On Unboundedness of Generalized Solution Sets for Interval Linear Systems*

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Abstract

The work presents a necessary and sufficient condition for the generalized AE-solution sets to interval linear systems of equations to be unbounded. The criterion requires solvability of one of the totality of 2^n linear inequalities constructed from the initial interval system. It is shown that the problem of recognizing the boundedness of AE-solution sets is NP-complete in general.

Keywords: interval linear equations, generalized solution sets, AE-solution sets, unboundedness, NP-hardness
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1 Introduction

We consider systems of interval linear algebraic equations of the form

$$\left\{\begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{m2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m \end{array}\right. \tag{1}$$

with interval coefficients $\boldsymbol{a}_{ij}$ and interval right-hand sides $\boldsymbol{b}_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is an interval $m \times n$-matrix, $x = (x_1, x_2, \ldots, x_n)^\top$ and $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m)^\top$ is an interval m-vector. The systems (1)–(2) are understood as families of point (noninterval) systems of linear algebraic equations of the same form $Ax = b$ with $m \times n$-matrices A from $\boldsymbol{A}$ and m-vectors b from $\boldsymbol{b}$.

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The notation of our paper adheres to the informal international standard [5]. Also, we denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$-matrices, while the inequalities between the matrices and the vectors are understood elementwise and coordinatewise, respectively.

As opposed to ordinary noninterval systems of equations, we can consider various *solution sets* to interval equations systems of the form (1)–(2). Historically, the *united solution set* (see [1, 20] and others)

$$\Xi_{uni}(\boldsymbol{A},\boldsymbol{b}) = \{\, x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(\, Ax = b\,)\,\}$$

was the first and remains the most investigated one so far. As the time passed, practical needs caused the introduction and investigation of another solution sets for the interval systems of equations.

In [19, 23], the set of inner solutions (later renamed to the *tolerable solution set*)

$$\Xi_{tol}(\boldsymbol{A},\boldsymbol{b}) = \{\, x \in \mathbb{R}^n \mid (\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(\, Ax = b\,)\,\}$$

was introduced and interpreted. In [7, 30], the *controllable solution set* was introduced as a result of solving the interval version of the automatic control problem. The controllable solution set is defined to be

$$\Xi_{ctr}(\boldsymbol{A},\boldsymbol{b}) = \{\, x \in \mathbb{R}^n \mid (\forall b \in \boldsymbol{b})(\exists A \in \boldsymbol{A})(\, Ax = b\,)\,\}.$$

From the above examples, the difference among various solution sets is that

(i) different quantifiers are attributed to different interval elements from the matrix $\boldsymbol{A}$ and right-hand side vector $\boldsymbol{b}$,

(ii) the orders of the quantifiers can vary in the logical formuli that stand after the vertical line in the definition of the solution set and "selects" appropriate points.

When carefully taking into account these differences, i.e., when attributing either existential or universal quantifier to each of the elements from $\boldsymbol{A}$ and $\boldsymbol{b}$, and choosing a certain order for these quantifiers, we obtain a large number of different concepts (forms of understanding) of the solution sets to (1)–(2). Such solution sets seem to have been appeared for the first time in [34] in connection with the game theory problems. Later, S.P. Shary continued investigation of such solution sets to interval systems of equations, and these results have been summarized in [31, 32].

The present paper avoids considering the most general definition of the solution sets that we can construct on this way. Our treating will be restricted to the case thoroughly examined by S.P. Shary [29, 31, 32]. Namely, we assume that a logical quantifier is attributed to each element from $\boldsymbol{A}$ and $\boldsymbol{b}$ and, furthermore, all the universal quantifiers precede all the occurrences of the existential quantifiers.

