A Comparative Study of Different Order Relations of Intervals*

Samiran Karmakar
Department of Business Mathematics and Statistics,
St. Xavier's College, Kolkata, India
skmath.rnc@gmail.com

A. K. Bhunia
Department of Mathematics, The University of Burdwan, Burdwan, India
math_akbhunia@buruniv.ac.in

Abstract

The objective of this paper is to study the existing definitions of interval order relations for comparing intervals in the context of decision-making problems. First, a detailed survey of existing definitions is presented, along with the advantages and drawbacks of each. Then, a global comparison is performed, taking the best order relations from each group. Finally, a conclusion is drawn about the best order relations.

Keywords: Interval numbers, Interval order relations, Optimization
AMS subject classifications: 90-02

1 Introduction

Decision making involves selecting the best alternative in conflicting situations arising in the different sectors of our daily life. It is also essential to study the different fields of optimization theory, operational research (O.R.) and management science, etc. A decision depends on various factors, such as uncertainty in the future or risk. Decision making is classified according to the scale of certainty, that ranges from full certainty to full uncertainty; this scalle is called the degree of certainty. There are several types of decisions, such as, (i) decisions under certainty, (ii) decisions under risk, (iii) decisions under conflicting situations, (iv) decisions under uncertain conditions, and others. Again, a decision under uncertainty is categorized into several types. Here, we focus mostly on optimistic and the pessimistic decision making. For optimistic decision making, the decision maker selects the best alternative, ignoring the uncertainty, whereas in the pessimistic case, the decision maker selects the best alternative with less uncertainty.

*Submitted: May 28, 2009; Revised: September 18, 2010; Accepted:October 17, 2011.

By real numbers, we generally represent the crisp or deterministic parameters of a particular mathematical model. These mathematical models actually describe some exact and certain problems in a traditional way. However, there are many unpredictable real life situations, especially in the cases of engineering problems or different branches of operations research and management science, where it is very difficult to assume the parameters as fixed real numbers. In these situations, the decision maker has to take decisions under uncertainty, and some imprecise or inexact elements automatically come into the existence in those models. To handle those imprecise or uncertain parameters, generally, O.R. practitioners and management authorities use either stochastic or fuzzy approaches. In the stochastic approach, imprecise parameters are considered as random variables following some known probability distributions. On the other hand, in fuzzy approaches, the uncertainty is viewed either as a fuzzy set with appropriate membership function or as a fuzzy number. Also, in some cases, both approaches are applied to tackle the impreciseness. In these approaches, there arises a question regarding the choice of probability distributions or the selection of membership functions. It is actually a formidable task for a decision maker in an inexact environment. To overcome this difficulty, recently, some researchers have used intervals to specify imprecise parameters. In decision making problems, the order relation between intervals plays an important role in selecting the best alternative. During the last few decades, several researchers have proposed the definitions of order relations of intervals using different mathematical approaches. Those approaches were mainly developed to reformulate or to solve various interval oriented optimization problems. Since their primary aim was to develop better solution techniques, in many cases rigorous discussions about the corresponding interval ordering definition have not been given. The researchers finished the discussions after fulfilling their purpose. However, our main intention here is to analyze various facets of those definitions and to discuss its general applicability in diverse fields. During our discussion, it will be seen that unlike real numbers the ranking of intervals is not symmetric. As a result, in many cases, their definitions cannot differentiate two intervals in general, even though they can be applied efficiently to solve the prescribed models.

In this paper, we present a comparative study of the existing definitions of order relations between intervals, analyzing level of acceptability and shortcomings from different points of view. The order relations we analyze were proposed by Moore [12], Ishibuchi and Tanaka [3], Chanas and Kuchta [1], Kundu [9], Zhang [18], Sengupta and Pal [14], Levin [10], Sevastjanov and Róg [15], Hu and Wang [2], Mahato and Bhunia [11] and Kulpa [8]. These order relations are discussed in detail and are compared on a set of selected pairs of intervals. Finally, an encouraging conclusion is drawn in this regard.

The rest of this article is constructed as follows: In Section 3, the basic features of the considered order relations are given. In Section 2 and 4, the basics of intervals, interval arithmetic and different types of intervals have been depicted. The detailed group wise investigations of the considered order relations have been given spreading over the sections 5 to 8. In Section 9, we have done a comparative study among the selected interval ranking definitions. Finally, the conclusion has been given in Section 10.

2 Interval Numbers

Interval numbers are a generalization of real numbers and are subsets of $\mathbb{R}$, the set of real numbers. Just as real numbers have an associated arithmetic and mathematical analysis, interval numbers have a distinct interval arithmetic and interval analysis [5, 4, 12, 13]. A new order relation and its application to uncertain data has been given in [2]. Kulpa [6, 7] described the space of intervals diagrammatically, and, applying this concept, he explained not only the interval arithmetic but also introduced a new paradigm for studying the interval relations. However, the ranking properties of intervals are not the same as real numbers.

Normally, an interval number A is defined as

$$A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in \mathbb{R}\}\,.$$

Here, $a_L, a_R \in \mathbb{R}$ are the lower and upper bounds of the interval A, respectively. Every real number $a \in \mathbb{R}$ is expressed as interval number $[a, a]$ with zero width. Alternatively, an interval number can also be expressed by its centre and radius. In this form, an interval number $A = [a_L, a_R]$ is denoted by $\langle a_C, a_W\rangle$, where $a_C = \frac{a_L+a_R}{2}$ and $a_W = \frac{a_R-a_L}{2}$ are known as the centre and radius of the interval, respectively.

3 Basic Features of the Order Relations Considered

The problem treated here deals with ordering of interval numbers in solving decision making problems. A number of ranking definitions prescribed in the literature are reviewed and tested on a set of selected pairs of intervals. A detailed analysis of the ranking definitions reveals that they were based on different mathematical backgrounds. Some of them were defined directly, i.e., just by observing we can rank intervals directly, while for the rest of the ranking definitions, intervals can be ranked with the help of either probabilistic or fuzzy concepts or with some specific functions or indices.

Here, the main features of all the considered order relations are given in chronological order. After the development of interval numbers, when the experts were recognizing the necessity of ranking of intervals to develop a well-organized and self-dependent theory of intervals, Moore [12], the pioneer of this study, defined two transitive order relations. One of them is dependent just on the bounds of the intervals and the other is the extension of the set inclusion property of intervals. Highlighting the disadvantages of the Moore's [12] definitions, Ishibuchi and Tanaka [3] defined much more efficient ranking definitions in the context of decision makers' point of view. They also presented the definitions using the bounds as well as the centre and radius of the intervals. The ranking definitions for maximization and minimization problems were offered differently in this case. To generalize the works of [3], Chanas and Kuchta [1] introduced the t_0, t_1-cut of the intervals and redefined the order relations of [3] by the same. However, all these definitions mentioned earlier have a lack of ability to answer the question "How much larger is the interval, if it is greater than the other?" raised by Sengupta and Pal [14]. Kundu [9] introduced a probabilistic approach to define the interval ranking relation named as "fuzzy leftness relation". Zhang et al. [18] introduced a new approach using the possibility degree of intervals to compare intervals. They defined two types of possibility degree for the intervals. Sengupta and Pal [14] pointed out two different approaches for comparing any two intervals. In the first approach, providing the definition of "Acceptability index" or "Value Judgment index", they compared any two intervals in the context of an optimistic decision maker's point of view, whereas in another approach, they suggested the "fuzzy preference ordering" between any two intervals in case of pessimistic decision making. However, their definitions fail in some cases. Levin [10] defined the order relations for intervals with the help of a remoteness function. He also used the generalization of some set theoretical and logical operations. However, the process is very complicated. Sevastjanov and Róg [15] also proposed the same using a probabilistic approach. In the last two approaches, the proposed definitions of the equality of two intervals are not generally applicable, unless $a_L = b_L$ and $a_R = b_R$. In the year 2006, another interval ranking definition was proposed by Hu and Wang [2] using the centre and radius of the intervals. Also, they pointed out the drawbacks of Kundu's [9] leftness relation and gave the modified fuzzy leftness relation. Almost at the same time, Mahato and Bhunia [11] proposed new definitions of order relations between two intervals to overcome the incompleteness of the definitions developed earlier. They defined the order relations "$\leq_{\text{omin}}$", "$<_{\text{omin}}$" and "$\geq_{\text{omax}}$", "$>_{\text{omax}}$" for optimistic and "$\leq_{\text{pmin}}$", "$<_{\text{pmin}}$" and "$\geq_{\text{pmax}}$", "$>_{\text{pmax}}$" for pessimistic decision making. Diagrammatic tools for describing the space of intervals and the corresponding interval relations have been given by Kulpa [8]. He used different types of diagrams, viz., *M R diagrams*, *W diagrams* etc. to serve the said purpose. The well known set inclusion relations ($\subseteq$ and $\supseteq$) and precedence relations ($\preceq$ and $\succeq$) were defined and represented diagrammatically as ordering relations on intervals. In this connection, he also defined and explained graphically the two closely related terms, viz., *in-between interval relation* and *Lozenge*.

4 Order Relations of Intervals

In contrast to real numbers, it is not straightforward to define a total order relation for intervals. As a result, researchers have defined order relations in different ways. Most of these definitions cannot specify the order relations properly for completely overlapping intervals. To put this into focus, we shall present a comparative study of existing order relations. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be a pair of arbitrary intervals. These can be classified as follows:

Type I: Non-overlapping intervals;

Type II: Partially overlapping intervals;

Type III: Completely overlapping intervals.

These three types of intervals are shown in Figure 1 for different situations.

From the existing literature, it is observed that several researchers have developed the definitions of order relations either based on set properties, or fuzzy applications, or probabilistic approaches, or value based approaches, or depending upon some specific indices or functions. For the sake of convenience, these definitions are divided into several groups, as follows:

Group 1: General definitions of interval ranking;

Group 2: Definitions depending upon some specific indices/functions;

Group 3: Interval ranking depending on probabilistic or fuzzy concepts;

Group 4: Diagrammatic representation of interval ranking.

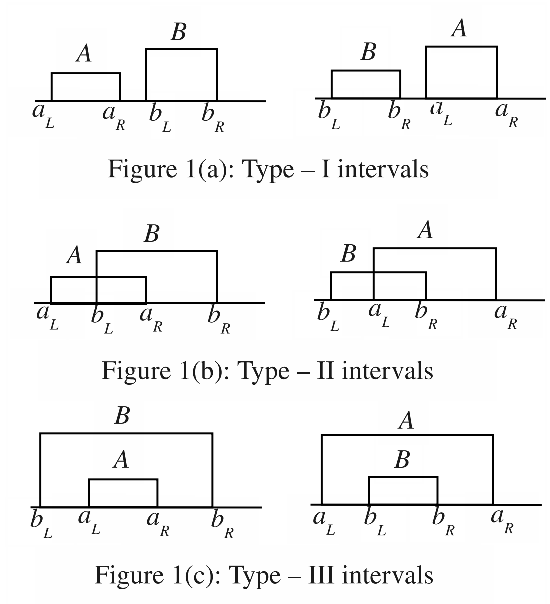

Figure 1: Different types of intervals

5 General Interval Ranking Definitions

Some basic interval ranking definitions were prescribed using the general concept of intervals. These were defined by means of upper bound, lower bound, centre and radius of the intervals.

5.1 Moore's Approach

Moore [12] is the pioneer of this study. He first gave two transitive order relations between two intervals A and B as follows:

(i) $A < B$ iff $a_R < b_L$,
(ii) $A \subseteq B$ iff $b_L \leq a_L$ and $a_R \leq b_R$.

