Ostrowski-Like Method for the Inclusion of a Single Complex Polynomial Zero*

Mimica R. Milošević†
Faculty of Science, Department of Mathematics and Informatics, University of Niš, 18 000 Niš, Serbia
mimica@pmf.ni.ac.rs

Miodrag S. Petković
Faculty of Electronic Engineering, Department of Mathematics, University of Niš, Serbia
msp@eunet.rs

Abstract

A new iterative method of Ostrowski's type for the inclusion of one isolated simple or multiple complex zero of a polynomial is established in circular complex arithmetic. Cubic convergence is proved and computationally verifiable initial condition that guarantees the convergence of this inclusion method is stated. In order to demonstrate convergence behavior of the proposed method, two numerical examples are given.

Keywords: Polynomial zeros, inclusion methods, convergence, circular arithmetic.
AMS subject classifications: 65H05, 65G20, 30C15.

1 Introduction

Moore's interval version of Newton's method [4, 5] for the inclusion of a simple real zero of a differentiable function can be applied only to real simple zeros, which limits the application of this method. In this paper we consider an iterative method of Ostrowski's type for the inclusion of one simple or multiple complex zero of a polynomial in circular complex arithmetic. Ostrowski-like algorithms for the simultaneous inclusion of all zeros of a polynomial, realized in circular complex arithmetic, are first investigated by I. Gargantini in [2] and by M. Petković in [6]. The proposed method produces disks that contain the sought zero in each iteration. In this manner, the information on the upper error bound of zero approximations, presented by centers of inclusion disks, is provided automatically, which is a significant advantage relative to other methods for finding a single zero. We also state computationally verifiable

*Submitted: January 19, 2012; Revised: March 24, 2012; Accepted: October 10, 2012.
†This research was supported by the Serbian Ministry of Science under grant number 174022.

initial condition that guarantees the convergence of the proposed method, which is of practical interest.

A general principle for constructing interval methods for finding polynomial zeros is based on *inclusion isotonicity property* (actually, subset property) and a suitable *zero-relation* which gives a relationship among polynomial zeros. Let $\zeta_1, \dots, \zeta_n$ be the zeros of a given polynomial and assume that we have found an array of disks (or rectangles, real intervals) $Z_1, \dots, Z_n$ such that $\zeta_i \in Z_i$ $(i = 1, \dots, n)$. A general form of a zero-relation is given by

$$\zeta_i = F(\zeta_1, \dots, \zeta_n; z_1, \dots, z_n), \quad z_j = \operatorname{mid} Z_j,\ i, j \in \{1, \dots, n\}. \tag{1}$$

Not all zeros or midpoints necessarily appear on the right side of (1). The zero-relation used in this paper is given by (10).

Substituting the zeros on the right side of (1) by their inclusion intervals and using the *inclusion property* , we obtain

$$\zeta_i \in \widehat{Z}_i := F(Z_1, \dots, Z_n; z_1, \dots, z_n), \quad z_j = \operatorname{mid} Z_j,\ i, j \in \{1, \dots, n\}, \tag{2}$$

where $\widehat{Z}_i$ is presumably smaller inclusion interval than Z_i. The relation (2) suggests an iterative interval method for the inclusion of polynomial zeros taking $(Z_1, \dots, Z_n) = (Z_1^{(0)}, \dots, Z_n^{(0)})$.

A special case of the relation (2) appears if we take that a single zero, say ζ_1, is isolated in an interval Z_1, and all remaining zeros lie in an open interval $\operatorname{ext} Z_1 = \{z : z \notin Z_1\}$. Then (2) becomes

$$\zeta_1 \in \widehat{Z}_1 := \widetilde{F}(\operatorname{ext} W_1, \operatorname{mid} Z_1). \tag{3}$$

Under suitable conditions (for instance, inversion of the open interval $\operatorname{ext} W_1$ is a finite interval, division by intervals containing 0 does not occur in (3)), we can construct an iterative method for the inclusion of a single (simple or multiple) zero of a given polynomial. This is the main goal of this paper.

A standard approach in finding polynomial zeros is the application of an iterative method for the simultaneous determination of *all* zeros. Motivation for the construction of interval methods for the inclusion of only one zero arises from specific requirements in real-life problems when only one zero, located in a region of interest, is requested. Besides, we emphasize that such methods are computationally much cheaper since not only considerably less numerical operations are needed in the implementation of the corresponding iterative formula but also due to the fact that the localization of only one zero is required, decreasing in this way computational costs.