Before turning to formal considerations, we should first give precise definitions of the necessary objects following the works [29, 31, 32]. We suppose that an $m \times n$-matrix $\Lambda = (\lambda_{ij})$, $\lambda_{ij} \in \{-1, 1\}$, $i = \overline{1,m}$, $j = \overline{1,n}$, and an m-vector $\beta = (\beta_1, \dots, \beta_n)^\top$, $\beta_i \in \{-1, 1\}$, $i = \overline{1,m}$, are given, along with the interval $m \times n$-matrix $\boldsymbol{A}$ and the interval m-vector $\boldsymbol{b}$. The matrix $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ is decomposed into two matrices $\boldsymbol{A}^\exists = (\boldsymbol{a}^\exists_{ij})$ and $\boldsymbol{A}^\forall = (\boldsymbol{a}^\forall_{ij})$ so that

$$\boldsymbol{a}^\exists_{ij} = \begin{cases} \boldsymbol{a}_{ij}, & \text{if } \lambda_{ij} = 1, \\ 0, & \text{if } \lambda_{ij} = -1, \end{cases} \qquad \boldsymbol{a}^\forall_{ij} = \begin{cases} 0, & \text{if } \lambda_{ij} = 1, \\ \boldsymbol{a}_{ij}, & \text{if } \lambda_{ij} = -1. \end{cases}$$

Similarly, we decompose the vector $\boldsymbol{b} = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_m)^\top$ into two vectors

$$\boldsymbol{b}^\exists = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_m)^\top \qquad \text{and} \qquad \boldsymbol{b}^\forall = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_m)^\top$$

such that

$$\boldsymbol{b}_i^{\exists} = \begin{cases} \boldsymbol{b}_i, & \text{if } \beta_i = 1, \\ 0, & \text{if } \beta_i = -1, \end{cases} \qquad \boldsymbol{b}_i^{\forall} = \begin{cases} 0, & \text{if } \beta_i = 1, \\ \boldsymbol{b}_i, & \text{if } \beta_i = -1. \end{cases}$$

It is obvious that $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$, $\boldsymbol{b} = \boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists}$.

Informally, we can speak that the matrix Λ and vector β describe the distribution of uncertainty types over the elements of the interval matrix $\boldsymbol{A}$ and interval vector $\boldsymbol{b}$. The matrices $\boldsymbol{A}^{\exists}$ and $\boldsymbol{A}^{\forall}$ and vectors $\boldsymbol{b}^{\exists}$ and $\boldsymbol{b}^{\forall}$ serve for the same purpose, but sometimes using them may prove more vivid and/or convenient.

Definition 1 (S.P. Shary [29, 31, 32]) Let us be given an interval system of linear algebraic equations $\boldsymbol{A}x = \boldsymbol{b}$. For a quantifier matrix Λ and a quantifier vector β and associated matrices $\boldsymbol{A}^{\exists}$, $\boldsymbol{A}^{\forall}$ and vectors $\boldsymbol{b}^{\exists}$, $\boldsymbol{b}^{\forall}$, the generalized *AE-solution set* of the type $\Lambda\beta$ to the system $\boldsymbol{A}x = \boldsymbol{b}$ is

$$\Xi_{\Lambda,\beta}(\boldsymbol{A}, \boldsymbol{b}) = \{\, x \in \mathbb{R}^n \mid (\forall A' \in \boldsymbol{A}^{\forall})(\forall b' \in \boldsymbol{b}^{\forall})(\exists A'' \in \boldsymbol{A}^{\exists})(\exists b'' \in \boldsymbol{b}^{\exists})(\,(A' + A'')\,x = b' + b''\,) \,\}. \quad (3)$$

The main purpose of our paper is to inquire into the algorithmic complexity (in the sense of [2]) of the problem relating to these sets:

> **Problem.** To determine whether the solution set (3) to an interval linear system is unbounded.

Apart from purely theoretical interest, the above problem has a certain practical value. In systems analysis, identification theory and data fitting, the fact that the set of feasible values of the parameters (which is often a solution set to an interval systems of equations) is unbounded shows that the original problem statement may be not completely correct. This leads to serious consequences in the general strategy of its solution.

In the sequel, we assume that the reader is familiar with the principal concepts of the computational complexity theory, such as polynomial time solvability, NP-hardness, NP-completeness, polynomial reducibility of a problem to the other one (see [2]). It is worth noting that the first result on NP-complexity of interval linear algebraic problems, the one concerning determination of singularity of an interval square matrix, seems to have been obtained in the work [22].