The second definition is known as the "set inclusion property" of the intervals. Moore also defined the equality of two intervals as

$$A = B \quad \text{iff} \quad a_L = b_L \ \text{ and } \ a_R = b_R.$$

Clearly, the first transitive order relation "$<$" is applicable only for Type - I intervals. This order relation is not a partial order. The second relation is the generalization of the definition of subsets for intervals. According to Sengupta and Pal [14], the second one describes only the condition that the interval A is nested in B, but it cannot make a value-based ordering of intervals. Nonetheless, this order relation is a partial order, since the traditional set operation "$\subseteq$" is a partial order.

5.2 Ishibuchi and Tanaka's Approach

It has been observed that the definitions given in [12] are not applicable for all pairs of intervals. Now we shall establish definitions for those intervals by considering different scenarios:

Example 5.2.1 *Let $A = [1, 6]$, $B = [2, 3]$ and $C = [4, 5]$ be three intervals. Now there is a question: which is the greater interval between A and B or between A and C? Moore's [12] definition does not answer this question. If A, B and C represent interval profits, then which interval represents the maximum profitable interval between A and C? We need further analysis to get satisfactory answers to these questions.*

In connection with the study of mathematical programming problems with interval coefficients, Ishibuchi and Tanaka [3] proposed some improved definitions of interval order relations in comparison to that defined in [12]. These order relations represent the decision makers' preference between intervals. They proposed ranking definitions separately for maximization and minimization problems. In these definitions, they used not only the lower and upper bound form of the intervals, but also the centre and radius form. The different definitions of Ishibuchi and Tanaka [3] are as follows:

5.2.1 Order relations for maximization problems

Definition 5.2.1 *If $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two interval profits, then the order relation $\leq_{LR}$ for maximization problems is defined as*

$$A \leq_{LR} B \quad \text{iff} \quad a_L \leq b_L \ \text{ and } \ a_R \leq b_R,$$

$$A <_{LR} B \quad \text{iff} \quad A \leq_{LR} B \ \text{ and } \ A \neq B.$$

Definition 5.2.2 *Let $A = \langle a_C, a_W \rangle$ and $B = \langle b_C, b_W \rangle$ be two intervals in centre and radius form, then the order relation $\leq_{CW}$ for maximization problems is defined as*

$$A \leq_{CW} B \quad \text{iff} \quad a_C \leq b_C \ \text{ and } \ a_W \geq b_W,$$

$$A <_{CW} B \quad \text{iff} \quad A \leq_{CW} B \ \text{ and } \ A \neq B.$$

Obviously, this order relation is a partial order.

5.2.2 Order relations for minimization problems

Definition 5.2.3 *If $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two interval costs/times then the order relation $\leq^*_{LR}$ is defined as*

$$A \leq^*_{LR} B \quad \text{iff} \quad a_L \leq b_L \ \text{ and } \ a_R \leq b_R,$$

$$A <^*_{LR} B \quad \text{iff} \quad A \leq^*_{LR} B \ \text{ and } \ A \neq B.$$

Clearly, the order relations $\leq^*_{\mathrm{LR}}$ and $\leq_{\mathrm{LR}}$ are the same.

Definition 5.2.4 *Another order relation $\leq^*_{LR}$ between the intervals $A = \langle a_C, a_W \rangle$ and $B = \langle b_C, b_W \rangle$ is as follows:*

$$A \leq^*_{CW} B \quad \text{iff} \quad a_C \leq b_C \ \text{ and } \ a_W \leq b_W,$$

$$A <^*_{CW} B \quad \text{iff} \quad A \leq^*_{CW} B \ \text{ and } \ A \neq B.$$

Obviously, this order relation is also a partial order.

Ishibuchi and Tanaka considered the centre and radius of the intervals as the expected value and the uncertainty of an imprecise parameter, respectively. These order relations select the alternatives having optimal expected value with less uncertainty. Hence, these represent the decision makers' preference in solving the optimization problems.

To convert linear programming problems involving objective functions with interval coefficients into bi-criterial linear programming problems, Ishibuchi and Tanaka defined two more order relations as follows.

Definition 5.2.5 *For maximization problems, the order relation* $\leq_{LC}$ *is defined by*

$$A \leq_{LC} B \quad \text{iff} \quad a_L \leq b_L \quad \text{and} \quad a_C \leq b_C,$$

$$A <_{LC} B \quad \text{iff} \quad A \leq_{LC} B \quad \text{and} \quad A \neq B.$$

Definition 5.2.6 *For minimization problems, the order relation* $\leq^*_{RC}$ *is defined by*

$$A \leq^*_{RC} B \quad \text{iff} \quad a_R \leq b_R \quad \text{and} \quad a_C \leq b_C,$$

$$A \leq^*_{RC} B \quad \text{iff} \quad A \leq^*_{RC} B \quad \text{and} \quad A \neq B.$$

Ishibuchi and Tanaka applied their definitions only to reformulate the aforementioned optimization problems in an alternative way, but their definitions can also be used to rank arbitrary intervals. For Type - I and Type - II intervals, the definitions hold very well. However, for Type - III intervals, these can only be applied partially. To see this, consider the following example.

Example 5.2.2 *Consider the Type - III pair of intervals* $A = [0, 10] = \langle 5, 5\rangle$ *and* $B = [4, 8] = \langle 6, 2\rangle$. *These are shown in Figure 2. For maximization problems, the interval* B *is preferred to* A. *Now, if we consider the ordering of the intervals* A *and* B *for minimization problems, we observe that both the definitions* $\leq^*_{CW}$ *and* $\leq^*_{RC}$ *fail to find the preferred interval from* A *and* B.

Sengupta and Pal [14] noticed some noteworthy drawbacks of these definitions with respect to the decision makers' point of view. For the maximization case, B is higher valued interval than A, but the answer to the question: "How much higher is the interval B?" can not be obtained. According to them, Ishibuchi and Tanaka [3] gave more emphasis to strict preference ordering rather than ranking of intervals in terms of values. In another viewpoint, let us consider a constraint relation $Ax \leq B$, where A and B are any two intervals. Using the definitions of [3], it is not possible to exploit the decision variable x comprehensively.

5.3 Chanas and Kuchta's Approach

In the work of Ishibuchi and Tanaka [3], it is seen that linear programming models with interval valued objective function are converted to bi-objective optimization problems by means of the several interval ranking techniques developed by them. Chanas and Kuchta [1] tried to generalize the work in [3] with the notion of t_0, t_1 - cut of the intervals, so they would be able to solve the said optimization model by changing to its parametric generalization instead of a multiobjective problem. In this circumstance, they had to generalize the interval ranking definitions given in [3]. However, a detailed

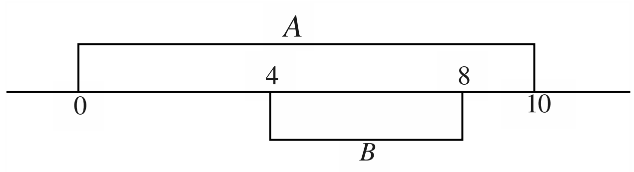

Figure 2: Ishibuchi and Tanaka's method partially applicable for this pair of intervals

analysis of the ranking was not given. It can be considered as the parametric generalization of the previous definitions. According to Chanas and Kuchta [1], the definition of the t_0, t_1 - cut of an interval is as follows:

Definition 5.3.1 *Let $A = [a_L, a_R]$ be any interval, t_0 and t_1 be any fixed numbers such that $0 \leq t_0 < t_1 \leq 1$. Then the t_0, t_1 - cut of the interval A, which is again an interval, is denoted by $A/_{[t_0,t_1]}$ and defined by*

$$A/_{[t_0,t_1]} = [a_L + t_0\,(a_R - a_L)\,, a_L + t_1\,(a_R - a_L)]\,.$$

Using this definition on intervals, Chanas and Kuchta modified the interval ranking definitions of [3]. For the maximization case, they considered definitions 5.2.1, 5.2.2 and 5.2.5 and redefined as follows:

$$\begin{array}{lllll} \text{(i)} & A \leq_{\mathrm{LR}}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} \leq_{\mathrm{LR}} B/_{[t_0,t_1]}, \\ & A <_{\mathrm{LR}}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} <_{\mathrm{LR}} B/_{[t_0,t_1]}, \\[1ex] \text{(ii)} & A \leq_{CW}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} \leq_{CW} B/_{[t_0,t_1]}, \\ & A <_{CW}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} <_{CW} B/_{[t_0,t_1]}, \\[1ex] \text{(iii)} & A \leq_{\mathrm{LC}}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} \leq_{\mathrm{LC}} B/_{[t_0,t_1]}, \\ & A <_{\mathrm{LC}}/_{[t_0,t_1]} B & \Leftrightarrow & A/_{[t_0,t_1]} <_{\mathrm{LC}} B/_{[t_0,t_1]}. \end{array}$$

One can similarly generalize definitions 5.2.3, 5.2.4 and 5.2.6 for minimization problems. For $t_0 = 0$ and $t_1 = 1$, definitions (i), (ii) and (iii) certainly lead to the order relations $\leq_{\mathrm{LR}}$, $\leq_{CW}$ and $\leq_{\mathrm{LC}}$ proposed in [3], respectively, and for $t_0 = 0$ and $t_1 = \frac{1}{2}$, definition (i) implies the order relation $\leq_{\mathrm{LC}}$.

Here, ranking decisions are taken by transforming into two new intervals with smaller widths using the t_0, t_1-cut of the interval. To illustrate the interval ranking definitions of Chanas and Kuchta [1], t_0, t_1-cuts of two given intervals are computed for different values of t_0 and t_1 and then ranking decisions have been taken. The results are displayed in Table 1, from which it is observed that for different pairs of t_0 and t_1, the ranking decisions between two intervals are different. This shows that the definitions of Chanas and Kuchta [1] do not give a concrete decision regarding the interval ranking between two arbitrary intervals.

5.4 Hu and Wang's Approach

A modified version of order relations for intervals has been proposed by Hu and Wang [2]. In their article, they have studied the incompleteness of interval ranking techniques

Table 1: Interval ranking depending on different t_0, t_1-cut values

$[t_0, t_1]$	$A/_{[t_0,t_1]}$	$B/_{[t_0,t_1]}$	Ranking
$[0,1]$	$A/_{[0,1]} = [10,30]$	$B/_{[0,1]} = [15,25]$	$A/_{[0,1]} \leq_{CW} B/_{[0,1]}$
$[.1,1]$	$A/_{[.1,1]} = [12,30]$	$B/_{[.1,1]} = [16,25]$	failed
$[.3,1]$	$A/_{[.3,1]} = [16,30]$	$B/_{[.3,1]} = [18,25]$	failed
$[.5,1]$	$A/_{[.5,1]} = [20,30]$	$B/_{[.5,1]} = [20,25]$	$B/_{[.5,1]} \leq_{LR} A/_{[.5,1]}$
$[.8,1]$	$A/_{[.8,1]} = [26,30]$	$B/_{[.8,1]} = [23,25]$	$B/_{[.8,1]} \leq_{LR} A/_{[.8,1]}$
$[.2,.8]$	$A/_{[.2,.8]} = [14,26]$	$B/_{[.2,.8]} = [17,23]$	$A/_{[.2,.8]} \leq_{CW} B/_{[.2,.8]}$
$[.2,.6]$	$A/_{[.2,.6]} = [14,22]$	$B/_{[.2,.6]} = [17,21]$	$A/_{[.2,.6]} \leq_{CW} B/_{[.2,.6]}$
$[.4,.9]$	$A/_{[.4,.9]} = [18,28]$	$B/_{[.4,.9]} = [19,24]$	failed
$[.5,.75]$	$A/_{[.5,.75]} = [20,25]$	$B/_{[.5,.75]} = [20,22.5]$	$B/_{[.5,.75]} \leq_{LR} A/_{[.5,.75]}$
$[.75,.85]$	$A/_{[.75,.85]} = [25,27]$	$B/_{[.75,.85]} = [22.5,23.5]$	$B/_{[.75,.85]} \leq_{LR} A/_{[.75,.85]}$

developed earlier. Introducing new approaches, they have tried to fulfill the shortcomings of the previous definitions. To introduce this, let us take a simple decision making situation regarding interval ranking.

