Theoretical Characterization of Enclosures*

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Abstract

We give a theoretical characterization of enclosures of the solution set of interval linear equations formulated in terms of components of the solution set of the "dual" Oettli-Prager inequality.

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1 Introduction and Notation

Seemingly, anyone interested in interval linear equations knows the inequality

$$|A_c x - b_c| \leq \Delta|x| + \delta;$$

this is the Oettli-Prager inequality [2] describing the solution set of a system of interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$ with $\boldsymbol{A} = [A_c - \Delta, A_c + \Delta] \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{b} = [b_c - \delta, b_c + \delta] \in \mathbb{IR}^n$. Very little, if anything at all, is known, however, of its "dual" inequality

$$|A_c x - b_c| \geq \Delta|x| + \delta.$$

In this note we show that these two inequalities are interrelated in a peculiar way. If $\boldsymbol{A}$ is regular and $\delta > 0$, then the solution set of the first inequality is connected whereas that one of the second inequality consists of exactly 2^n components (nonempty connected subsets maximal with respect to inclusion), and an interval vector encloses the solution set of the first inequality if and only if it intersects all the 2^n components of the solution set of the second inequality. It is just this result that we call the "theoretical characterization of enclosures". The proof employs two nontrivial results from [3], [4], of which particularly the second one is little known.

Notation used: $Y = \{-1, 1\}^n$ is the set of all ±1-vectors in $\mathbb{R}^n$, and T_y denotes the diagonal matrix with diagonal vector y (used for $y \in Y$ only). The sign vector $z = \mathrm{sgn}(x)$ of a vector $x \in \mathbb{R}^n$ is defined by $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ otherwise ($i = 1, \ldots, n$).

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2 The Result and its Consequences

Denote

$$\boldsymbol{X}(\boldsymbol{A},\boldsymbol{b}) = \{\, x \mid |A_c x - b_c| \leq \Delta|x| + \delta \,\}$$

and

$$\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b}) = \{\, x \mid |A_c x - b_c| \geq \Delta|x| + \delta \,\}.$$

Then we have the following main result.

Theorem 2.1. *Let $\boldsymbol{A}$ be regular and let $\delta > 0$. Then an interval vector $[\underline{x}, \overline{x}]$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A},\boldsymbol{b})$ if and only if it intersects all components of $\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$.*

Proof: The proof proceeds in three steps.

(a) For each $y \in Y$ define a set X_y by

$$X_y = \{\, x \mid T_y A_c x - \Delta t \geq T_y b_c + \delta,\ -t \leq x \leq t \text{ for some } t \in \mathbb{R}^n \,\}. \tag{1}$$

The set described by the right-hand side system of linear inequalities is a convex polyhedron, therefore X_y, as its projection onto the x-subspace, is again a convex polyhedron. Next we prove that $X_y \subseteq \boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$. Let $x \in X_y$, then it satisfies

$$T_y(A_c x - b_c) \geq \Delta t + \delta, \quad t \geq |x|, \tag{2}$$

hence

$$T_y(A_c x - b_c) \geq \Delta|x| + \delta \tag{3}$$

which by virtue of nonnegativity of the right-hand side implies that $T_y(A_c x - b_c) \geq 0$, thus $T_y(A_c x - b_c) = |A_c x - b_c|$, and (3) turns into

$$|A_c x - b_c| \geq \Delta|x| + \delta \tag{4}$$

which means that $x \in \boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$. Thus, $\bigcup_{y \in Y} X_y \subseteq \boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$. To prove the converse inclusion, take $x \in \boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$. Then it satisfies (4), thus also (3) for $y = \operatorname{sgn}(A_c x - b_c)$, and taking $t = |x|$, we see that it also satisfies (2), so that $x \in X_y$. In this way we have proved that

$$\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b}) = \bigcup_{y \in Y} X_y. \tag{5}$$

Finally we prove that all the X_y's are mutually disjoint. Suppose it is not so, so that $x \in X_y \cap X_{y'}$ for some $y \neq y'$, where $y_i = 1$ and $y'_i = -1$ for some i. Then from (3) we obtain both $(A_c x - b_c)_i \geq 0$ and $-(A_c x - b_c)_i \geq 0$, hence $(A_c x - b_c)_i = 0$ implying $(\Delta|x| + \delta)_i = 0$ which is a contradiction because $\delta > 0$ by assumption. Hence, (5) is a decomposition of $\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$ into a union of mutually disjoint convex (i.e., connected) polyhedra which, in turn, means that each X_y is a component of $\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$ (we shall see later that all the X_y's are nonempty, so that there are exactly 2^n of them).