The works [3, 8, 9, 11, 12, 14, 15, 16, 24, 25, 27] are devoted to investigation of computational complexity of the problems arising in connection with various solution sets to interval linear systems of equations. These are problems of recognizing nonemptyness of the solution sets and computing their enclosures, including the particular cases under additional conditions on the system (boundedness, finiteness, positivity, etc.). We note that the work [13] studied the complexity of estimation of the generalized solution set to interval linear systems.

A large number of other NP-complete (and NP-hard) problems that naturally arise in connection with the interval computations can be found in the book [10].

2 Characterization of Generalized Solution Sets

This section derives an Oettli-Prager-type description of the generalized solution sets, which will be needed further in our considerations. In doing this, we rely upon the characterization of the generalized solution sets to interval linear systems suggested by S.P. Shary in [29, 31, 32].

In our work, we use traditional definitions of the interval operations and relations for both scalars and matrices (vectors), see [17, 18, 32]. For the interval $m \times n$-matrix $\boldsymbol{A}$ and n-vector $x \in \mathbb{R}^n$, the product $\boldsymbol{A}x$ is defined as the natural interval extension of the usual point matrix-vector product [17, 18, 32]. On the other hand, the following useful equality is valid in this particular case:

$$\boldsymbol{A}x = \{\, Ax \mid A \in \boldsymbol{A} \,\}.$$

As a sequence, the following statement can de derived:

Theorem 2.1 (S.P. Shary [29, 31, 32]) *For any quantifiers $\varLambda$ and β of the same sizes as $\boldsymbol{A}$ and $\boldsymbol{b}$, respectively, the following equivalent representation of the AE-solution set holds:*

$$\varXi_{\varLambda,\beta}(\boldsymbol{A},\boldsymbol{b}) = \bigl\{\, x \in \mathbb{R}^n \mid \boldsymbol{A}^{\forall}x - \boldsymbol{b}^{\forall} \subseteq \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists}x \,\bigr\}.$$

In the rest of the paper, for two $m \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, we will denote by $A \circ B$ their Hadamard product $A \circ B = (a_{ij}b_{ij})$ (see e.g., [4]). Using Theorem 2.1 along with the well-known Oettli-Prager theorem (see e.g., [1, 18, 21, 32]), it is possible to obtain the following result:

Theorem 2.2 (J. Rohn [26]) *For any quantifiers $\varLambda$ and β of the same sizes as $\boldsymbol{A}$ and $\boldsymbol{b}$, respectively, the equality*

$$\varXi_{\varLambda,\beta}(\boldsymbol{A},\boldsymbol{b}) = \bigl\{\, x \in \mathbb{R}^n \mid |A_c x - b_c| \le (\varLambda \circ \Delta)|x| + \beta \circ \delta \,\bigr\} \tag{4}$$

is true, where $A_c = \frac{1}{2}(\underline{\boldsymbol{A}} + \overline{\boldsymbol{A}})$, $\Delta = \frac{1}{2}(\overline{\boldsymbol{A}} - \underline{\boldsymbol{A}})$, $b_c = \frac{1}{2}(\underline{\boldsymbol{b}} + \overline{\boldsymbol{b}})$, $\delta = \frac{1}{2}(\overline{\boldsymbol{b}} - \underline{\boldsymbol{b}})$.

Using the description (4), we can substantiate the following assertion:

Theorem 2.3 *Let $T_y = \operatorname{Diag}\{y_1, \dots, y_n\}$. The solution set $\varXi_{\varLambda,\beta}(\boldsymbol{A},\boldsymbol{b})$ is unbounded if and only if, for some $y \in Q = \bigl\{\, x \in \mathbb{R}^n \mid x_i \in \{-1, 1\},\ i = \overline{1,n} \bigr\}$, there exists a solution to the system of linear inequalities*

$$\begin{cases} -(\varLambda \circ \Delta)T_y x - \beta \circ \delta \le A_c x - b_c \le (\varLambda \circ \Delta)T_y x + \beta \circ \delta, \\ T_y x \ge 0, \\ -(\varLambda \circ \Delta)T_y z \le A_c z \le (\varLambda \circ \Delta)T_y z, \\ T_y z \ge 0, \\ \displaystyle\sum_{i=1}^{n} y_i z_i \ge 1. \end{cases} \tag{5}$$

