability:**

Definition 4.4. *A matrix* A *is called* inverse stable *if it is nonsingular and* $0 <| A^{-1} |$.

By the continuity of the determinant a matrix interval is inverse stable if it is nonsingular and each entry of the inverse stays either positive or negative through the entire matrix interval.

IP 4.4 [39, Theorem 2.1]: *The set of the inverse stable matrices with identical sign pattern of their inverses has the interval property.*

4.3 Properties Requiring at Most 2^{n-1} or 2^n Vertex Matrices

In this subsection we consider in parts a) and b) the vertex matrices A_{yz} introduced in Subsection 4.2 with $y = z$. In part c) we employ their dual vertex matrices $A_{-z,z}$. In both cases, the cardinality of $V([A])$ is reduced to at most 2^{n-1}. In part d) we use the matrices $A_{\pm z,z}$; thus the cardinality of $V([A])$ is at most 2^n.

a) P**-matrices:**

Definition 4.5. *A matrix is called* $P(P_0)$-matrix *if all its principal minors are positive (nonnegative).*

Instances of the P-matrices considered so far in this paper are the M-matrices, the $NsTN$ matrices, the inverse M-matrices [24, Corollary 1], and the diagonally stable matrices. Inspection of the proof to [11, Theorem 1 (i)] shows that the matrices used therein are just the matrices A_{zz} and we have the following interval property.

IP 4.4 [11, Theorem 1 (i) and Remark (b)]: *The set of the* $P(P_0)$*-matrices has the interval property.*

[11, Theorem 2] shows that for the P-matrices the set $V([A])$ cannot be replaced by a nonempty proper subset. For the interval property of matrices with alternating sign of their principal minors see [11, Remark (b)].

b) **Positive (semi)definiteness**[1]:

IP 4.5 [40, Theorem 2]: *The set of the positive (semi)definite matrices has the interval property.*

In [29, Theorem 2.2] it was shown that in the positive definite case the set $V([A])$ cannot be replaced by a nonempty proper subset.

We consider now symmetric positive (semi)definite matrices and consequently only those matrices in the given matrix interval $[A]$ which are symmetric; this set is denoted by $[A]_{sym}$. We also require that $[A]$ is *symmetric* by which we mean in mild abuse of our definition at the very end of Subsection 2.2 that the two corner matrices of $[A]$ are symmetric. Note that then each matrix A_{zz} is symmetric, too. Since a symmetric positive (semi)definite matrix is a $P(P_0)$-matrix, we may also use IP 4.4 to obtain immediately the following theorem.

Theorem 4.2. [11, p.40] *Let $[A]$ be a symmetric matrix interval. Then $[A]_{sym}$ contains only positive (semi)definite matrices if and only if all the vertex matrices from $V([A])$ are positive (semi)definite.*

In passing we mention a conjecture related to [39, Theorem 1.2] and Theorem 4.2 on the square of the first pivot in the Cholesky decomposition (which is identical to the reciprocal value of the entry in the bottom right position of A^{-1}).

Conjecture 4.1. [18, Conjecture 1] *Let $[A] \in \mathbb{I}(\mathbb{R}^{n,n})$ be symmetric and $[A]_{sym}$ contain only positive definite matrices. Then the function* $\det A/\det A[1,\ldots,n-1]$ *attains its minimum value on $[A]_{sym}$ at a matrix A_{zz} with $z \in Y_n$.*

c) **Hurwitz stability**:

Definition 4.6. *A matrix is called* Hurwitz stable *if all its eigenvalues have negative real parts.*

It is well-known that the Hurwitz stability of a matrix interval cannot in general be inferred from the Hurwitz stability of all of its vertex matrices, see [19, p.395] and [40, p.181]. However, if a matrix A is symmetric then A is Hurwitz stable if and only if $-A$ is positive definite. Using this fact, the following theorem can be shown.