Example 5.4.1 *Let $A = [1,5]$ and $B = [2,4]$ be two intervals with the same mid point. Which is the more acceptable interval between A and B? Most decision makers cannot make a decision regarding the best interval in this case. In this context, Hu and Wang's [2] definition is useful. The definitions of [3] can also be applied for this judgment.*

Hu and Wang also introduced some novel interval arithmetic operations and proved that their ranking definitions satisfy some basic properties (such as reflexivity, anti-symmetry, comparability etc.) with the help of the newly developed arithmetic operations. Their interval ranking relation "$\prec_=$" is defined as follows:

Definition 5.4.1 *For any two intervals $A = [a_L, a_R] = \langle a_C, a_W\rangle$ and $B = [b_L, b_R] = \langle b_C, b_W\rangle$,*

$$A \prec_= B \text{ iff } \begin{cases} a_C < b_C & \text{whenever } a_C \neq b_C \\ a_W \geq b_W & \text{whenever } a_C = b_C. \end{cases}$$

Furthermore

$$A \prec B \text{ iff } A \prec_= B \text{ and } A \neq B.$$

The centre and radius of the intervals are regarded as the expected value and the uncertainty of the parameters, respectively, as we have seen previously in [3]. So, whenever the centers of two intervals are the same, Hu and Wang [2] emphasized the radii of the intervals for ordering.

The relation "$\prec_=$" satisfies the following relational properties:

(i) $A \prec_= A$ for any interval A (*Reflexivity*)

(ii) $A \prec_= B$ and $B \prec_= A$ then $A = B$ for any two intervals A and B (*Anti-symmetry*)

(iii) $A \prec_= B$ and $B \prec_= C$ then $A \prec_= C$ for any three intervals A, B and C (*Transitivity*)

(iv) $A \prec_= B$ or $B \prec_= A$ holds for any two intervals A and B (*Comparability*)

Now, we present two examples to demonstrate the efficiency of the interval ranking definitions due to Hu and Wang [2].

Example 5.4.2 *Let* $A = [5, 15] = \langle 10, 5\rangle$ *and* $B = [8, 10] = \langle 9, 1\rangle$. *Since* $a_C = 10 > 9 = b_C$ *so* $B \prec_= A$.

Example 5.4.3 *Let* $A = [5, 15] = \langle 10, 5\rangle$ *and* $B = [8, 12] = \langle 10, 2\rangle$. *Here* $a_C = b_C = 10$ *but* $a_W = 5 > 2 = b_W$. *According to Hu and Wang's definition, the interval* B *is preferable to* A *for any type of optimization problem.*

5.5 Mahato and Bhunia's Approach

At the same time, Mahato and Bhunia [11] proposed another class of definitions of interval order relations which place more emphasis on the decision makers' preference. They first pointed out the incompleteness of the aforementioned interval ranking definitions with respect to the decision makers' point of view. To clarify, let us consider an example with a pair of intervals of Type-III:

Example 5.5.1 *Let* $A = [10, 50] = \langle 30, 20\rangle$ *and* $B = [25, 45] = \langle 35, 10\rangle$ *be two intervals representing the profits in the case of maximization problems and time/cost intervals in the case of minimization problems. It is obvious that an optimistic decision maker will always prefer the interval* A *to* B *for both maximization and minimization problems. However, the job is not so easy for a pessimistic decision maker. For maximization problems, pessimists may choose the interval* B *as a most profitable interval and for minimization problems, they select the lower cost/time interval* A.

Mahato and Bhunia [11] proposed a modified version of Ishibuchi and Tanaka's [3] interval ranking definition according to the decision makers' point of view. Only upper bound-lower bound form and centre-radius form are used to define this order relation.

Let $A = [a_L, a_R] = \langle a_C, a_W\rangle$ and $B = [b_L, b_R] = \langle b_C, b_W\rangle$ be two interval costs/times for minimization problems and interval profits for maximization problems.

5.5.1 Optimistic decision-making

Definition 5.5.1 *For minimization problems, the order relation* $\leq_{omin}$ *between the intervals* A *and* B *is*

$$A \leq_{omin} B \quad \text{iff} \quad a_L \leq b_L,$$
$$A <_{omin} B \quad \text{iff} \quad A \leq_{omin} B \quad \text{and} \quad A \neq B.$$

This implies that A is superior to B and A is accepted. This order relation is not symmetric.

Definition 5.5.2 *For maximization problems, the order relation* $\geq_{omax}$ *between the intervals* A *and* B *is*

$$A \geq_{omax} B \quad \text{iff} \quad a_R \geq b_R,$$
$$A >_{omax} B \quad \text{iff} \quad A \geq_{omax} B \quad \text{and} \quad A \neq B.$$

This implies that A is superior to B and optimistic decision makers accept the profit interval A. Here also, the order relation $\geq_{\mathrm{omax}}$ is not symmetric.

5.5.2 Pessimistic decision-making

In this case, the decision maker determines the minimum cost/time for minimization problems and the maximum profit for maximization problems according to the principle "Less uncertainty is better than more uncertainty".

Definition 5.5.3 *For minimization problems, the order relation $<_{pmin}$ between the intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ for a pessimistic decision maker are*

(i) $A <_{pmin} B$ iff $a_C < b_C$, for Type - I and Type - II intervals,

(ii) $A <_{pmin} B$ iff $a_C \le b_C$ and $a_W < b_W$, for Type - III intervals.

However, for Type - III intervals with $a_C < b_C$ and $a_W > b_W$, pessimistic decisions cannot be determined. In this case, the optimistic decision is to be considered.

Definition 5.5.4 *For maximization problems, the order relation $>_{pmax}$ between the intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ for a pessimistic decision maker are*

(i) $A >_{pmax} B$ iff $a_C > b_C$, for Type - I and Type - II intervals,

(ii) $A >_{pmax} B$ iff $a_C \ge b_C$ and $a_W < b_W$, for Type - III intervals.

However, for Type - III intervals with $a_C > b_C$ and $a_W > b_W$, a pessimistic decision cannot be taken. In this case, the optimistic decision is to be taken.

5.6 Comparative Examples

The comparison has been done among the interval ranking approaches for the first group of definitions. A set of 10 pairs of intervals has been chosen. Using these definitions, we have tried to find the ranking of those pair of intervals; the results are shown in Tables 2 and 3. Here, both Tables 2 and 3 shows the ranking for maximization problems. The illustrative examples (Examples 1 to 10) are taken from Type - I, Type - II and Type - III intervals and are of different degrees of complexity.

The simplest interval ranking definitions given by Moore [12] do not specify the ranking of intervals except for one or two simple and non-questionable examples. The first transitive order relation "$<$" can be applied only for Example 1, i.e., for Type - I intervals. Another order relation, viz., "Set inclusion property" can be applied for Examples 4 to 10, but no decision regarding the ordering of intervals can be drawn. For Examples 2 and 3, Moore's definitions [12] are not applicable.

Ishibuchi and Tanaka's [3] definitions are more widely applicable than those of Moore [12]. Ishibuchi and Tanaka's definitions are applicable (partially or fully) for almost all the examples. For Examples 1 to 4 and for Example 10 (i.e., for Type - I and Type - II intervals), the order relations "$<_{\mathrm{LR}}$" and "$\le_{\mathrm{LR}}$" (or "$<^*_{\mathrm{LR}}$" and "$\le^*_{\mathrm{LR}}$" for minimization problems) are fully applicable and directly find the optimum interval. However, for Examples 5 to 9 (i.e., for Type - III intervals), the order relations "$<_{CW}$" and "$\le_{CW}$" are only partially applicable. For Examples 5 and 6, the optimal interval can be chosen using "$\le_{CW}$" for maximization problems, whereas for Examples 8 and 9, the minimum interval can be chosen using the order relation "$\le^*_{CW}$". Ishibuchi and Tanaka's definitions are applicable either for maximization or minimization problems, but are applicable for both types of problems on Example 7.

Table 2: Comparative studies for maximization problems

Ex.	Intervals	Moore [12]	Ishibuchi & Tanaka [3]	Hu & Wang [2]	Mahato & Bhunia (Pessimistic Case) [11]
1	$A = [10, 20] = \langle 15, 5\rangle$, $B = [21, 23] = \langle 22, 1\rangle$.	$A < B$, B is accepted.	$A <_{\mathrm{LR}} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\mathrm{pmax}} A$, B is accepted.
2	$A = [10, 20] = \langle 15, 5\rangle$, $B = [20, 22] = \langle 21, 1\rangle$.	Not Applicable.	$A <_{\mathrm{LR}} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\mathrm{pmax}} A$, B is accepted.
3	$A = [10, 20] = \langle 15, 5\rangle$, $B = [19, 21] = \langle 20, 1\rangle$.	Not Applicable.	$A <_{\mathrm{LR}} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\mathrm{pmax}} A$, B is accepted.
4	$A = [10, 20] = \langle 15, 5\rangle$, $B = [18, 20] = \langle 19, 1\rangle$.	$B \subseteq A$, decision cannot be taken.	$A <_{\mathrm{LR}} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\mathrm{pmax}} A$, B is accepted.
5	$A = [10, 20] = \langle 15, 5\rangle$, $B = [17, 19] = \langle 18, 1\rangle$.	$B \subseteq A$, decision cannot be taken.	$A <_{CW} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\mathrm{pmax}} A$, B is accepted.

From the comparison tables, it is seen that Hu and Wang's [2] definitions can successfully be applied to all the examples.

The interval ranking definitions due to Mahato and Bhunia [11] are also applicable for all the types of examples considered. In our comparative study, we have considered the order relation $<_{\mathrm{pmax}}$ with respect to pessimistic decision makers' point of view. For Examples 1 to 7, the definition $<_{\mathrm{pmax}}$ finds the ranking of intervals; however, for Examples 8 to 10, the order relation $<_{\mathrm{pmax}}$ fails to find the same. In these cases, Mahato and Bhunia proposed to use the order relation from an optimistic decision maker's point of view. A similar situation occurs if we consider the ranking of intervals for finding the interval for minimization problems. From the above consideration, it can be concluded that the ranking definitions due to Hu and Wang [2] and Mahato and Bhunia [11] are equally widely applicable.

6 Definitions Depending on Specific Indices or Functions

The ranking definitions discussed to this point depend only either on the values of upper-lower bound or centre-radius form of the intervals. On the other hand, several researchers have prescribed ordering definitions that depend on some particular indices or specified functions. Now we shall discuss those definitions.

Table 3: Comparative studies for maximization problems

Ex.	Intervals	Moore [12]	Ishibuchi & Tanaka [3]	Hu & Wang [2]	Mahato & Bhunia (Pessimistic Case) [11]
6	$A = [10, 20] = \langle 15, 5\rangle$, $B = [15, 17] = \langle 16, 1\rangle$.	$B \subseteq A$, decision cannot be taken.	$A <_{CW} B$, B is accepted.	$A \prec B$, B is accepted.	$B >_{\text{pmax}} A$, B is accepted.
7	$A = [10, 20] = \langle 15, 5\rangle$, $B = [14, 16] = \langle 15, 1\rangle$.	$B \subseteq A$, decision cannot be taken.	$a_C = b_C$ and $a_W > b_W$. So, $A \leq_{CW} B$ is satisfied and B is chosen.	$a_C = b_C$ but $a_W > b_W$. So, $A \prec B$, B is accepted.	$B >_{\text{pmax}} A$, B is accepted.
8	$A = [10, 20] = \langle 15, 5\rangle$, $B = [12, 16] = \langle 14, 2\rangle$.	$B \subseteq A$, decision cannot be taken.	$\leq_{CW}$ failed. $B \leq^*_{CW} A$ is satisfied for minimization problems.	$B \prec A$, A is accepted.	$>_{\text{pmax}}$ fails, but $A >_{\text{omax}} B$ and A is accepted.
9	$A = [10, 20] = \langle 15, 5\rangle$, $B = [11, 14] = \langle 12.5, 1.5\rangle$.	$B \subseteq A$, decision cannot be taken.	$\leq_{CW}$ failed. $B \leq^*_{CW} A$ is satisfied for minimization problems.	$B \prec A$, A is accepted.	$>_{\text{pmax}}$ fails, but $A >_{\text{omax}} B$ and A is accepted.
10	$A = [10, 20] = \langle 15, 5\rangle$, $B = [10, 12] = \langle 11, 1\rangle$.	$B \subseteq A$, decision cannot be taken.	$B <_{\text{LR}} A$, A is accepted.	$B \prec A$, A is accepted.	$>_{\text{pmax}}$ fails, but $A >_{\text{omax}} B$ and A is accepted.