Proof: It follows from the formula (4) that, for any $y \in Q$, the intersection of the solution set $\varXi_{\varLambda,\beta}(\boldsymbol{A}, \boldsymbol{b})$ with the orthant $\mathbb{R}^n_y = \{x \in \mathbb{R}^n \mid T_y x \geq 0\}$ is a convex polyhedron.

Hence, the set $\varXi_{\varLambda,\beta}(\boldsymbol{A}, \boldsymbol{b})$ is unbounded if and only if some of these polyhedrons are unbounded (for a certain $y \in Q$). On the other hand, we are reminded [28] that a nonempty polyhedron is unbounded if and only if its characteristic cone is non-zero.

In the system (5), solvability of the first two inequalities is equivalent to the polyhedron being nonempty, while solvability of the rest inequalities is equivalent to the fact that its characteristic cone is non-zero. ■

3 Computational Complexity

To state the problems of interest correctly, we shall assume that, for each m and n, there is a fixed $m \times n$-matrix $\varLambda(m,n) = (\lambda_{ij}(m,n))$ such that $\lambda_{ij}(m,n) \in \{-1, 1\}$, $i = \overline{1,m}$, $j = \overline{1,n}$, and an m-vector $\beta(m) = (\beta_1(m), \ldots, \beta_m(m))^T$ such that $\beta_i(m) \in \{-1, 1\}$, $i = \overline{1,m}$. In other words, the matrix-valued function $\varLambda$ and the vector-valued β are defined such that the function $\varLambda$ determines a correspondence between $m \times n$-matrices of $\{-1, 1\}$ and the pairs of natural numbers (m, n), $(m \geq 1, n \geq 1)$, while the function β sets a correspondence between the m-vectors of $\{-1, 1\}$ and the natural numbers m.

In the rest of the paper, we use the following standard yet helpful notation. For a real number λ, we denote the positive part of λ by $\lambda^+ = \max\{0, \lambda\}$, and the negative part of λ by $\lambda^- = \max\{0, -\lambda\}$ respectively. For the $m \times n$-matrices $\varLambda$ and the m-vector β, the positive parts $\varLambda^+$ and β^+ and the negative parts $\varLambda^-$ and β^- will be understood elementwise and componentwise, respectively.

Hence, for any interval system of the form (1)–(2) having m equations for n variables, it is possible to define the solution set $\varXi_{\varLambda(m,n),\beta(m)}(\boldsymbol{A}, \boldsymbol{b})$. Furthermore, we assume that the matrix $\varLambda(m,n)$ and the vector $\beta(m)$ are "easily computable" in the following sense.

Definition 2 [13] The functions $\varLambda$ and β are called *easily computable* if there exists a pseudo-polynomial time algorithm computing the matrix $\varLambda(m,n)$ and the vector $\beta(m)$, i.e., an algorithm whose processing time is limited by a polynomial in m and n.

Also, we will say that an interval matrix $\boldsymbol{A}$ is *integer* if the endpoints of its entries are integer numbers. The main problem that we deal with in the present paper is

Problem UNB$(\varLambda, \beta)$
(checking unboundedness of the AE-solution sets $\varXi_{\varLambda,\beta}(\boldsymbol{A}, \boldsymbol{b})$)

Given. Integer interval $m \times n$-matrices $\boldsymbol{A} = [A_c - \Delta,\ A_c + \Delta]$ and integer interval m-vectors $\boldsymbol{b} = [b_c - \delta,\ b_c + \delta]$. Functions $\varLambda(m,n)$ and $\beta(m)$ described above.

Question. Is it true that the solution sets $\varXi_{\varLambda(m,n),\beta(m)}(\boldsymbol{A}, \boldsymbol{b})$ to the interval linear systems $\boldsymbol{A}x = \boldsymbol{b}$ are unbounded?