In this paper we consider cubically convergent interval Ostrowski-like method. This method shows good convergent characteristics (see [2]) and it is relatively fast, as demonstrated by a number of numerical examples. Numerical examples also showed that the presented method in most cases are better than other interval methods of third order. We did not consider Weierstrass-like interval method of the second order (studied, e.g., in [1, Ch. 8], [8, Sec. 3.1]) for two reasons: this method is computationally very expensive (see [8, Ch. 6]) and the convergence of its version for a single zero is only *linear*.

The presentation of the paper is organized as follows. Some basic definitions and operations of circular complex interval arithmetic, necessary for the construction and the convergence analysis of inclusion method, are given at the end of Introduction. The derivation of the Ostrowski-like method for the inclusion of one simple or multiple complex zero is given in Section 2 and the convergence analysis is presented in

Section 3. Numerical examples, presented in Section 4, were realized in INTLAB [13], the Matlab toolbox for reliable computing and self-validating algorithms. Note that INTLAB runs in double precision arithmetic by default, giving approximately 16 significant decimal digits without any specific declaration of variables. In this way the enclosure of the zeros in the presence of rounding errors is provided.

The construction of the mentioned inclusion method and its convergence analysis, presented in this paper, need the basic properties of the so-called circular complex arithmetic introduced by Gargantini and Henrici [3]. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ is denoted by the parametric notation $Z := \{c; r\}$. The following basic circular arithmetic operations are defined as follows:

$$\begin{aligned} \alpha + \{c;r\} &= \{\alpha + c; r\}, \\ \alpha\{c;r\} &= \{\alpha c; |\alpha| r\} \quad (\alpha \in \mathbb{C}), \\ \{c_1;r_1\} \pm \{c_2;r_2\} &= \{c_1 \pm c_2; r_1 + r_2\}. \end{aligned}$$

The inversion of a non-zero disk Z is defined by the Möbius transformation,

$$Z^{-1} = \left\{\frac{1}{z} : z \in Z\right\} = \left\{\frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2}\right\} \quad (|c| > r, \text{ that is, } 0 \notin Z). \tag{4}$$

Addition, subtraction and inversion Z^{-1} are exact operations.

Let us define a disk $\{z : |z-a| \le R\}$, denoted by $\{a; R\}$, and its exterior $W = \{z : |z-a| > R\}$. If $z \notin W$ (that is, $|z-a| \le R$), then the inversion of the region

$$z - W = \{w : w - (z-a) > R\}$$

is the closed interior of a circle given by

$$V(z) = (z - W)^{-1} = \left\{w : \left|w + \frac{\bar{z} - \bar{a}}{R^2 - |z-a|^2}\right| \le \frac{R}{R^2 - |z-a|^2}\right\} =: \{h(z); d(z)\}, \tag{5}$$

where

$$h(z) = \operatorname{mid} V(z) = \frac{\bar{a} - \bar{z}}{R^2 - |z-a|^2}$$

and

$$d(z) = \operatorname{rad} V(z) = \frac{R}{R^2 - |z-a|^2}$$

(see [12]).

The set $\{z_1 z_2 : z_1 \in Z_1, z_2 \in Z_2\}$, in general, is not a disk. In order to remain within the realm of disks, Gargantini and Henrici [3] introduced the multiplication by

$$Z_1 \cdot Z_2 := \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\} \supseteq \{z_1 z_2 : z_1 \in Z_1, z_2 \in Z_2\}. \tag{6}$$

Then the division is defined by

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{-1}.$$

The square root of a disk $\{c; r\}$ that does not contains the origin, where $c = |c| e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [2]):

$$\{c;r\}^{1/2} := \left\{\sqrt{|c|} e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}}\right\} \bigcup \left\{-\sqrt{|c|} e^{i\theta/2}; \frac{r}{\sqrt{|c|} + \sqrt{|c| - r}}\right\}. \tag{7}$$

In this paper we will use the following obvious properties:

$$z \in \{c; r\} \iff |z - c| \le r,$$
$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \iff |c_1 - c_2| > r_1 + r_2.$$

More details about circular arithmetic can be found in the books [1] and [12].