Theorem 4.3. [40, Theorem 6] *Let $[A]$ be a symmetric matrix interval. Then $[A]$ is Hurwitz stable if and only if each vertex matrix $A_{-z,z}$, $z \in Y_n$, is Hurwitz stable.*

In [29, Theorem 2.2] it was shown that a further reduction of the set Y_n is impossible: without checking all 2^{n-1} matrices $A_{-z,z}$ we cannot guarantee that all $A \in [A]$ are Hurwitz stable. In [42] matrices are considered which are connected with mathematical models of ecosystems describing the effects a species may have on itself and its surrounding species. It is demonstrated on some examples that a few vertex matrices of this type may suffice to conclude that the entire matrix interval is Hurwitz stable.

d) **Schur stability**:

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[1] See Definition 4.2.

Definition 4.7. *A matrix is called* Schur stable *if the modulus of all its eigenvalues is less than* 1.

It is well-known that the Schur stability of the vertex matrices of a matrix interval does not imply the Schur stability of the entire matrix interval, see, e.g., [35]. In the symmetric case, however, we have the following result. In contrast to Theorem 4.3 the conclusion concerns only $[A]_{sym}$.

Theorem 4.4. [21, Corollary 2] *Let* $[A]$ *be a symmetric matrix interval. Then* $[A]_{sym}$ *contains only Schur stable matrices if and only if each vertex matrix* $A_{\pm z,z}$, $z \in Y_n$, *is Schur stable.*

For a survey of 'interval properties' of polynomial families related to stability and further applications see [19].

5 Related Problems

In this last section we consider a related problem, viz. to find for the single entries of a matrix A exhibiting a certain property an (respectively, the maximum) allowable perturbation such that this property (or related properties) is retained for all perturbed matrices.

In [5, 6] the first two authors of the present paper solve this problem for two subclasses of the TN matrices. Specifically, they give in [5] for a tridiagonal (not necessarily nonsingular) TN matrix the largest amount by which each of its single entries (inside the tridiagonal band and on the second sub- and superdiagonal) can be perturbed such that the resulting matrix remains TN. In [6] for each single entry of a TP matrix the largest amount for the persistence of total positivity is provided. For both classes of matrices the maximum allowable perturbation is presented in terms of ratios of minors of the unperturbed matrix.

Next we consider the problem of allowable perturbation of the single entries of a tridiagonal M-matrix. A perturbation which retains the M-matrix sign pattern leads to an M-matrix if the (generalized) strict diagonal dominance is maintained. Any perturbation inside the tridiagonal band which destroys the M-matrix sign pattern results in a matrix which is not inverse nonnegative [28, Theorem 5]. In, e.g., [28, 22] the problem of a positive entry-wise perturbation outside the tridiagonal band is considered. Such matrices are no longer M-matrices but may indeed be inverse nonnegative. In [22] the maximum allowable perturbation for each entry outside the tridiagonal band is presented, provided in terms of ratios of entries on the first sub- and superdiagonal and principal minors of the given matrix. It is noted that if the column index of the perturbed entry above (below) the tridiagonal band is increased (decreased) than the actual maximum allowable perturbation decreases. Generally speaking, the farther the perturbed entry is away from the main diagonal, the smaller the maximum allowable perturbation. Specification to the case of a tridiagonal M-matrix with Toeplitz structure, i.e., the entries along each diagonal are identical, is given, too. Furthermore, the persistence of inverse nonnegativity under simultaneous perturbation of more than one entry is considered therein.

Finally, we mention that in [32] a class of inverse nonnegative matrices is considered which cannot be entry-wise increased without losing the property of being inverse nonnegative. On the other hand, it is shown therein that each entry of an inverse nonnegative matrix can be decreased by a sufficiently small positive amount without destroying inverse nonnegativity.

Presistence of diagonal stability under entry-wise perturbation is considered in [13, Section V]

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References

[1] Álvaro Barreras and Juan M. Peña. Intervals of structured matrices. In Chérif Amrouche et al, editor, *Monografías Matemáticas García de Galdeano, (to appear)*. Prensas Univ. Zaragoza, 2015. Thirteenth International Conference Zaragoza-Pau on Mathematics and its Applications.