From the foregoing considerations, the computational complexity of this problem is substantially determined by the number of the existential quantifiers in the definition of $\Xi_{\Lambda,\beta}(\boldsymbol{A}, \boldsymbol{b})$, i. e., by the number of (+1)'s in the matrix $\Lambda(m,n)$ and in the vector $\beta(m)$. Roughly speaking, if the number of existential quantifiers is "large enough", that is, a sufficiently large number of the columns of the matrix Λ contain at least one (+1), and a sufficiently large number of rows of the extended matrix $(\Lambda\beta)$ contain at least one (+1), then the problem formulated above is NP-complete. If the total number of (+1)'s in the matrix Λ grows slowly in comparison with the number mn (specifically, it has the order of $\log_2(mn)$), then the above problem can be solved in polynomial time.

To formulate what is meant by the term "sufficiently many existential quantifiers", we need additional clarification. When defining the term precisely, we will use usual notation for submatrices of a matrix [4], i.e. if $\Lambda = (\lambda_{ij})$ is an $m \times n$-matrix and $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_l\}$, $(1 \le i_1 < i_2 \ldots < i_k \le m,\ 1 \le j_1 < j_2 \ldots < j_l \le n$, then we denote the $k \times l$-matrix located at the intersections of the rows with the numbers $i_1, \ldots, i_k$ and the columns with the numbers $j_1, \ldots, j_l$ by $\Lambda(I|J)$. Similarly, for the m-vector β and $I = \{i_1, \ldots, i_k\}$, $1 \le i_1 < \ldots < i_k \le m$, we denote the k-vector with the corresponding components as $\beta(I)$.

Definition 3 [13] The functions Λ and β are said to be *computationally 1-saturate* (or, briefly, 1-saturate) if there exists an algorithm allowing the numbers m, n, k, l and two submatrices Λ_0, Λ_1 of the matrix $\Lambda(m,n)$ of dimensions $k \times s$ and $s \times l$, respectively, to be found for any natural number s, so that the following conditions hold:

1) the running time of the algorithms is restricted by a polynomial in s (that is, similar to Definition 1, the algorithm is quasi-polynomial with respect to s);
2) $m \ge k + l + s + 1,\ \ n \ge l + s$;
3) if $\Lambda_0 = \Lambda(m,n)(K \mid J)$, $\Lambda_1 = \Lambda(m,n)(I \mid L)$, then $K \cap I = J \cap L = \varnothing$, i. e., submatrices are located in different rows and different columns;
4) each of the columns in the submatrix Λ_0 contains at least one element (+1);
5) each of the rows in the submatrix $(\Lambda_1\gamma)$ obtained by adding of the column $\gamma = \beta(m)(I)$ to the submatrix Λ contains at least one element (+1);

In other words, up to within the transposition of rows and columns, the extended matrix $(\Lambda\beta)$ has the form

$$\big(\Lambda(m,n)\beta(m)\big) = \begin{pmatrix} \Lambda_0 & * & * & * \\ * & \Lambda_1 & * & \gamma \\ * & * & * & * \end{pmatrix}, \tag{6}$$

where the submatrices Λ_0 and Λ_1 possess the properties 4 and 5 from Definition 3.

Comment. Denote by $U_\Lambda(m,n)$ the number of (+1)'s in the the matrix $\Lambda(m,n)$. If the functions Λ, β are 1-saturate, then, from the condition 1 of Definition 3 and from the fact that the complexity of construction of the matrix $\Lambda(m,n)$ is greater or equal to mn, it follows that there exist such constants $C > 1$, $M > 1$ that $mn \le Cs^M$.

Since $U_\Lambda(m,n) \ge s$ according to the condition 4 of Definition 3, we arrive at the estimate

$$U_\Lambda(m,n) \ge \left(\frac{1}{C}\right)^{\frac{1}{M}} (mn)^{\frac{1}{M}}, \tag{7}$$

i.e., in this case the relation

$$\limsup_{m,n\to\infty} \frac{U_\Lambda(m,n)}{\sqrt[M]{mn}} > 0.$$

holds true for some $M > 1$. Therefore, the condition (7) is at least necessary for the functions Λ, β to be 1-saturate. It imposes a restriction from below on the order of growth of $U_\Lambda(m,n)$.