2 Ostrowski-Like Method

Let

$$P(z) = \prod_{j=1}^{n} (z - \zeta_j)^{\mu_j} \tag{8}$$

be a monic polynomial of degree $N \ge 3$ with n $(2 \le n \le N)$ distinct real or complex zeros $\zeta_1, \dots, \zeta_n$ of respective multiplicities $\mu_1, \dots, \mu_n$, where $\mu_1 + \cdots + \mu_n = N$ and let

$$\delta_1(z) = \frac{P'(z)}{P(z)}, \quad \delta_2(z) = \frac{P'(z)^2 - P(z)P''(z)}{P(z)^2}$$

and

$$s_{2,i}(z) = \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} \frac{\mu_j}{(z - \zeta_j)^2} \quad (\boldsymbol{I}_n := \{1, \dots, n\}). \tag{9}$$

From the factorization (8) we find

$$\delta_2(z) = -\frac{d^2}{dz^2}\Big(\log P(z)\Big) = \sum_{j=1}^{n} \frac{\mu_j}{(z - \zeta_j)^2} = \frac{\mu_i}{(z - \zeta_i)^2} + s_{2,i}(z).$$

Solving the last equation in ζ_i we obtain the following zero-relation

$$\zeta_i = z - \frac{\sqrt{\mu_i}}{\Big[\delta_2(z) - s_{2,i}(z)\Big]_*^{1/2}}. \tag{10}$$

It is assumed that only one complex value (of two) of the square root has to be taken in the last formula, which is indicated by the symbol $*$. This value is chosen in such a way that the right-hand side reduces to ζ_i.

Assume that we have found the inclusion disk $\{z : |z - a| \le R\}$ with the center a and the radius R containing only one zero ζ_i of P. All other zeros are supposed to lie in the region $W = \{z : |z - a| > R\}$. Using the inclusion isotonicity property we obtain for $z \in \{a; R\}$

$$(z - \zeta_j)^{-1} \in \Big(\frac{1}{z - W}\Big) \quad (j \in \boldsymbol{I}_n \setminus \{i\}). \tag{11}$$

If $z \notin W$, by (5) we obtain the inversion of the region

$$V(z) = (z - W)^{-1} = \{h(z); d(z)\}. \tag{12}$$

Using the inclusion isotonicity property, and (9), (11) and (12), we have

$$s_{2,i}(z) \in \sum_{j \in \boldsymbol{I}_n \setminus \{i\}} (z - W)^{-2} = (N - \mu_i)\, V(z)^2 =: S_{2,i}(z) \quad (i \in \boldsymbol{I}_n).$$

According to the last relation, from (10) we get for $z = z_i$

$$\zeta_i \in z_i - \frac{\sqrt{\mu_i}}{\left[\delta_2(z_i) - S_{2,i}(z_i)\right]_*^{1/2}} =: \hat{Z}_i. \tag{13}$$

Assuming that the disk under the square root does not contain 0, $\hat{Z}_i$ is a new outer approximation to the zero ζ_i.

In our consideration only one zero is requested so that, without the loss of generality, we may adopt that this zero is denoted with ζ_1 and suppose that all other zeros $\zeta_2, \ldots, \zeta_n$ lie in the exterior of $\{a; R\}$. Moreover, for brevity, we will write ζ instead of ζ_1, and also s_2 and S_2 instead of $s_{2,1}$ and $S_{2,1}$.

Let $Z^{(m)} = \{z^{(m)}; r^{(m)}\}$ be the disk with the center $z^{(m)} = \operatorname{mid} Z^{(m)}$ and the radius $r^{(m)} = \operatorname{rad} Z^{(m)}$ for $m = 0, 1, \ldots$. For the initial inclusion disk $Z^{(0)}$ we have $Z^{(0)} = \{a; R\}$, i.e., $z^{(0)} = a$, $r^{(0)} = R$. Besides, we introduce the notation

$$V^{(m)} := V(z^{(m)}) = \left(z^{(m)} - W\right)^{-1} = \left\{h\left(z^{(m)}\right); d\left(z^{(m)}\right)\right\}$$

and

$$S_2^{(m)} = (N - \mu)\left(V^{(m)}\right)^2.$$

The relation (13) suggests the following iterative method for the inclusion of a simple or multiple complex zero of a polynomial P, starting from the isolated initial inclusion disk $Z^{(0)} = \{a; R\}$,

$$Z^{(m+1)} = z^{(m)} - \frac{\sqrt{\mu}}{\left[\delta_2\left(z^{(m)}\right) - (N-\mu)\left\{h\left(z^{(m)}\right); d\left(z^{(m)}\right)\right\}^2\right]_*^{1/2}} \tag{14}$$

for $m = 0, 1, \ldots$.