[2] Mohammad Adm. *Perturbation and Intervals of Totally Nonnegative Matrices and Related Properties of Sign Regular Matrices*. PhD thesis, University of Konstanz, Konstanz, Germany, 2016.

[3] Mohammad Adm and Jürgen Garloff. Intervals of totally nonnegative matrices. *Linear Algebra Appl.*, 439:3796–3806, 2013.

[4] Mohammad Adm and Jürgen Garloff. Improved tests and characterizations of totally nonnegative matrices. *Electron. J. Linear Algebra*, 27:588–610, 2014.

[5] Mohammad Adm and Jürgen Garloff. Invariance of total nonnegativity of a tridiagonal matrix under element-wise perturbation. *Oper. Matrices*, 8(1):129–137, 2014.

[6] Mohammad Adm and Jürgen Garloff. Invariance of total positivity of a matrix under entry-wise perturbation and completion problems. In *Contemporary Mathematics,* ed. by Carlos M. da Fonseca et al.*,* Amer. Math. Soc., Providence, RI, 2016, to appear.

[7] Mohammad Adm and Jürgen Garloff. Intervals of special sign regular matrices. *Linear Multilinear Algebra*, 2016, to appear, doi:10.1080/03081087.2015.1090388.

[8] Mohammad Adm, Jürgen Garloff, and Jihad Titi. Total nonnegativity of matrices related to polynomial roots and poles of rational functions. *J. Math. Anal. Appl.*, 434(1):780–797, 2016.

[9] Wihelm Barth and Erich Nuding. Optimale Lösung von Intervallgleichungssystemen. *Computing*, 12:117–125, 1974.

[10] Martin Baumann. A regularity criterion for interval matrices. In: Jürgen Garloff et al., editors., *Collection of Scientific Papers Honoring Prof. Dr. Karl Nickel on the Occasion of his 60th Birthday Vol. I,* Institute for Applied Mathematics, University of Freiburg, Freiburg, pages 45–49, 1984.

[11] Stauisław Białas and Jürgen Garloff. Intervals of P-matrices and related matrices. *Linear Algebra Appl.*, 58:33–41, 1984.

[12] Shaun M. Fallat and Charles R. Johnson. *Totally Nonnegative Matrices.* Princeton Ser. Appl. Math., Princeton University Press, Princeton and Oxford, 2011.

[13] Mauro Forti and Alberto Tesi. New conditions for global stability of neural networks with application to linear and quadratic programming problems. *IEEE Trans. Circuits Syst.−1: Fundam. Theory Applic.*, 42(7):354–366, 1995.

[14] Feliks R. Gantmacher and Mark G. Krein. *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme.* Mathematische Lehrbücher und Monographien, I. Bd. V. Abteilung, Akademie-Verlag, Berlin, 1960.

[15] Jürgen Garloff. Criteria for sign regularity of sets of matrices. *Linear Algebra Appl.*, 44:153–160, 1982.

[16] Jürgen Garloff. Vertex implications for totally nonnegative matrices. In: Mariano Gasca and Charles A. Micchelli, editors. *Total Positivity and its Applications,* Kluwer Academic Publishers, Dordrecht, pages 103–107, 1996.

[17] Jürgen Garloff. Intervals of almost totally positive matrices. *Linear Algebra Appl.*, 363:103–108, 2003.

[18] Jürgen Garloff. Pivot tightening for direct methods for solving symmetric positive definite systems of linear interval equations. *Computing*, 94(2-4):97–107, 2012.

[19] Jürgen Garloff and Nirmal K. Bose. Boundary implications for stability properties. In: Ramon E. Moore, editor. *Reliability in Computing,* Academic Press, Boston, San Diego, New York, pages 391–402, 1988.