(b) Next we prove that if $[\underline{x}, \overline{x}]$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A},\boldsymbol{b})$, then it intersects all the components X_y, $y \in Y$. To see this, take an arbitrary $y \in Y$ and consider the absolute value equation

$$A_c x - T_y \Delta|x| = b_c + T_y \delta. \tag{6}$$

Since $\boldsymbol{A}$ is regular by assumption, by Theorem 2.2 in [3] the equation (6) has exactly one solution x_y which belongs to $\boldsymbol{X}(\boldsymbol{A},\boldsymbol{b})$ and thus also to $[\underline{x}, \overline{x}]$. Rearranging the equation (6) to the form

$$T_y(A_c x - b_c) = \Delta|x| + \delta,$$

we can see that x_y satisfies (3) and (2), hence $x_y \in X_y$. Thus $x_y \in [\underline{x}, \overline{x}] \cap X_y$ for each $y \in Y$, so that $[\underline{x}, \overline{x}]$ intersects all the components of $\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b})$.

(c) Finally we shall prove that if $[\underline{x}, \overline{x}] \cap X_y \neq \emptyset$ for each $y \in Y$, then $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) \subseteq [\underline{x}, \overline{x}]$. Take $\tilde{x}_y \in [\underline{x}, \overline{x}] \cap X_y$ for each $y \in Y$ and let $x \in \boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$. To prove that $x \in [\underline{x}, \overline{x}]$, we proceed as follows. Since $x \in \boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$, by definition of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ there exist $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ such that $Ax = b$. Now we have

$$|T_y(A\tilde{x}_y - b) - T_y(A_c\tilde{x}_y - b_c)| = |(A - A_c)\tilde{x}_y + (b_c - b)| \leq \Delta|\tilde{x}_y| + \delta,$$

hence

$$T_y(A\tilde{x}_y - b) \geq T_y(A_c\tilde{x}_y - b_c) - \Delta|\tilde{x}_y| - \delta \geq 0,$$

the nonnegativity being a consequence of (3) because $\tilde{x}_y \in X_y$. Thus we have proved that

$$T_y(A\tilde{x}_y - b) \geq 0,$$

which can be rewritten as

$$T_{-y}A\tilde{x}_y \leq T_{-y}b,$$

for each $y \in Y$. Now Theorem 2 in [4] tells us that this property implies existence of an x^* such that $Ax^* = b$, and x^* belongs to the convex hull of the points $\tilde{x}_y$, $y \in Y$. Since each $\tilde{x}_y$, $y \in Y$, belongs to the convex set $[\underline{x}, \overline{x}]$, its convex hull is a part of $[\underline{x}, \overline{x}]$, hence $x^* \in [\underline{x}, \overline{x}]$. But since $Ax^* = b$ and $Ax = b$ and A is nonsingular, it must be $x^* = x$, hence $x \in [\underline{x}, \overline{x}]$. In this way we finally have that $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b}) \subseteq [\underline{x}, \overline{x}]$, which was to be proved. □

We have proved that

$$\boldsymbol{X_d}(\boldsymbol{A}, \boldsymbol{b}) = \bigcup_{y \in Y} X_y,$$

where

$$\begin{aligned} X_y &= \{\, x \mid T_yA_cx - \Delta t \geq T_yb_c + \delta,\ -t \leq x \leq t \text{ for some } t \,\} && (7)\\ &= \{\, x \mid T_y(A_cx - b_c) \geq \Delta|x| + \delta \,\}. && (8) \end{aligned}$$

In the proof we have used the former description to demonstrate that X_y is a convex polyhedron; now we shall employ the latter to show that under mild additional assumption this polyhedron is unbounded for each $y \in Y$.

Theorem 2.2. *If, additionally, $\Delta \neq 0$, then all the components X_y, $y \in Y$, are unbounded.*

Proof: Let $y \in Y$. According to Theorem 5.1, (B4) in [3], regularity of $\boldsymbol{A}$ implies existence of a positive vector r_y satisfying

$$|A_c^{-1}T_y\Delta r_y| < r_y.$$

Then, with x_y being the unique solution of (6), for each $\lambda \geq 0$ we have

$$\begin{aligned} T_y(A_c(x_y + \lambda A_c^{-1}T_y\Delta r_y) - b_c) &= \Delta|x_y| + \delta + \lambda\Delta r_y \\ &= \Delta(|x_y| + \lambda r_y) + \delta \\ &\geq \Delta(|x_y| + \lambda|A_c^{-1}T_y\Delta r_y|) + \delta \\ &\geq \Delta|x_y + \lambda A_c^{-1}T_y\Delta r_y| + \delta \end{aligned}$$

which in the light of (8) shows that $x_y + \lambda A_c^{-1} T_y \Delta r_y \in X_y$ for each $\lambda \geq 0$ and since $A_c^{-1} T_y \Delta r_y \neq 0$ because of $\Delta \neq 0$ and $r_y > 0$, this means that the whole half-ray