According to (7), the square root of a disk in (14) produces two disks; the symbol $*$ indicates that one of these disks has to be chosen. The criterion for the choice of a proper disk is considered in [2] (see also [6]) and reads:

Let $\left[\delta_2\left(z^{(m)}\right) - (N-\mu)\left\{h\left(z^{(m)}\right); d\left(z^{(m)}\right)\right\}^2\right]^{1/2} = D_1^{(m)} \bigcup D_2^{(m)}$. *Among the disks* $D_1^{(m)}$ *and* $D_2^{(m)}$ *one has to choose that disk whose center minimizes*

$$\left|\frac{P'\left(z^{(m)}\right)}{\mu P\left(z^{(m)}\right)} - \operatorname{mid} D_p^{(m)}\right| \quad (p = 1, 2).$$

3 Convergence Analysis

In this section we will give a convergence analysis of the iterative method (14). The iterative method (14) can be expressed in the form

$$Z^{(m+1)} = z^{(m)} - \frac{\sqrt{\mu}}{\left\{c\left(z^{(m)}\right); \eta\left(z^{(m)}\right)\right\}_*^{1/2}} = z^{(m)} - \frac{\sqrt{\mu}}{\left\{\sqrt{c\left(z^{(m)}\right)}; u\left(z^{(m)}\right)\right\}}, \tag{15}$$

where

$$\begin{aligned} c(z) &= \delta_2(z) - (N-\mu)h(z)^2 = \delta_2(z) - (N-\mu)\frac{(\bar{a}-\bar{z})^2}{\left(R^2 - |z-a|^2\right)^2}, \\ \eta(z) &= (N-\mu)\left(2|h(z)|d(z) + d(z)^2\right) = (N-\mu)\frac{2|a-z|R + R^2}{\left(R^2 - |z-a|^2\right)^2}, \\ u(z) &= \frac{\eta(z)}{|c(z)| + \sqrt{|c(z)| - \eta(z)}}. \end{aligned}$$

Assume that we have found an initial disk $Z^{(0)} = \{a; R\}$ so that the condition

$$|\delta_2(a)| > \frac{5(N-\mu)^2\mu}{2R^2} \tag{16}$$

is satisfied. Also, for $m = 1, 2, \ldots$ let us introduce

$$\rho^{(m)} = R - \left|z^{(m)} - a\right|.$$

At the beginning, let us consider the first iteration ($m = 0$). Using the inversion (5) we obtain

$$\begin{aligned} \{c(a); \eta(a)\} &= \delta_2(a) - (N-\mu)\left\{h(a); d(a)\right\}^2 = \delta_2(a) - (N-\mu)\left\{0; \frac{1}{R}\right\}^2 \\ &= \left\{\delta_2(a); \frac{N-\mu}{R^2}\right\}. \end{aligned} \tag{17}$$

Since $(N-\mu)\mu \geq 2$, from (17) we estimate

$$\eta(a) = \frac{N-\mu}{R^2} \leq \frac{(N-\mu)^2\mu}{2R^2}. \tag{18}$$

According to the inequalities (16) and (18) and the equality $c(a) = \delta_2(a)$ (see (17)), we obtain that

$$|c(a)| > \frac{5(N-\mu)^2\mu}{2R^2} > \frac{(N-\mu)^2\mu}{2R^2} \geq \eta(a),$$

which mean that the disk $\left\{c(a); \eta(a)\right\}$ does not contain the origin when $m = 0$ and we can compute the square root

$$\left\{c(a); \eta(a)\right\}_*^{1/2} = \left\{\sqrt{c(a)}; u(a)\right\}.$$

Using (7) and the estimates (16) and (18) we have

$$\begin{aligned} u(a) &= \frac{\eta(a)}{\sqrt{|c(a)|} + \sqrt{|c(a)| - \eta(a)}} < \frac{\dfrac{(N-\mu)^2\mu}{2R^2}}{\sqrt{\dfrac{5(N-\mu)^2\mu}{2R^2}} + \sqrt{\dfrac{2(N-\mu)^2\mu}{R^2}}} \\ &< \frac{17}{100} \cdot \frac{(N-\mu)\sqrt{\mu}}{R}. \end{aligned} \tag{19}$$