6.1 Sengupta and Pal's Approach

Two valuable interval ranking approaches have been developed by Sengupta and Pal [14]. They proposed two different ranking definitions — the first is the "acceptability index" or the "value judgment index" and the second is the "fuzzy preference ordering". Both the methods depend on the decision maker's point of view. "Fuzzy preference ordering" will be discussed later. The acceptability index is defined in terms of values using the centre and radius form of the intervals.

Definition 6.1.1 *Let $\mathbb{IR}$ be the set of all closed intervals on the real line $\mathbb{R}$. The acceptability function $\mathscr{A} : \mathbb{IR} \times \mathbb{IR} \to [0, \infty)$ for the intervals $A, B \in \mathbb{IR}$ with $a_C \leq b_C$ is denoted by $\mathscr{A}(A, B)$ or $\mathscr{A}_{<}(A, B)$ and is defined by*

$$\mathscr{A}(A, B) = \frac{b_C - a_C}{b_W + a_W}, \ \ b_W + a_W \neq 0 \text{ and } b_C \geq a_C.$$

$\mathscr{A}(A, B)$ is considered as the grade of acceptability of "the interval A to be inferior to the interval B". Here, the terms "inferior" and "superior" are equivalent to the terms "smaller" and "greater" respectively.

The values of the grade of acceptability $\mathscr{A}(A, B)$ of the intervals A and B are given by

$$\mathscr{A}(A, B) \begin{cases} = 0 & & if \quad a_C = b_C, & \\ = m & (0 < m < 1) & if \quad a_C < b_C & \text{and} \quad a_R > b_L \\ \geq 1 & & if \quad a_C < b_C & \text{and} \quad a_R \leq b_L. \end{cases}$$

If $\mathscr{A}(A, B) \geq 1$, then for minimization problems the interval A and for maximization problems the interval B is accepted with full satisfaction. If $0 < \mathscr{A}(A, B) < 1$ then for minimization problems A is accepted (obviously B is accepted for maximization problems) with the grade of acceptability $\mathscr{A}(A, B)$. Again, if $\mathscr{A}(A, B) = 0$, then neither the interval A nor B is accepted. For this situation Sengupta and Pal [14] suggest that the less uncertain interval would be the better choice for any type of optimization problem.

To verify the applicability of the acceptability index, let us consider the intervals $A = [10, 30] = \langle 20, 10 \rangle$, $B = [12, 16] = \langle 14, 2 \rangle$, $C = [15, 25] = \langle 20, 5 \rangle$, $D = [24, 29] = \langle 26.5, 2.5 \rangle$ and $E = [32, 40] = \langle 36, 4 \rangle$. These intervals are represented geometrically in Figure 3.

Now, $\mathscr{A}(B, A) = 0.5$, i.e., $0 < \mathscr{A}(A, B) < 1$,
$\mathscr{A}(A, C) = \mathscr{A}(C, A) = 0$,
$\mathscr{A}(A, D) = 0.52$ i.e., $0 < \mathscr{A}(A, D) < 1$,
$\mathscr{A}(A, E) = 1.142857$,
$\mathscr{A}(C, D) = 0.8667$, i.e., $0 < \mathscr{A}(C, D) < 1$.

These results clarify the change of the values of the acceptability indices for different pairs of intervals with respect to the change of their relative positions. The decision maker here is provided with the grade of inferiority (or superiority), i.e., how much one interval is inferior (or superior) to the other. Here, the working principle of this method is quite compatible with our intuition. In addition, we can use this index method to exploit the decision variable x from the interval valued constraint $Ax \leq B$, where A and B are intervals, within the range of the intervals. This is extremely useful in solving interval valued constrained optimization problems.

However, this technique has some drawbacks. According to Sengupta and Pal [14], the acceptability index is only a value based ranking index, and it can only be

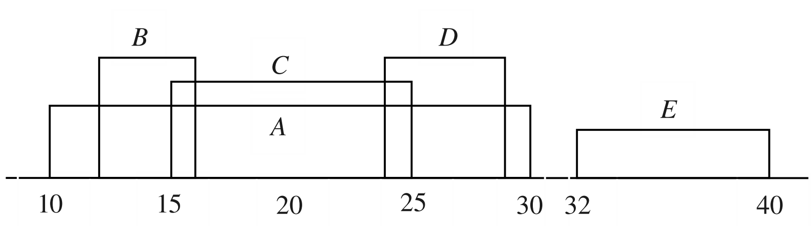

Figure 3: Different intervals

applied partially to select the better alternative from the pessimistic point of view of the decision maker. So, only the optimistic decision maker can use it completely.

6.2 Levin's Approach

A noteworthy interval ordering was given by Levin [10]. First, he defined a specific function known as "remoteness function", then he prescribed comparison rule for intervals.

Definition 6.2.1 *The* remoteness function $U(A,B)$ *of two intervals* A *and* B *is defined by the proximity of intervals or by the dual measure of their remoteness. Mathematically,*

$$U(A,B) = |A \setminus B| + |B \setminus A| + |P|.$$

The remoteness function represents the total length of all the subintervals that make A and B different from each other, including a subinterval P between A and B in cases where A and B do not partially or completely overlap. "$|P|$" represents the length of the interval P (or subinterval). Let us calculate the values of the remoteness function for different pairs of intervals.

Example 6.2.1 *If* $A = [2,8]$ *and* $B = [4,8]$ *(shown in Figure 4), then*

$$U(A,B) = |A \setminus B| + |B \setminus A| + |P| = 2 + 0 + 0 = 2.$$

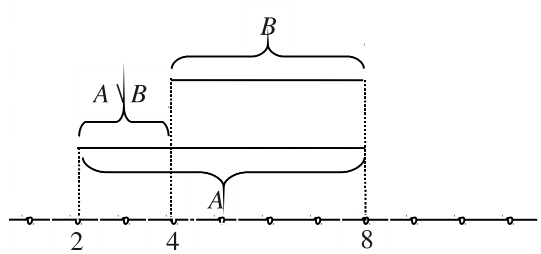

Figure 4: Intervals $A = [2,8]$ and $B = [4,8]$

Example 6.2.2 *If* $A = [2, 5]$ *and* $B = [2, 8]$ *(shown in Figure 5), then*

$$U(A, B) = |A \setminus B| + |B \setminus A| + |P| = 0 + 3 + 0 = 3.$$

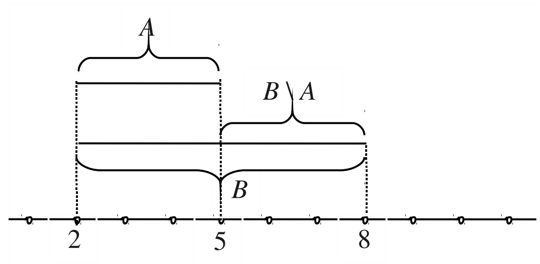

Figure 5: Intervals $A = [2, 5]$ and $B = [2, 8]$

Example 6.2.3 *If* $A = [2, 5]$ *and* $B = [3, 8]$ *(shown in Figure 6), then*

$$U(A, B) = |A \setminus B| + |B \setminus A| + |P| = 1 + 3 + 0 = 4.$$

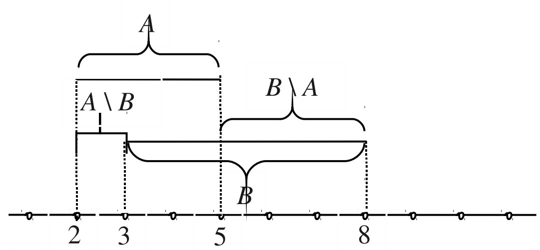

Figure 6: Intervals $A = [2, 5]$ and $B = [3, 8]$

Example 6.2.4 *If* $A = [2, 5]$ *and* $B = [8, 10]$ *(shown in Figure 7), then*

$$U(A, B) = |A \setminus B| + |B \setminus A| + |P| = 3 + 2 + 3 = 8.$$

Example 6.2.5 *If* $A = [2, 10]$ *and* $B = [4, 7]$ *(shown in Figure 8), then*

$$U(A, B) = |A \setminus B| + |B \setminus A| + |P| = (2 + 3) + 0 + 0 = 5.$$

The concepts of colloquial logical and set theoretical operations, viz., disjunction ($\vee$) and conjunction ($\wedge$) operations also appear, and are defined as

$$A \vee B = [a_L, a_R] \vee [b_L, b_R] = [a_L \vee b_L, a_R \vee b_R];$$

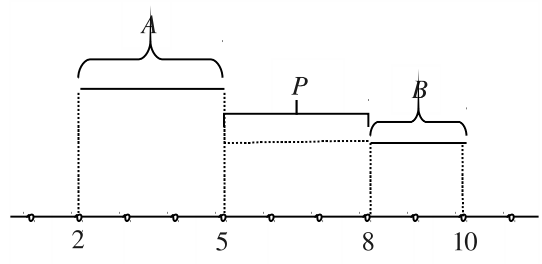

Figure 7: Intervals $A = [2, 5]$ and $B = [8, 10]$

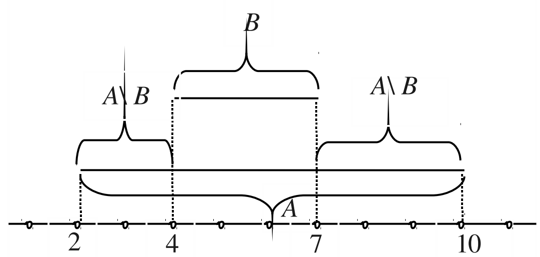

Figure 8: Intervals $A = [2, 10]$ and $B = [4, 7]$

$$A \wedge B = [a_L, a_R] \wedge [b_L, b_R] = [a_L \wedge b_L, a_R \wedge b_R],$$

where $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two arbitrary intervals. The operations $\vee = \max$ and $\wedge = \min$ over the intervals A and B were defined as the set theoretical generalizations. Using the remoteness function, the comparability relations "$>$","$\geq$" and "$=$" for arbitrary intervals as defined by Levin are given as follows:

$$A \geq B \Leftrightarrow [U(A, A \vee B) \leq U(B, A \vee B), U(A, A \wedge B) \geq U(B, A \wedge B)],$$

$$A > B \Leftrightarrow [U(A, A \vee B) < U(B, A \vee B), U(A, A \wedge B) > U(B, A \wedge B)],$$

$$A = B \Leftrightarrow [U(A, A \vee B) = U(B, A \vee B), U(A, A \wedge B) = U(B, A \wedge B)].$$

Now we illustrate how the remoteness function oriented interval ranking procedure works with the following examples.

Example 6.2.6 *Let us consider the previous example, 6.2.3, where* $A = [2, 5]$ *and* $B = [3, 8]$. *We first calculate*

$A \vee B = [2, 5] \vee [3, 8] = [3, 8]$;
$A \wedge B = [2, 5] \wedge [3, 8] = [2, 5]$;
$U(A, A \vee B) = 4; U(B, A \vee B) = 0$;
$U(A, A \wedge B) = 0; U(B, A \wedge B) = 4$;
$\Rightarrow U(A, A \vee B) > U(B, A \vee B)$ *and* $U(A, A \wedge B) < U(B, A \wedge B)$.
$\Rightarrow B > A$, *so* B *is greater than* A.