$$\{\, x_y + \lambda A_c^{-1} T_y \Delta r_y \mid \lambda \geq 0 \,\}$$

belongs to X_y which is thereby unbounded. □

In a special case of $\Delta = 0$ we have from (8) that

$$X_y = \{\, x \mid T_y(A_c x - b_c) \geq \delta \,\}.$$

If we write $T_y(A_c x - b_c) = \delta + h$, where $h \geq 0$, then $x = A_c^{-1}(b_c + T_y \delta) + A_c^{-1} T_y h = x_y + A_c^{-1} T_y h$, where x_y is the solution to (6). In this way we obtain the description

$$X_y = \{\, x_y + A_c^{-1} T_y h \mid h \geq 0 \,\}$$

which shows that in this case X_y is a closed convex cone emanating from the point x_y.

Next we shall show that enclosures can also be characterized without resorting to the notion of component.

Theorem 2.3. *Let $\boldsymbol{A}$ be regular and $\delta > 0$. Then an interval vector $\boldsymbol{x}$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ if and only if for each $y \in Y$ it contains a vector $\tilde{x}_y$ satisfying*

$$T_y(A_c \tilde{x}_y - b_c) \geq \Delta |\tilde{x}_y| + \delta. \tag{9}$$

Proof: If an interval vector $\boldsymbol{x}$ is an enclosure of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$, then for each $y \in Y$ the unique solution x_y of (6) belongs to $\boldsymbol{x}$ and satisfies (9). Conversely, if an interval vector $\boldsymbol{x}$ for each $y \in Y$ contains a solution $\tilde{x}_y$ of (9), then $\boldsymbol{x}$ intersects all components of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ and thus is an enclosure of it by Theorem 2.1. □

Finally we show how an enclosure can be constructed directly from solutions of (9).

Theorem 2.4. *Let $\boldsymbol{A}$ be regular and $\delta > 0$. If for each $y \in Y$ the inequality (9) has a solution $\tilde{x}_y$, then*

$$\boldsymbol{x} = [\min_{y \in Y} \tilde{x}_y, \max_{y \in Y} \tilde{x}_y] \tag{10}$$

(entrywise minimum/maximum) is an enclosure of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$.

Proof: Obviously, $\tilde{x}_y \in \boldsymbol{x}$ for each $y \in Y$, hence $\boldsymbol{x}$ is an enclosure by Theorem 2.3. □

Theorem 2.4 in [3] shows that if for each $y \in Y$ the inequality (9) holds as equation, then $\boldsymbol{x}$ given by (10) is the interval hull of $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$.

3 Example

Consider the example by Hansen [1]

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [\ 0, 120] \\ [60, 240] \end{pmatrix}. \tag{11}$$

In Fig. 1 the solution set $\boldsymbol{X}(\boldsymbol{A}, \boldsymbol{b})$ is depicted in green and the four unbounded components of $\boldsymbol{X}_{\boldsymbol{d}}(\boldsymbol{A}, \boldsymbol{b})$ in red. The components touch the solution set just in the points x_y (solutions of (6)). Observe that, indeed, an interval vector encloses the solution set if and only if it intersects all four components.

The picture was made with the help of the software package Intlininc2d by I. A. Sharaya [6], [5].

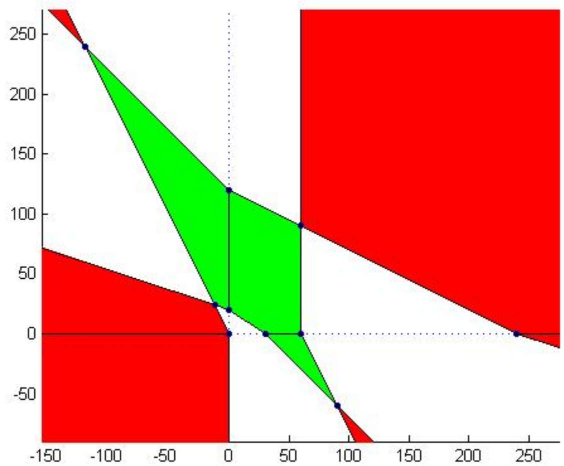

Figure 1: Example (11), sets $\boldsymbol{X}(\boldsymbol{A},\boldsymbol{b})$ (in green) and $\boldsymbol{X_d}(\boldsymbol{A},\boldsymbol{b})$ (in red).

4 Conclusion

The main result remains highly theoretical because in practice we will hardly ever be able to check that an interval vector intersects 2^n sets. But it is of certain interest because of its three features: first, that such a characterization exists at all; second, due to a special way in which inequalities $|A_c x - b_c| \le \Delta|x| + \delta$ and $|A_c x - b_c| \ge \Delta|x| + \delta$ are interrelated; and third, due to the sole fact that the solution set of $|A_c x - b_c| \ge \Delta|x| + \delta$ has exactly 2^n components that are explicitly described by (7), (8).

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