Theorem 3.1 *If the functions Λ, β are easily computable and 1-saturate, then the Problem* UNB(Λ,β) *is NP-complete.*

Proof: The fact that the problem lies in the class NP follows from the description of the solution set $\Xi_{\Lambda,\beta}$ given in Theorem 2.2 and Theorem 2.3. Indeed, it follows from the description that the intersection of $\Xi_{\Lambda,\beta}$ with any orthant may be defined as a solution set to the system of $2m + 2n + 1$ linear inequalities of $2n$ variables, whose solvability can be determined with the use of the polynomial ellipsoid algorithm for linear programming problem [6].

We demonstrate now that a classical NP-complete problem called Partition (see details in [2]) is polynomially reducible to our problem under study. The problem Partition can be formulated as follows:

> **Given.** Positive integer numbers $d_1, \ldots, d_s$, $s > 1$.
> **Question.** Is there a sequence of signs $\varepsilon_l, \ldots, \varepsilon_s \in \{-1, 1\}$ such that $\varepsilon_l d_1 + \ldots + \varepsilon_s d_s = 0$.

Let us reduce this problem to the problem UNB(Λ,β). For given positive integer numbers $d_1,\ldots,d_s$, we choose $m = m_{s+2}$, $n = n_{s+2}$, $k = k_{s+2}$, $l = l_{s+2}$ so that the conditions of Definition 3 (in which $(s+2)$ is substituted for s) are satisfied. Furthermore, without loss in generality, we can assume that the extended matrix $(\Lambda\beta)$ has the form (6).

We denote

$$l_i = \sum_{j=1}^{s} \lambda_{ij}^+$$

for $i = \overline{1,k}$ and

$$l_i = \sum_{j=s+1}^{s+l} \lambda_{ij}^+$$

for $i = \overline{k+1,k+s}$ and consider the following system composed of m_{s+2} interval equations of n_{s+2} variables:

$$\begin{cases} \sum_{j=1}^{s} [-\lambda_{ij}^+, \lambda_{ij}^+] x_j = l_i, & \text{for} \quad i = \overline{1,k}, \\ (l_i + \beta_i^+) x_{i-k} + \sum_{j=s+1}^{s+l} [-\lambda_{ij}^+, \lambda_{ij}^+] x_j = \beta_i^+, & \text{for} \quad i = \overline{k+1,k+s}, \\ x_{i-k} = 1, & \text{for} \quad i = \overline{k+s+1,k+s+l}, \\ x_1 d_1 + \ldots + x_s d_s + x_{s+l+1} - x_{s+l+2} = 0, & \text{for} \quad i = k+s+l+1, \\ x_{s+l+1} - x_{s+l+2} = 0, & \text{for} \quad i = k+s+l+2, \\ 0 = 0, & \text{for} \quad i > k+s+l+2. \end{cases} \tag{8}$$

From Theorem 2.2, we can conclude that a vector $x \in \mathbb{R}^{n_{s+2}}$ belongs to the solution set $\Xi_{\Lambda,\beta}$ of the system (8) if and only if it satisfies the following system of inequalities:

$$\begin{cases} l_i \le \sum\limits_{j=1}^{s} \lambda_{ij}^{+}|x_j|, & \text{for} \quad i = \overline{1,k}, \\ (l_i + \beta_i^{+})|x_{i-k}| \le \sum\limits_{j=s+1}^{s+l} \lambda_{ij}^{+}|x_j| + \beta_i^{+}, & \text{for} \quad i = \overline{k+1, k+s}, \\ |x_{i-k} - 1| \le 0, & \text{for} \quad i = \overline{k+s+1, k+s+l}, \\ |x_1 d_1 + \ldots + x_s d_s + x_{s+l+1} - x_{s+l+2}| \le 0, & \text{for} \quad i = k+s+l+1, \\ |x_{s+l+1} - x_{s+l+2}| \le 0, & \text{for} \quad i = k+s+l+2. \end{cases} \tag{9}$$