Table 4: Comparative studies for maximization problems

Ex.	Intervals	Sengupta and Pal's Acceptability Index [14]	Levin [10]
1	$A = [10, 20] = \langle 15, 5\rangle$, $B = [21, 23] = \langle 22, 1\rangle$.	$\mathscr{A}(A, B) = 1.1667 > 1$, B is accepted with full satisfaction.	$B > A$, B is accepted.
2	$A = [10, 20] = \langle 15, 5\rangle$, $B = [20, 22] = \langle 21, 1\rangle$.	$\mathscr{A}(A, B) = 1$, B is accepted with full satisfaction.	$B > A$, B is accepted.
3	$A = [10, 20] = \langle 15, 5\rangle$, $B = [19, 21] = \langle 20, 1\rangle$.	$\mathscr{A}(A, B) = 0.8267$, B is accepted with grade of acceptability 0.8267.	$B > A$, B is accepted.
4	$A = [10, 20] = \langle 15, 5\rangle$, $B = [18, 20] = \langle 19, 1\rangle$.	$\mathscr{A}(A, B) = 0.66$, B is accepted with grade of acceptability 0.66.	$B > A$, B is accepted.
5	$A = [10, 20] = \langle 15, 5\rangle$, $B = [17, 19] = \langle 18, 1\rangle$.	$\mathscr{A}(A, B) = 0.5$, B is accepted with grade of acceptability 0.5.	$B > A$, B is accepted.

In addition, Levin [10] gave another set of order relations between two intervals using the centre and radius form:

$$A \geq B \Leftrightarrow a_C \geq b_C,$$
$$A \leq B \Leftrightarrow a_C \leq b_C,$$
$$A = B \Leftrightarrow a_C = b_C.$$

However, this definition is restricted to intervals with different centre, and not applicable to intervals with the same centre. Let us consider the following example:

Example 6.2.7 *Let $A = [0, 10] = \langle 5, 5\rangle$ and $B = [4, 6] = \langle 5, 1\rangle$ be two intervals. The consecutive steps of Levin's [10] interval ranking process are as follows:*

$A \vee B = [0, 10] \vee [4, 6] = [4, 10]$;

$A \wedge B = [0, 10] \wedge [4, 6] = [0, 6]$;

$U(A, A \vee B) = 4; U(B, A \vee B) = 4$;

$U(A, A \wedge B) = 4; U(B, A \wedge B) = 4$.

Clearly, $U(A, A \vee B) = U(B, A \vee B)$ and $U(A, A \wedge B) = U(B, A \wedge B)$. Hence, Levin's [10] remoteness function gives $A = B$, which is not true in this case. It is thus concluded that Levin's [10] approach is not a complete one, in general.

6.3 Comparative Examples

Two interval ranking definitions belong to this group. As with the first group, a comparative study has been made, taking the same set of 10 pairs of intervals as considered in the first group. Here, the decision will be taken in the context of maximization problems. A summary appears in Tables 4 and 5.

Table 5: Comparative studies for maximization problems

Ex.	Intervals	Sengupta and Pal's Acceptability Index [14]	Levin [10]
6	$A = [10, 20] = \langle 15, 5\rangle$, $B = [15, 17] = \langle 16, 1\rangle$.	$\mathscr{A}(A, B) = 0.1667 > 0$, B is accepted with grade of acceptability 0.1667.	$B > A$, B is accepted.
7	$A = [10, 20] = \langle 15, 5\rangle$, $B = [14, 16] = \langle 15, 1\rangle$.	$\mathscr{A}(A, B) = 0$. Here the interval with less uncertainty is chosen.	$B = A$, Method fails to find the ranking.
8	$A = [10, 20] = \langle 15, 5\rangle$, $B = [12, 16] = \langle 14, 2\rangle$.	$\mathscr{A}(B, A) = 0.14286$, A is accepted.	$B < A$, A is accepted.
9	$A = [10, 20] = \langle 15, 5\rangle$, $B = [11, 14] = \langle 12.5, 1.5\rangle$.	$\mathscr{A}(B, A) = 0.3846$, A is accepted with grade of acceptability 0.3846.	$B < A$, A is accepted.
10	$A = [10, 20] = \langle 15, 5\rangle$, $B = [10, 12] = \langle 11, 1\rangle$.	$\mathscr{A}(B, A) = 0.667$, A is accepted with grade of acceptability 0.667.	$B < A$, A is accepted.

Sengupta and Pal's [14] acceptability index is an efficient interval ordering technique. It ranks the intervals depending upon the values of the acceptability function $\mathscr{A}(B, A)$. For comparatively simple examples (Examples 1 and 2), the value of the acceptability function is greater than or equal to 1, and the optimum interval is selected with full satisfaction. For the rest of the examples, which are of Type - III intervals, the index value lies between 0 and 1 (i.e., $0 < \mathscr{A}(B, A) < 1$), and the preference interval is selected with the grade of acceptability $\mathscr{A}(B, A)$. A greater value of the acceptability function gives the preference intervals with a more satisfactory level. However, when the value of the acceptability function is 0 (for Example 7), the optimum interval is selected depending upon the width of the intervals. Levin's [10] method is a very interesting interval ranking technique, but the process is too complicated. Almost all the examples are tackled very efficiently by this definition. However, for Example 7, this definition fails to find a ranking. It is seen that for the intervals with same centre the definitions are facing a little problem. However, Sengupta and Pal [14] manage the situation by choosing the less uncertain intervals as optimal. Hence, Sengupta and Pal's [14] acceptability index is more generally applicable than Levin's [10].

7 Ordering Depending on Probabilistic Concepts

Now, we shall discuss some ranking definitions depending on probabilistic measurements. Sometimes these definitions are very effective for ranking of intervals.

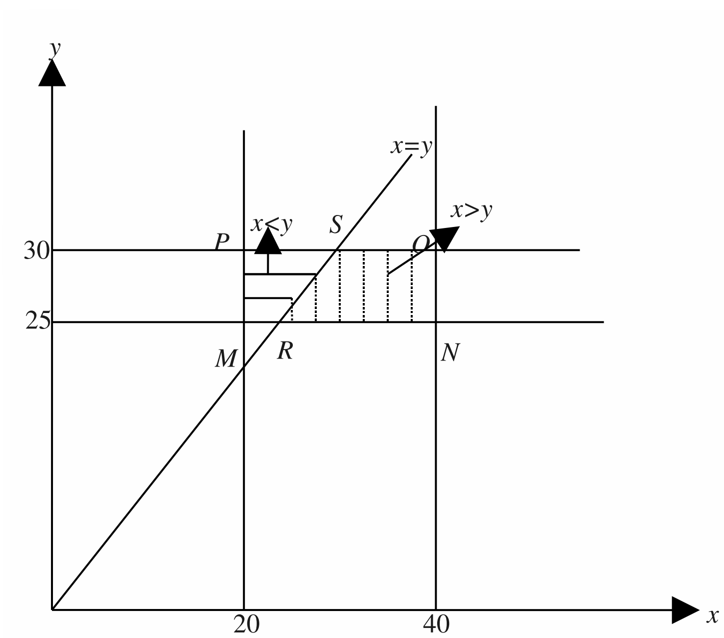

Figure 9: Graphical representation for Kundu's leftness relation

7.1 Kundu's Approach

The preceding definitions of interval order relations (except for Sengupta and Pal [14]) do not provide the answer to the question "How much larger is the interval is, if it is known to be larger than the other?" Kundu [9] introduced a probabilistic approach to define the order relations between two intervals. The answer to the above question can be extracted from this ranking definition. First of all, he defined the fuzzy leftness relation Left(A, B) between the interval A and B on the real line.

Definition 7.1.1 *For any two intervals A and B, Left(A, B) is defined by*

$$\textit{Left}(A, B) = \max\{0, P(x < y) - P(x > y)\},$$

where $P(x < y)$ denotes the probability that $x < y$ for $x \in A$, $y \in B$ uniformly and independently distributed in the intervals A and B respectively.

This preference relation is useful to select the most or the least preferred intervals from a set of alternatives. Left$(A, B) > 0$ gives the least preferred choice A. For most preferred choice, the relation was reformulated as

$$\text{Right}(A, B) = \max\{0, P(x > y) - P(x < y)\}.$$

Example 7.1.1 *Let us consider two intervals $A = [20, 40]$ and $B = [25, 30]$. Let $x \in A$ and $y \in B$ be independently and uniformly distributed. The graphical representation* *of the above two intervals is given in Figure 9. From the figure, it is clear that*

$$P(x < y) = \frac{Area\,MRSP}{Area\,MNOP} = \frac{37.5}{100} = 0.375,$$

$$P(x > y) = \frac{Area\,RNOS}{Area\,MNOP} = \frac{62.5}{100} = 0.625.$$

Hence,

$$\begin{aligned} Left(A,B) &= \max\{0, P(x<y) - P(x>y)\} \\ &= \max\{0, 0.375 - 0.625\} = 0. \end{aligned}$$

However,

$$\begin{aligned} Right(A,B) &= \max\{0, P(x>y) - P(x<y)\} \\ &= \max\{0, 0.625 - 0.375\} = 0.25 > 0. \end{aligned}$$

Thus, the results imply that the interval B is left to the interval A and A is the best choice for the maximization problem. This is quite reasonable. In addition, it is evident that we may get a measurement of largeness of one interval over the other.

Now, let us consider another interval $C = [25, 35]$ and try to find the best alternative between the intervals A and C applying the above probabilistic method. Then, we have the following results: $\text{Left}(A, C) = 0$ and $\text{Right}(A, C) = 0$. Hence, both A and C are optimal choices, which is inconsistent with a rational decision maker's goal. Therefore, this approach is also restricted to intervals with different centres.

7.2 Hu and Wang's Modified Leftness Relation

The deficiency of Kundu's [9] approach is clear from the preceding discussion. For the intervals with same mid point, the definition is unable to differentiate the intervals, i.e., the decision maker cannot find the preference intervals. In this perspective, a modified fuzzy leftness relation has been given by Hu and Wang [2] that can compensate for the previous drawback of Kundu's [9] method.The modified leftness relation is defined as follows:

Definition 7.2.1 *Let* $A = [a_L, a_R] = \langle a_C, a_W\rangle$ *and* $B = [b_L, b_R] = \langle b_C, b_W\rangle$ *be two intervals such that* $a_L \neq a_R$ *and* $b_L \neq b_R$*. Then*

$$Left_H(A, B) = \begin{cases} Left\,(A,\, B) & a_C \neq b_C \\ \max\left\{0,\ \dfrac{a_R - b_R}{2\max\{a_W,\, b_W\}}\right\} & a_C = b_C. \end{cases}$$

We have seen that Kundu's [9] definition fails to distinguish intervals with the same centres. However, this modified definition gives a slightly extended form of Kundu's, and makes it a complete fuzzy leftness relation. Hu and Wang also prove that this relation $\text{Left}_H(A, B)$ follows same basic properties, which are very important to solve interval oriented optimization problems or mathematical models.