[20] Ken R. Goodearl, Stephane Launois, and Tom H. Lenagan. Totally nonnegative cells and matrix Poisson varieties. *Adv. Math.*, 226:779–826, 2011.

[21] David Hertz. The extreme eigenvalues and stability of real symmetric interval matrices. *IEEE Trans. Automat. Control*, 37(4):532–535, 1992.

[22] Jie Huang, Ronald D. Haynes, and Ting-Zhu Huang. Monotonicity of perturbed tridiagonal M-matrices. *SIAM J. Matrix Anal. Appl.*, 33(2):681–700, 2012.

[23] Rong Huang, Jianzhou Liu, and Li Zhu. Nonsingular almost strictly sign regular matrices. *Linear Algebra Appl.*, 436:4179–4192, 2012.

[24] Charles R. Johnson. Inverse M-matrices. *Linear Algebra Appl.*, 47:195–216, 1982.

[25] Charles R. Johnson and Ronald L. Smith. Intervals of inverse M-matrices. *Reliab. Comput.*, 8(3):239–243, 2002.

[26] Charles R. Johnson and Ronald L. Smith. Inverse M-matrices, II. *Linear Algebra Appl.*, 435(5):953–983, 2011.

[27] Eugenius Kaszkurewicz and Amit Bhaya. *Matrix Diagonal Stability in Systems and Computation*. Birkhäuser, Boston, Basel, Berlin, 2000.

[28] Shannon C. Kennedy and Ronald D. Haynes. Inverse positivity of perturbed tridiagonal M-matrices. *Linear Algebra Appl.*, 430:2312–2323, 2009.

[29] Vladik Kreinovich. Optimal finite characterization of linear problems with inexact data. *Reliab. Comput.*, 11(6):479–489, 2005.

[30] James R. Kuttler. A fourth-order finite-difference approximation for the fixed membrane eigenproblem. *Math. Comp.*, 25(114):237–256, 1971.

[31] Stephane Launois and Tom H. Lenagan. Efficient recognition of totally nonnegative matrix cells. *Found. Comput. Math.*, 14(2):371–387, 2014.

[32] Linzhang Lu and Michael K. Ng. Maximum inverse positive matrices. *Appl. Math. Lett.*, 20:65–69, 2007.

[33] Kurt Metelmann. *Inverspositive Bandmatrizen und totalnichtnegative Green'sche Matrizen*. PhD thesis, University of Cologne, Cologne, Germany, 1972.

[34] Kurt Metelmann. Ein Kriterium für den Nachweis der Totalnichtnegativität von Bandmatrizen. *Linear Algebra Appl.*, 7:163–171, 1973.

[35] Takehiro Mori and Hideki Kokame. Convergence property of interval matrices and interval polynomials. *Int. J. Control*, 45(2):481–483, 1987.

[36] Juan M. Peña, editor. *Shape Preserving Representations in Computer-aided Geometric Design*. Nova Science Publishers, Inc., New York, 1999.

[37] Allan Pinkus. *Totally Positive Matrices*. Cambridge Tracts in Mathematics 181, Cambridge Univ. Press, Cambridge, UK, 2010.

[38] Jiří Rohn. Systems of linear interval equations. *Linear Algebra Appl.*, 126:39–78, 1989.

[39] Jiří Rohn. Inverse interval matrix. *SIAM J. Numer. Anal.*, 30:864–870, 1993.

[40] Jiří Rohn. Positive definiteness and stability of interval matrices. *SIAM J. Matrix Anal. Appl.*, 15(1):175–184, 1994.

[41] Jiří Rohn. Forty necessary and sufficient conditions for regularity of interval matrices: A survey. *Electron. J. Linear Algebra*, 18:500–512, 2009.

[42] Rama K. Yedavalli and Nagini Devarakonda. Sufficiency of vertex matrix check for robust stability of interval matrices via the concept of qualitative robustness. In: *Proceedings 2013 European Control Conference*, July 17-19, 2013, Zürich, Switzerland, IEEE, pages 2351–2356, 2013.