Furthermore, starting from (15) and using (4), we find the upper bound for $r^{(1)}$ employing the inequalities (16) and (19),

$$\begin{aligned} r^{(1)} &= \operatorname{rad} Z^{(1)} = \operatorname{rad} \frac{\sqrt{\mu}}{\left\{\sqrt{c(a)}; u(a)\right\}} = \sqrt{\mu}\,\frac{u(a)}{|c(a)| - u(a)^2} \\ &< \sqrt{\mu}\,\frac{\dfrac{17(N-\mu)\sqrt{\mu}}{100R}}{\dfrac{5(N-\mu)^2\mu}{2R^2} - \dfrac{17^2(N-\mu)^2\mu}{10^4R^2}}, \end{aligned}$$

whence

$$r^{(1)} < \frac{7}{100} \cdot \frac{R}{N-\mu}. \tag{20}$$

Using the same estimates, we find

$$\left|z^{(1)} - a\right| = \sqrt{\mu}\,\frac{|\sqrt{c(a)}|}{|c(a)| - u(a)^2} < \sqrt{\mu}\,\frac{\sqrt{\dfrac{5(N-\mu)^2\mu}{2R^2}}}{\dfrac{5(N-\mu)^2\mu}{2R^2} - \dfrac{17^2(N-\mu)^2\mu}{10^4R^2}},$$

wherefrom

$$\left|z^{(1)} - a\right| < \frac{16}{25} \cdot \frac{R}{N-\mu}. \tag{21}$$

Now we prove that the condition (16) implies the inequality

$$\rho^{(1)} > 5(N-\mu)r^{(1)}. \tag{22}$$

Using the inequality (21) we find

$$\rho^{(1)} = R - \left|z^{(1)} - a\right| > R - \frac{16R}{25(N-\mu)} = R\left[1 - \frac{16}{25(N-\mu)}\right],$$

so that, according to (20) and (22), it is sufficient to show that

$$R\left[1 - \frac{16}{25(N-\mu)}\right] > 5(N-\mu)\frac{7R}{100(N-\mu)} = \frac{35R}{100}.$$

The last inequality is obvious because of

$$\min_{1\le\mu<N}\left(1 - \frac{16}{25(N-\mu)}\right) = \frac{9}{25} = \frac{36}{100}.$$

The analysis of the first iterative step shows that

(i) a new disk approximation $Z^{(1)}$ includes the zero ζ;

(ii) this disk is contracted because of

$$r^{(1)} = \operatorname{rad} Z^{(1)} < \frac{7R}{100}. \tag{23}$$

Besides, the initial condition (16) induces the condition (22).

Now, we can analyze the iterative process (15) beginning with $m \geq 1$ and starting from the inclusion disk $Z^{(1)}$ with the assumption that the condition (22) holds. For simplicity, in our further analysis we omit the iteration index always when the possibility of any confusion does not exist.

Lemma 1. *If the inequality*

$$\rho > 5(N-\mu)r \tag{24}$$

holds, then $0 \notin \{c(z); \eta(z)\}$ *and*

$$\sqrt{\mu}\,\frac{\sqrt{|c(z)|}}{|c(z)|-u(z)^2} < \frac{13}{10}\,r. \tag{25}$$

Proof. First, since $|z-\zeta_j| > \rho$, for all $j = 2,\ldots,n$, under the condition (24) we estimate

$$\begin{aligned}\left|\frac{P(z)}{P'(z)}\right| &= \left|\sum_{j=1}^{n}\frac{\mu_j}{z-\zeta_j}\right|^{-1} \leq \left(\frac{\mu}{|z-\zeta|} - \sum_{j=2}^{n}\frac{\mu_j}{|z-\zeta_j|}\right)^{-1} < \left(\frac{\mu}{r} - \frac{N-\mu}{\rho}\right)^{-1}\\ &= \frac{r}{\mu-(N-\mu)\dfrac{r}{\rho}} < \frac{r}{\mu-\dfrac{1}{5}} = \frac{5r}{5\mu-1} \quad (\mu_1 = \mu)\end{aligned}$$