Now, we shall explain the working principle of modified leftness relation with the help of the following example:

Example 7.2.1 *Let* $A = [4, 10] = \langle 7, 3\rangle$ *and* $B = [6, 8] = \langle 7, 1\rangle$*. Clearly,* $Left(A, B) = 0$*. But* $Left_H(A, B) = \max\left\{0, \frac{10-8}{2\max\{3,1\}}\right\} = \frac{1}{3} > 0$*. Hence, it is concluded that the interval A is left of B, and B is the most preferred interval in the case of maximization.*

7.3 Zhang's Approach

Another interval ranking method has been introduced by Zhang et al. [18] using the possibility degree of intervals. Like probability theory and fuzzy set theory, possibility theory is another technique to handle uncertainty. It uses a pair of dual set functions — possibility and necessity measures [16, 17]. Here, possibility degree means a certain degree that one interval is larger or smaller than the other. Zhang et al defined the possibility degrees $P_{A\geq B}$ and $P_{B\geq A}$ for the intervals A and B (for three different types of intervals) in the following way:

$$P_{B\geq A} = \begin{cases} 1 & \text{if} \quad b_L \geq a_R \\ \dfrac{b_R - a_R}{b_R - b_L} + \dfrac{a_R - b_L}{b_R - b_L}.\dfrac{b_L - a_L}{a_R - a_L} & \text{if} \quad a_L \leq b_L < a_R \leq b_R \\ \qquad +0.5\dfrac{a_R - b_L}{b_R - b_L}.\dfrac{a_R - b_L}{a_R - a_L} & \\ \dfrac{b_R - a_R}{b_R - b_L} + 0.5\dfrac{a_R - a_L}{b_R - b_L} & \text{if} \quad b_L < a_L < a_R \leq b_R \end{cases}$$

$$P_{A\geq B} = \begin{cases} 0 & \text{if} \quad b_L \geq a_R \\ 0.5\dfrac{a_R - b_L}{b_R - b_L}.\dfrac{a_R - b_L}{a_R - a_L} & \text{if} \quad a_L \leq b_L < a_R \leq b_R \\ \dfrac{a_L - b_L}{b_R - b_L} + 0.5\dfrac{a_R - a_L}{b_R - b_L} & \text{if} \quad b_L < a_L < a_R \leq b_R \end{cases}$$

Zhang et al. [18] considered the basic three types of intervals presented in Figure 1. The intervals A and B are regarded as random variables a and b with uniform distributions in their intervals. The possibility degrees $P_{A\geq B}$ and $P_{B\geq A}$ are equal to the respective probabilities that the random variable a is larger or smaller than b.

The following example illustrates the working principle of the possibility degree ranking.

Example 7.3.1 *Let us consider the three intervals as* $A = [10, 20], B = [16, 19]$ *and* $C = [11, 14]$. *Here, our objective is to rank the intervals A and B and the intervals A and C. For this purpose, we have to calculate* $P_{A\geq B}$, $P_{B\geq A}$ *and* $P_{A\geq C}$, $P_{C\geq A}$. *Now for A and B,* $P_{B\geq A} = 0.75$ *and* $P_{A\geq B} = 0.25$, *which imply that the interval B is larger than A. Again, for A and C,* $P_{C\geq A} = 0.75$ *and* $P_{A\geq C} = 0.25$, *which compel a decision maker to choose the interval C as better than A. Here, if we take another interval* $D = [14, 16]$, *then* $P_{D\geq A} = 0.5$ *and* $P_{A\geq D} = 0.5$. *It is obvious from these results that a decision concerning the superiority of the intervals A and D can not be made. Hence, this method is also not suitable for intervals with the same centre.*

7.4 Sengupta and Pal's Fuzzy Preference Ordering

Another ordering definition using fuzzy set theory and fuzzy logic was proposed by Sengupta and Pal [14] from the pessimistic decision maker's point of view. Risk averseness is the basic character of pessimistic decision makers, i.e., they take the decision with the principle "more uncertainty is worse than less uncertainty".

The class of intervals with $\mathscr{A}(A, B) \geq 0$ and $a_W < b_W$ is considered here. For intervals not belonging to this class, i.e., for intervals with $\mathscr{A}(A, B) \geq 0$ and $a_W \geq b_W$,

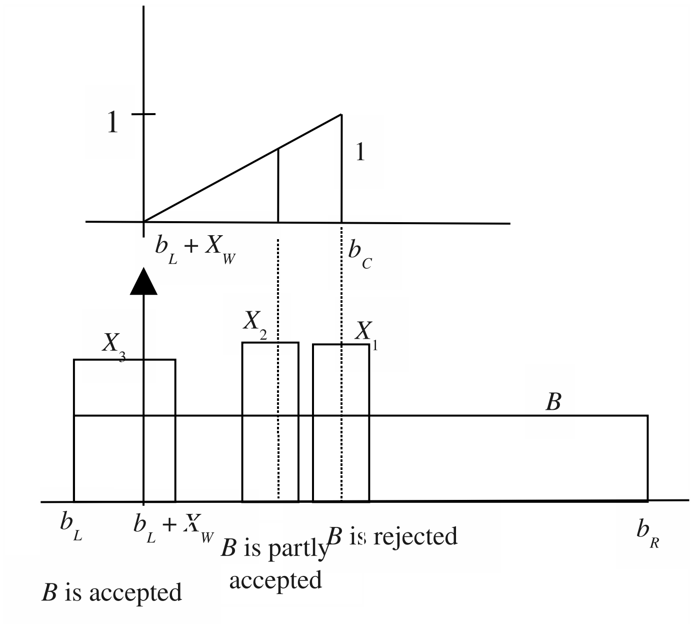

Figure 10: Representation of membership function

the choice of interval for any decision maker is straight forward. Thus, they considered only the former class of intervals. Among these intervals,

if $\mathscr{A}(A,B) \geq 1 \Rightarrow B$ is strictly preferred to A,
if $\mathscr{A}(A,B) = 0 \Rightarrow A$ is strictly preferred to B,
if $\mathscr{A}(A,B) \in]0,1[\Rightarrow \exists$ a fuzzy preference between A and B.

Definition 7.4.1 *Sengupta and Pal [14] defined the fuzzy set B^* as*

$$B^* = \text{Rejection of } B = \{(X,B) : \mathscr{A}(X,B) \geq 0 \text{ and } x_W < b_W\}$$

and the membership function $\mu_{B^*}(X)$ of the set B^* is defined by

$$\mu_{B^*}(X) = \begin{cases} 1 & \textit{if} \quad x_C = b_C \\ \max\left\{0, \dfrac{x_C - (b_L + x_W)}{b_C - (b_L + x_W)}\right\} & \textit{if} \quad b_C \geq x_C \geq b_L + x_W \\ 0 & \textit{otherwise}, \end{cases}$$

where X is the variable interval and $X = [x_L, x_R] = \langle x_C, x_W \rangle$.

The values of the membership function B^* lie between 0 and 1. A graphical representation of the membership function μ_{B^*} appears in Figure 10.

A complication in pessimistic decision theory arises when the intensity of the pessimism is changed. Let us consider a linguistically termed set of different degrees of pessimisms as {..., very very low, very low, low, moderate, high, very high, very very high,...}. Sengupta and Pal [14] modified their fuzzy preference ordering according to different degrees of pessimism. They defined a modified nonlinear membership

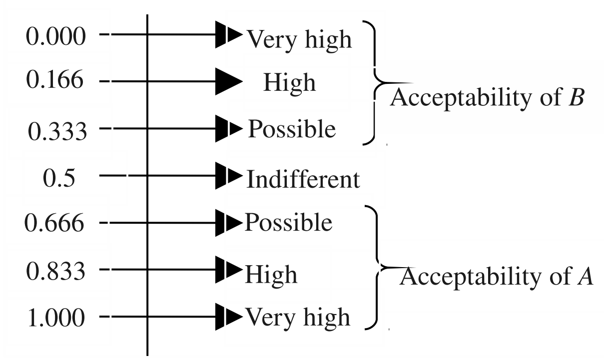

Figure 11: Linguistic scale of different degrees of pessimism

function as

$$\pi_{B^*} = (\mu_{B^*})^p \tag{1}$$

where p is a crisp number that represents the level of pessimism of the parameter for any decision maker and $p \in [1/m, m]$, where m is a finite large number. The quantity p takes the values $1/m$ and m whenever the decision maker becomes absolutely pessimistic or absolutely optimistic, respectively. However, a decision maker never becomes an absolute pessimist or absolute optimist. A linguistic scale was defined by them on the basis of different degrees of pessimism as given in Figure 11.

From the above scale, it is clear that when the values of the nonlinear membership function given in Equation (1) lie between [0.3333, 0.6667], the decision maker with any preference level wants to remain indifferent to reject or accept B. Again, some difficulties arise when applying this preference relation to distinguish between two arbitrary intervals. Since the modified membership function is dependent on the degree of pessimism p, for different degrees of pessimism, we get different values of the membership function and therefore, the preference ordering will also fluctuate frequently according to the values of p. This makes the method difficult to generalize, and thus it is also inconsistent with our requirement. This is the disadvantage of the above preference ordering. Though there are some drawbacks pointed out in both the techniques introduced by Sengupta and Pal [14], the acceptability index and fuzzy preference ordering methods are considered as powerful as well as valuable interval ranking methods in decision making problems.

7.5 Sevastjanov and Róg's Approach

A compact interval ranking definition has been presented by Sevastjanov and Róg [15] using the probabilistic concept. They have actually pointed out two separate ranking definitions - one for crisp intervals and the other for fuzzy intervals. Here we shall discuss only the ordering of crisp intervals. In fuzzy interval ordering, they assumed

α -level sets (which are the crisp intervals) corresponding to each fuzzy interval and used the interval ordering definition developed for crisp intervals. However, a detailed discussion of this case is out of the scope of this paper. In order to define their ranking definition, they considered not only ranking of interval numbers, but also the critical cases of comparisons of real numbers with intervals. They spontaneously excluded the cases of Type - I intervals and those where the real numbers are situated outside the intervals. These are the easiest cases in which we can select our optimal choice without any hesitation.

We now present the actual definitions of Sevastjanov and Róg [15]. Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two intervals. The interval $A = [a_L, a_R]$ is a degenerate real number $a \in \mathbb{R}$ if $a_L = a_R = a$. As for the preceding two probabilistic approaches (viz., Zhang et al. [18] and Kundu's [9]), we assume that the two random variables $x \in [a_L, a_R]$ and $y \in [b_L, b_R]$ are uniformly and independently distributed. The interval ordering is as follows:

$$P(B > A) = \begin{cases} 0 & \text{if } b_L \le a_L \le b_R \le a_R \\ \dfrac{b_L - a_L}{a_R - a_L} & \text{if } a_L \le b_L \text{ and } b_R \le a_R \\ \dfrac{b_L - a_L}{a_R - a_L} & \text{if } a_L < b_L = b_R < a_R, \end{cases}$$

$$P(B < A) = \begin{cases} 1 - \dfrac{(b_L - a_L)^2}{(a_R - a_L)(b_R - b_L)} & \text{if } b_L \le a_L \le b_R \le a_R \\ \dfrac{a_R - b_R}{a_R - a_L} & \text{if } a_L \le b_L \text{ and } b_R \le a_R \\ \dfrac{a_R - b_L}{a_R - a_L} & \text{if } a_L < b_L = b_R < a_R, \end{cases}$$

$$P(B = A) = \begin{cases} \dfrac{(b_R - a_L)^2}{(a_R - a_L)(b_R - b_L)} & \text{if } b_L \le a_L \le b_R \le a_R \\ \dfrac{b_R - b_L}{a_R - a_L} & \text{if } a_L \le b_L \text{ and } b_R \le a_R \\ 0 & \text{if } a_L < b_L = b_R < a_R. \end{cases}$$

The following examples illustrate the working principle of the above interval ranking method.

Example 7.5.1 *For $A = [4, 8]$ and $B = [2, 5]$, the above probability based ranking definitions give $P(B > A) = 0$; $P(B < A) = 0.91667$; $P(B = A) = 0.0833$. These results indicate that the interval B is less than A, and the interval A is selected for maximization problems.*

Example 7.5.2 *If $A = [2, 10]$ and $B = [3, 4]$ then $P(B > A) = 0.125$; $P(B < A) = 0.75$; $P(B = A) = 0.125$. These results also show that for maximization problems the interval A is accepted, while B is selected for minimization problems.*

The following examples reveal an interesting result regarding the order relations between a real number and an interval number and also between two intervals.