Next, we demonstrate that the vector $x = (x_1, \ldots, x_{n_{s+2}})^{\top} \in \mathbb{R}^{n_{s+2}}$ satisfies the system (9) if and only if

$$\begin{cases} x_{s+1} = \ldots = x_{s+l} = 1, \qquad x_{s+l+1} = x_{s+l+2}, \\ x_1, \ldots, x_s \in \{-1, 1\}, \\ x_1 d_1 + \ldots + x_s d_s = 0. \end{cases} \tag{10}$$

The fact that a vector $x \in \mathbb{R}^{n_{s+2}}$ satisfies the system (9) under the conditions (10) can be verified easily by straightforward substitution of x into the system.

To prove the converse implication, let us imagine that for $x \in \mathbb{R}^{n_{s+2}}$, the inequality (9) is satisfied. Then the first and last of the equalities (10) follow from the third, fourth and fifth inequalities of (9), respectively. From the first equality of (10), we obtain for $i = \overline{k+1, k+s}$

$$\sum_{j=s+1}^{s+l} \lambda_{ij}^{+}|x_j| = \sum_{j=s+1}^{s+l} \lambda_{ij}^{+} = l_i,$$

and then, from the second inequality of (9), we get that $(l_i + \beta_i^{+})|x_{i-k}| \le (l_i + \beta_i^{+})$. Since due to the condition 5 of Definition 3 it follows that $l_i + \beta_i^{+} \ge 1$ for $i = \overline{k+1, k+s}$, then

$$|x_i| \le 1 \qquad \text{for} \quad i = \overline{1,s}. \tag{11}$$

Let

$$k_j = \sum_{i=1}^{k} \lambda_{ij}^{+}$$

for $j = \overline{1,s}$. Notice that $\sum\limits_{j=1}^{s} k_j = \sum\limits_{i=1}^{k} l_i$ and, due to the condition 4 of Definition 3, $k_j \ge 1$, $j = \overline{1,s}$.

Further, by adding together all the first inequalities from (9), we obtain that

$$\sum_{j=1}^{s} k_j \; = \; \sum_{i=1}^{k} l_i \le \sum_{i=1}^{k}\sum_{j=1}^{s} \lambda_{ij}^{+}|x_j| \; = \; \sum_{j=1}^{s} \left(\sum_{i=1}^{k} \lambda_{ij}^{+} \right) |x_j| \; = \; \sum_{j=1}^{s} k_j |x_j|,$$

i.e.,

$$\sum_{j=1}^{s} k_j \le \sum_{j=1}^{s} k_j |x_j|.$$

It follows from the latter inequality and the inequality (11) that $|x_j| = 1$ for all $j = \overline{1, s}$. Hence, the second relation from (10) is valid.

Next, we can conclude that, for the system (8), the solution set $\Xi_{\Lambda,\beta}$ is unbounded if and only if there exists a solution to the problem Partition for the given $d_1, \dots, d_s$. Furthermore, due to the conditions of the theorem, the system (8) is constructed by applying $d_1, \dots, d_s$ times the algorithms which are polynomial with respect to the length of the input. Therefore, the problem Partition is polynomially reducible to the problem UNB(Λ, β). ∎

In the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ of the form (8) constructed in the proof of Theorem 3.1, the $m \times n$-matrices $\boldsymbol{A}$ and the interval right-hand side m-vectors $\boldsymbol{b}$ meet the additional requirement

$$\Lambda^- \circ \Delta = 0_{m,n}, \qquad \beta^- \circ \delta = 0_m,$$

where $0_{m,n}$ is zero $m \times n$-matrix, 0_m is zero m-vector. Hence, the equality $\Xi_{\Lambda,\beta}(\boldsymbol{A}, \boldsymbol{b}) = \Xi_{uni}(\boldsymbol{A}, \boldsymbol{b})$ holds true for them. consequently, it is possible to use the technique [16] to reduce the problem UNB(Λ, β) to the solution of interval linear systems with positive interval matrices.