Example 7.5.3 *Let* $A = [0, 10]$ *and* $b = 7$. *Then,* $P(b > A) = 0.7; P(b < A) = 0.3; P(b = A) = 0$. *So,* A *is selected for minimization problem and* b *is selected for maximization problems.*

However, Sevastjanov and Róg's ranking fails for the following two examples.

Example 7.5.4 *For* $A = [0, 10]$ *and* $B = [4, 6]$. $P(B > A) = 0.4$*;* $P(B < A) = 0.4$*;* $P(B = A) = 0.2$.

Example 7.5.5 *For* $A = [10, 15]$ *and* $b = 12.5$ *then* $P(b > A) = 0.5$*;* $P(b < A) = 0.5$*;* $P(b = A) = 0$.

From the probability measures of the above two pairs of intervals as given in the last two examples, no decisions can be taken about the choice of optimal intervals, i.e., we cannot take any decision that either the interval A or B is accepted for a maximization problem. Hence, the above ranking definition is also not applicable for intervals having the same centre.

7.6 Comparative Examples

We have compared the interval ranking definitions for maximization problems for the probabilistic group of rankings with the same set of 10 pairs of intervals as the comparisons of the general rankings and rankings depending on indices. The results appear in Tables 6 and 7.

Using Kundu's [9] leftness relation Left(A, B), we have obtained preference intervals for each example except Example 7. For Type - I and Type - II intervals (here, Example 1 and 2), the value of the leftness relation is 1 and the preference intervals are selected easily. For other types of intervals, the values of the leftness relation lies between 0 and 1. The choice of better interval is easier when the values of the relation are nearly 1 (for Examples 3, 4 and 10). For other examples, we also get our optimal choice but not so firmly. For Example 7, the value of the leftness function is 0 and the definition fails accordingly. Hence, a decision maker cannot use this definition when the intervals have the same centre.

Next, Sengupta and Pal's [14] fuzzy preference ordering, the complement of the acceptability index method, is applicable only for examples 8, 9 and 10, as we know that it is relevant to the class of intervals with $\mathscr{A}(A, B) \geq 0$ and $a_W < b_W$. In each case, the rejection value A^* is calculated for the interval with greater width. In Example 10 the rejection value for the interval A is 0, so we cannot reject the interval A, i.e., A can be accepted as preferable to B for maximization problems. The rejection values for Example 9 and Example 8 for the interval A are 0.286 and 0.6667 respectively. Here also we can accept the interval A. The fuzzy preference method is well accepted by pessimistic decision makers.

Zhang's [18] method needs to calculate the values of the possibility degrees $P_{A \geq B}$ and $P_{B \geq A}$ for the intervals A and B. For the simpler examples Example 1 to 5 and examples 9 and 10, the differences between the possibility degrees are comparatively high and hence, in these cases, we can easily select the preference interval. For examples 6 and 8, we agree with $A \geq B$ or $B \geq A$ according to the corresponding greater

Table 6: Comparative studies for maximization problems

Ex.	Intervals	Kundu's Leftness relation [9]	Sengupta and Pal's Fuzzy Preference relation [14]	Zhang et al. [18]	Sevastjanov and Róg [15]	Hu and Wang's modified leftness relation[2]
1	$A = [10, 20] = \langle 15, 5\rangle$, $B = [21, 23] = \langle 22, 1\rangle$.	Left$(A, B) =$ 1, A is left to B and B is accepted.	$\mathscr{A}(A, B) =$ 1.1667 $>$ 1, B is accepted with full satisfaction.	$P_{A\geq B} =$ 0, $P_{B\geq A} =$ 1, B is accepted.	B is accepted without any test.	Left$_H(A, B)$ = 1, A is left to B and B is accepted.
2	$A = [10, 20] = \langle 15, 5\rangle$, $B = [20, 22] = \langle 21, 1\rangle$.	Left$(A, B) =$ 1, A is left to B and B is accepted.	$\mathscr{A}(A, B) =$ 1.0, B is accepted with full satisfaction.	$P_{A\geq B} =$ 0, $P_{B\geq A} =$ 1 B is accepted.	B is accepted without any test.	Left$_H(A, B)$ =1, A is left to B and B is accepted.
3	$A = [10, 20] = \langle 15, 5\rangle$, $B = [19, 21] = \langle 20, 1\rangle$.	Left$(A, B) =$ 0.95, A is left to B and B is accepted.	$\mathscr{A}(A, B) =$ 0.8267, B is accepted with grade of acceptability 0.8267.	$P_{A\geq B} =$ 0.025, $P_{B\geq A} =$ 0.975 B is accepted.	$P(A > B) = 0$, $P(A < B) =$ 0.95, $P(A = B) =$ 0.05, B is selected.	Left$_H(A, B)$ =0.95, A is left to B and B is accepted.
4	$A = [10, 20] = \langle 15, 5\rangle$, $B = [18, 20] = \langle 19, 1\rangle$.	Left$(A, B) =$ 0.8, A is left to B and B is accepted.	$\mathscr{A}(A, B) =$ 0.66, B is accepted with grade of acceptability 0.66.	$P_{A\geq B} =$ 0.1, $P_{B\geq A} =$ 0.9 B is accepted.	$P(A > B) = 0$, $P(A < B) =$ 0.8, $P(A = B) =$ 0.2, B is selected.	Left$_H(A, B)$ =0.8, A is left to B and B is accepted.
5	$A = [10, 20] = \langle 15, 5\rangle$, $B = [17, 19] = \langle 18, 1\rangle$.	Left$(A, B) =$ 0.6, A is left to B and B is accepted.	$\mathscr{A}(A, B) =$ 0.5, B is accepted with grade of acceptability 0.5.	$P_{A\geq B} =$ 0.2, $P_{B\geq A} =$ 0.8 B is accepted.	$P(A > B) =$ 1, $P(A < B) =$ 0.7, $P(A = B) =$ 0.2, B is selected.	Left$_H(A, B)$ =0.6, A is left to B and B is accepted.

Table 7: Comparative studies for maximization problems

Ex.	Intervals	Kundu's Leftness relation [9]	Sengupta and Pal's Fuzzy Preference relation [14]	Zhang et al. [18]	Sevastjanov and Róg [15]	Hu and Wang's modified leftness relation[2]
6	$A = [10, 20] = \langle 15, 5\rangle$, $B = [15, 17] = \langle 16, 1\rangle$.	Left$(A, B) = 0.2$, A is left to B and B is accepted.	$\mathscr{A}(A, B) = 0.1667 > 0$, B is accepted with grade of acceptability 0.1667.	$P_{A\geq B} = 0.4$, $P_{B\geq A} = 0.6$ B is accepted.	$P(A > B) = 0.3$, $P(A < B) = 0.5$, $P(A = B) = 0.2$, B is selected.	Left$_H(A, B)$ =0.2, A is left to B and B is accepted.
7	$A = [10, 20] = \langle 15, 5\rangle$, $B = [14, 16] = \langle 15, 1\rangle$.	Left$(A, B) = 0.0 =$ Left(B, A), method fails.	$\mathscr{A}(A, B) = 0.0$, here the interval with less uncertainty is chosen.	$P_{A\geq B} = 0.5$, $P_{B\geq A} = 0.5$ method fails.	$P(A > B) = 0.4$, $P(A < B) = 0.4$, $P(A = B) = .2$, method fails.	Left$_H(A, B)$ =.4, A is left to B and B is accepted.
8	$A = [10, 20] = \langle 15, 5\rangle$, $B = [12, 16] = \langle 14, 2\rangle$.	Left$(B, A) = 0.2$, B is left to A and A is accepted.	A^*= Rejection of A. Its membership function value $\mu_{A^*} = 0.6667$.	$P_{A\geq B} = 0.6$, $P_{B\geq A} = 0.4$ A is accepted.	$P(A > B) = 0.4$, $P(A < B) = 0.2$, $P(A = B) = 0.4$, Decision cannot be taken.	Left$_H(B, A)$ =.2, B is left to A and A is accepted.
9	$A = [10, 20] = \langle 15, 5\rangle$, $B = [11, 14] = \langle 12.5, 1.5\rangle$.	Left$(B, A) = 0.5$, B is left to A and A is accepted.	A^*= Rejection of A. Its membership function value $\mu_{A^*} = 0.286$.	$P_{A\geq B} = 0.75$, $P_{B\geq A} = 0.25$ A is accepted.	$P(A > B) = 0.6$, $P(A < B) = 0.1$, $P(A = B) = 0.3$, A is selected.	Left$_H(B, A)$ =.5, B is left to A and A is accepted.
10	$A = [10, 20] = \langle 15, 5\rangle$, $B = [10, 12] = \langle 11, 1\rangle$.	Left$(B, A) = 0.8$, B is left to A and A is accepted.	A^*= Rejection of A. Its membership function value $\mu_{A^*} = 0.0$.	$P_{A\geq B} = 0.9$, $P_{B\geq A} = 0.1$ A is accepted.	$P(A > B) = 0$, $P(A < B) = 0.8$, $P(A = B) = 0.2$, A is selected.	Left$_H(B, A)$ =.8, B is left to A and A is accepted.

possibility degrees. However, for Example 7, when the possibility degrees are equal, the definition completely fails to rank the intervals.

Now, we analyze the ranking definition due to Sevastjanov and Róg's [15]. Here, we have to calculate three probabilities $P(A > B)$, $P(A = B)$ and $P(A < B)$ for each example. For examples 1 and 2, there is no need to apply this definition. For the other examples, the optimum interval is selected according to the greatest value among the three probabilities. However, for Example 7, the probabilities $P(A > B)$ and $P(A < B)$ are same and hence, the decision regarding interval ranking is not possible. A similar situation also occurs for Example 8, as the probabilities $P(A > B)$ and $P(A = B)$ are the same.

The last ranking definition of this group is the modified leftness relation due to Hu and Wang [2]. Actually, this is an extension of Kundu's [9] leftness relation. For Example 7, i.e., in case of intervals with the same center, Kundu's [9] leftness relations $\text{Left}(A, B) = 0.0$ and $\text{Left}(B, A) = 0.0$, but the modified leftness relation $\text{Left}_H(A, B) = 0.4$. Hence, B is the selected interval. For other examples, Kundu's [9] leftness relation and Hu and Wang's [2] modified leftness relation follow the same formula. In this group, Sengupta and Pal's [14] method and Hu and Wang's [2] method are working with the same efficiency level but, the latter one can be handled easily and more efficiently in application areas.

8 Diagrammatic Representation of Interval Ordering

To this point we have discussed interval orderings depending on mathematical formulae. Now we shall discuss the ordering of intervals diagrammatically due to Kulpa [8]. An extensive exposition on the diagrammatic representation of a number of interval relations has been given by Kulpa [8]. In fact he developed the ways of representing the space of intervals in diagrams. The diagrams were configured on a *two-dimensional* Euclidean space. The different types of diagrams developed by him are *M R-diagrams* (i.e., mid point-radius diagrams), *W-diagrams* (the shape of the diagram looks like the letter "W') etc. To interpret the ordering of intervals, *M R-diagrams* are essential.

8.1 M R-Diagrams

Generally, the diagrammatic representation of intervals is done by *E-diagrams* (i.e., End points diagrams). If an interval is given in its upper-lower bound form then *E-diagrams* are useful, but E-diagrams are too complicated to handle for the mid point-radius form. The *M R-diagrams* are then much more easily applicable. As the name implies, the diagrams are drawn using the mid points and radii of the intervals. In *two-dimensional* Euclidean space, the horizontal axis represents the mid points and the vertical axis represents the radii of the intervals; i.e., the intervals are uniquely represented by the points of the M R-diagram space. For an interval $A = [a_L, a_R] = \langle a_C, a_W \rangle$, the M R-diagram is shown in Figure 12. The end points of A can also be easily obtained from the diagram, which is evident from the figure. Thus, any pair of the four basic quantities a_L, a_R, a_C and a_W of an interval A uniquely determine the point representing that interval. There are some dotted diagonal lines in the diagram. These diagonal lines indicate that all the intervals lying on those have the same value of their *beginning*, or *end* respectively. Details concerning M R- and

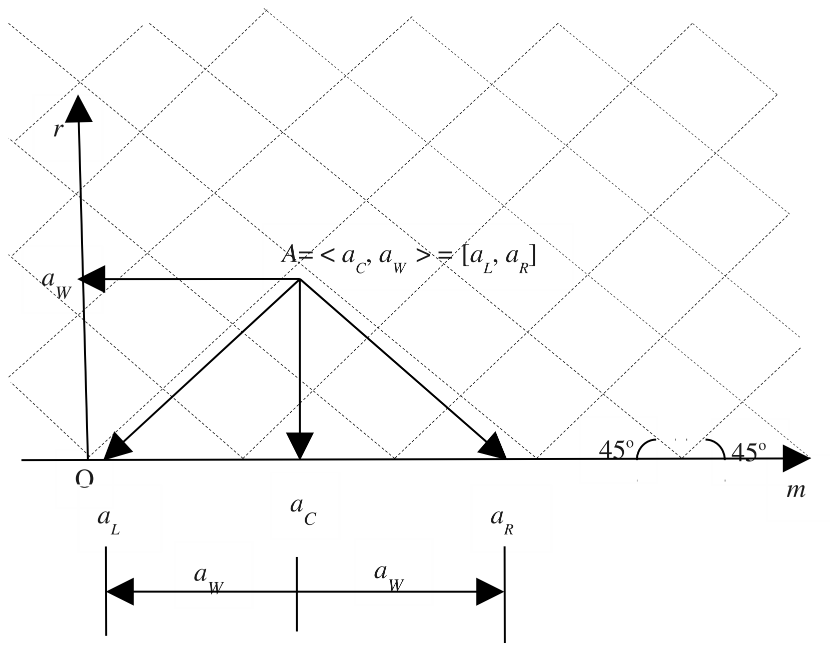

Figure 12: The M R diagram for the interval space

other diagrams are available in [8].

8.2 Symbols for Different Types of Intervals

Generally, the set of intervals is divided into three types: (i) completely disjoint, (ii) partially overlapping and (iii) fully overlapping. Nonetheless, Kulpa [8] divided the aggregate of intervals into seven different types, and to indicate these types he introduced different interval relation symbols and conjunction diagrams. For simplicity, we shall use other symbols than those used by Kulpa [8], without using the conjunction diagrams. Our notations and symbols are depicted in Table 8.

8.3 Interval Order Relations

Kulpa [8] defined the order relations of interval numbers in several ways. The basic interval order relations, viz., "<" and ">" are applicable only for disjoint intervals, and these are too restrictive in many situations. Before giving other order relations, he defined two related notions — *In between interval relation* and *Lozenge*.

Definition 8.3.1 *(In between interval relation) Let $A = [a_L, a_R]$ and $B = [b_L, b_R]$ be two intervals. A variable interval $U = [u_L, u_R]$ is said to lie in-between A and B if*

$$\min\{a_L, b_L\} \le u_L \le \max\{a_L, b_L\} \ \text{ and } \ \min\{a_R, b_R\} \le u_R \le \max\{a_R, b_R\}.$$

Table 8: Basic interval relations and their symbols

Symbol names	Intervals	Relation between the interval parameters	Nature of the relation
T_1	A : ------ B : ---------	$a_R < b_L$	Completely disjoint.
T_2	A : ------- B : ------	$a_R = b_L$	Partially overlapping just touching at one of the end points.
T_3	A : ------ B : ----------	$a_L < b_L < a_R < b_R$	Partially overlapping.
T_4	A : ------ B : -----------	$a_L = b_L$ but $a_R < b_R$	Overlapping but the lower end points are equal.
T_5	A : ------ B : -------------	$a_L > b_L$ and $a_R < b_R$	Fully overlapping.
T_6	A : ------ B : --------------	$a_L > b_L$ and $a_R = b_R$	Overlapping but the upper end points are equal.
T_7	A : ------ B : ------	$a_L = b_L$ and $a_R = b_R$	Equal intervals.

Definition 8.3.2 *(Lozenge) A Lozenge, denoted by $\ll A, B \gg$ for a pair of intervals $\langle A, B\rangle$, is the set of intervals that lie in-between some given intervals. Mathematically,*

$$\ll A, B \gg \;=\; \{U = [u_L, u_R] : (\min\{a_L, b_L\} \le u_L \le \max\{a_L, b_L\}) \;\; \textit{and} \;\; (\min\{a_R, b_R\} \le u_R \le \max\{a_R, b_R\})\}.$$

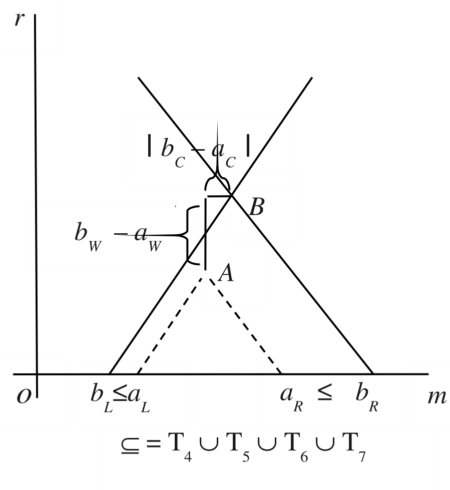

Figure 13: M R-diagram for set inclusion property

Kulpa [8] defined a pair of interval orderings. Among these, one is the traditional *set inclusion relation* "$\subseteq$" considering the intervals as sets, while the other is the *precedence relation* denoted by "$\preceq$". The relations can be defined as follows:

(i) $A \subseteq B \Leftrightarrow (b_L \le a_L \text{ and } a_R \le b_R) \Leftrightarrow |b_C - a_C| \le b_W - a_W$,
$A \supseteq B \Leftrightarrow B \subseteq A$,

(ii) $A \preceq B \Leftrightarrow (a_L \le b_L \text{ and } a_R \le b_R) \Leftrightarrow |b_W - a_W| \le b_C - a_C$,
$A \succeq B \Leftrightarrow B \preceq A$.

The *M R-diagrams* of these interval relations are shown in Figure 13 and Figure 14. Figure 13 represents the diagrammatic representation of *set inclusion relation*, while Figure 14 shows the same for the *precedence relation*. The diagonal border lines shown in the figures have extra significance: Exactly one of the four ordering relations $\subseteq, \supseteq, \preceq, \succeq$ must hold between the intervals A and B when they do not lie on the same diagonal line. For the intervals which lie on the same diagonal line, exactly two relations i.e., either any one between $\subseteq$ and $\supseteq$ or between $\preceq$ and $\succeq$ hold.

9 Global Comparison

In this section, a comparison of the best interval ranking definitions from the groups in sections 5, 6, 7 and 8 is made. For this purpose, we have selected 5 pairs of different

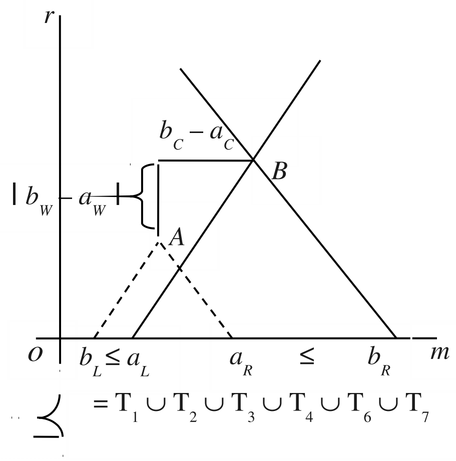

Figure 14: M R-diagram for set precedence property

types of intervals (for maximization problems). From the first group, any one of the definitions of Hu and Wang [2] and Mahato and Bhunia [11] can be considered, since they work with the same efficiency level. Here we have taken Mahato and Bhunia's [11] definition for global comparison. From the second and third groups, the best selected ordering definitions are Sengupta and Pal's [14] acceptability index method and Hu and Wang's [2] modified leftness relation. In fourth group, there is only one definition, viz., Kulpa's [8] diagrammatic definition, which has been taken for comparison. The comparison is displayed in Table 9.

From the table, it is evident that for examples 1 and 5, the same intervals (interval B for Example 1 and interval A for Example 5) have been selected by all the definitions, whereas in Example 2 and 4, except for Kulpa's scheme [8], all other definitions select the same intervals. For Example 3, Sengupta and Pal's [14] acceptability index and Kulpa's [8] diagrammatic approach fail to select the better interval. Hence, it is concluded that the definitions due to Mahato and Bhunia [11] and Hu and Wang [2] from first group and Hu and Wang's [2] modified leftness relation from the third group can be considered for ranking the interval numbers.

10 Concluding Remarks

In this paper, we have discussed the existing definitions of interval order relations along with their advantages and shortcomings. These definitions are based on set properties, fuzzy applications, probabilistic approaches, or value-based approaches, or depending upon some specific indices/functions. To serve the purpose, the ranking definitions have first been categorized into four groups based on different properties. In each group, we have done comparative studies with a set of pairs of intervals, then the best definition has been selected from each group. A global comparison has then been performed on the selected definitions. For future research, one may use any one of the group wise selected definitions for solving continuous optimization problems

Table 9: Comparative studies for maximization problems

Ex.	Intervals	Group I (Mahato & Bhunia) [11]	Group II (Sengupta-Pal's Acceptability Index)[14]	Group III (Hu & Wang's leftness relation)[2]	Group IV (Kulpa)[8]
1	$A = [0, 5] = \langle 2.5, 2.5\rangle$, $B = [0, 10] = \langle 5, 5\rangle$.	B is accepted.	$\mathscr{A}(B, A) = 0.333$, B is accepted.	$\text{Left}_H(A, B) = 0.5$, A is left to B and B is accepted.	T_4 type and $B \supseteq A$. B will be selected.
2	$A = [2, 6] = \langle 4, 2\rangle$, $B = [0, 10] = \langle 5, 5\rangle$.	B is accepted.	$\mathscr{A}(B, A) = 0.14286$, B is accepted.	$\text{Left}_H(A, B) = 0.2$, A is left to B and B is accepted.	T_5 type and $B \supseteq A$. Decision cannot be taken.
3	$A = [4, 6] = \langle 5, 2\rangle$, $B = [0, 10] = \langle 5, 5\rangle$.	A is accepted.	$\mathscr{A}(B, A) = 0.0 = \mathscr{A}(A, B)$, method fails.	$\text{Left}_H(B, A) = 0.4$, B is left to A and A is accepted.	T_5 type and $B \supseteq A$. Decision cannot be taken.
4	$A = [4, 8] = \langle 6, 2\rangle$, $B = [0, 10] = \langle 5, 5\rangle$.	A is accepted.	$\mathscr{A}(A, B) = 0.14286$, A is accepted.	$\text{Left}_H(B, A) = 0.2$, B is left to A and A is accepted.	T_5 type and $B \supseteq A$. Decision cannot be taken.
5	$A = [5, 10] = \langle 7.5, 2.5\rangle$, $B = [0, 10] = \langle 5, 5\rangle$.	A is accepted.	$\mathscr{A}(A, B) = 0.333$, A is accepted.	$\text{Left}_H(B, A) = 0.5$, B is left to A and A is accepted.	T_6 type and $B \supseteq A$. A will be selected.

with real/interval coefficients.

Acknowledgement

The authors are grateful to the responsible Managing Editor and the anonymous referees for their valuable comments and suggestions, which have greatly improved the earlier versions of this paper.

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