More precisely, let us call the system (1)–(2) *strongly positive* if $\underline{A} > 0_{m,n}$ and $\underline{b} > 0_m$. Then the following statement holds.

Corollary If the functions Λ and β are easily computable and 1-saturate, then the problem UNB(Λ, β) for strongly positive interval systems is NP-complete.

Let us now show that if the number of (+1)'s in the matrix $\Lambda(m, n)$ is "not too large", then the problem UNB(Λ, β) is polynomially solvable.

Theorem 3.2 *If the functions Λ and β are easily computable, and the condition*

$$\limsup_{m,n \to \infty} \frac{U_\Lambda(m, n)}{\log_2(mn)} \le C$$

is satisfied for a fixed integer C, then there exist polynomial time algorithms that solve the problem UNB(Λ, β).

Proof: By Theorem 2.2 and Theorem 2.3, the problem UNB(Λ, β) is equivalent to the problem of checking the unboundedness of the solution set to the system

$$|\, A_c x - b_c| \le (\Lambda \circ \Delta)\, |x| + \beta \circ \delta. \tag{12}$$

By decomposing Λ into the positive and negative parts $\Lambda = \Lambda^+ - \Lambda^-$, inequality (12) can be rewritten in the form

$$|\, A_c x - b_c| + (\Lambda^- \circ \Delta)\, |x| \le (\Lambda^+ \circ \Delta)|x| + \beta \circ \delta. \tag{13}$$

Note that, since $U_\Lambda(m, n) \le C \log_2(mn)$, we have no more than $C \log_2(mn)$ nonzero coefficients at the components of the vector x in the right-hand side of this inequality. Since Λ is easily computable, we can find these components in polynomial time. Also, we assume that, up to a renumeration of the components, the right-hand side of (13) contains only the components $|x_1|, \dots, |x_k|$ with non-zero coefficients, where $k \le C \log_2(mn)$.

Next, it can be shown that the system (13) is equivalent to

$$\begin{cases} A_c x - b_c = u_1 - u_2, \\ x = v_1 - v_2, \\ u_1 \geq 0,\ u_2 \geq 0,\ v_1 \geq 0,\ v_2 \geq 0, \\ u_1 + u_2 + (\mathit{\Lambda}^- \circ \Delta)(v_1 + v_2) \ \leq (\mathit{\Lambda}^+ \circ \Delta)|x| + \beta \circ \delta. \end{cases} \tag{14}$$

If we fix the signs of the first k coordinates of the vector x in the system (14), it transforms into a system of linear equations and inequalities, while its solvability can be determined in polynomial time [6]. Therefore, to verify the unboundedness of (13), we only need to write all the possible distributions of the signs for the first k coordinates of the vector x (2^k totally), and to investigate the system of linear equations and inequalities for each one of them. Since the number of such systems is $2^k \leq 2^{C \log_2 mn} = (mn)^C$, we also can answer the question on unboundedness of the solution set (13) in polynomial time. Hence, the problem $\mathrm{UNB}(\mathit{\Lambda}, \beta)$ is polynomially solvable. ■

NP-hardness (intractability) of the problem of unboundedness recognition of the AE-solution sets means that in practice, in addition to exponentially complex recognition procedures, it is desirable to have, at our disposal, a range of efficient (easily computable) sufficient criteria. One of the possible simple ways to determine whether a solution set is bounded can be based on examination of the united solution set $\mathit{\Xi}_{uni}(\boldsymbol{A}, \boldsymbol{b})$, which is the widest of the AE-solution sets, i. e., $\mathit{\Xi}_{uni}(\boldsymbol{A}, \boldsymbol{b}) \supseteq \mathit{\Xi}_{\mathit{\Lambda}\beta}(\boldsymbol{A}, \boldsymbol{b})$ for any $\mathit{\Lambda}$ and β. In its turn, boundedness of the united solution sets can be recognized by examination of the rank of the interval matrix $\boldsymbol{A}$: if $\boldsymbol{A}$ has full column rank, then $\mathit{\Xi}_{uni}(\boldsymbol{A}, \boldsymbol{b})$ is bounded. See details in [33].

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