and

$$\frac{|a-z|}{R^2-|z-a|^2} = \frac{R-\rho}{R^2-(R-\rho)^2} < \frac{1}{\rho}.$$

Therefore,

$$(N-\mu)\left(\frac{|a-z|}{R^2-|z-a|^2}\right)^2 < \frac{N-\mu}{\rho^2}.$$

Simple transformations give

$$\begin{aligned}g(z) :&= \left|\frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)}\right| = \left|\left(\frac{P'(z)^2-P''(z)P(z)}{P(z)^2}\right)\Big/\left(\frac{P'(z)}{P(z)}\right)\right|\\ &= \left|\left(\sum_{j=1}^{n}\frac{\mu_j}{(z-\zeta_j)^2}\right)\Big/\left(\sum_{j=1}^{n}\frac{\mu_j}{z-\zeta_j}\right)\right| = \left|\frac{1}{z-\zeta}\cdot\frac{1+\beta(z)}{1+\gamma(z)}\right|,\end{aligned}$$

where we put

$$\beta(z) = \frac{(z-\zeta)^2}{\mu}\sum_{j=2}^{n}\frac{\mu_j}{(z-\zeta_j)^2}, \quad \gamma(z) = \frac{z-\zeta}{\mu}\sum_{j=2}^{n}\frac{\mu_j}{z-\zeta_j}.$$

Using the inequality (24) we find

$$|\beta(z)| < \frac{r^2}{\mu}\sum_{j=2}^{n}\frac{\mu_j}{|z-\zeta_j|^2} < \frac{r^2}{\mu}\cdot\frac{N-\mu}{\rho^2} < \frac{1}{25\mu(N-\mu)} \leq \frac{1}{50}$$

and

$$|\gamma(z)| < \frac{r}{\mu}\sum_{j=2}^{n}\frac{\mu_j}{|z-\zeta_j|} < \frac{r}{\mu}\cdot\frac{N-\mu}{\rho} < \frac{1}{5},$$

so that

$$\left|\frac{1+\beta(z)}{1+\gamma(z)}\right| > \frac{1-\frac{1}{50}}{1+\frac{1}{5}} = \frac{49}{60}.$$

According to the last estimate and the inequality $|z-\zeta| \le r$, we have

$$g(z) = \left|\frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)}\right| > \frac{1}{|z-\zeta|}\left|\frac{1+\beta(z)}{1+\gamma(z)}\right| > \frac{49}{60r}.$$

Taking into account the previous bounds, we obtain

$$\begin{aligned} |c(z)| &= \left|\frac{P'(z)}{P(z)}\left(\frac{P'(z)}{P(z)} - \frac{P''(z)}{P'(z)}\right) - (N-\mu)\left(\frac{|a-z|}{R^2-|z-a|^2}\right)^2\right| \\ &> \frac{5\mu-1}{5r}\cdot\frac{49}{60r} - \frac{N-\mu}{\rho^2}, \end{aligned}$$

wherefrom

$$|c(z)| > \frac{49}{60}\left(\mu - \frac{1}{4}\right)\frac{1}{r^2} > \frac{3}{5r^2}. \tag{26}$$

The upper bound of $\eta(z)$ is given by

$$\eta(z) = (N-\mu)\frac{2|a-z|R+R^2}{(R^2-|z-a|^2)^2} < \frac{N-\mu}{\rho^2} < \frac{1}{25r^2}. \tag{27}$$

From the inequalities (26) and (27) we conclude that

$$|c(z)| > \frac{3}{5r^2} > \frac{1}{25r^2} > \eta(z)$$

and whence $0 \notin \{c(z); \eta(z)\}$, which proves the first part of the lemma.

Now we will prove the inequality (25). First, using (24), (26) and (27), we estimate

$$\begin{aligned} u(z) &= \frac{\eta(z)}{\sqrt{|c(z)|} + \sqrt{|c(z)| - \eta(z)}} < \frac{\dfrac{N-\mu}{\rho^2}}{\sqrt{\dfrac{3}{5r^2}} + \sqrt{\dfrac{3}{5r^2} - \dfrac{1}{25r^2}}} \\ &< \frac{2}{3}\,\frac{(N-\mu)r}{\rho^2} < \frac{2}{75r}, \end{aligned} \tag{28}$$

and finally

$$\sqrt{\mu}\,\frac{\sqrt{|c(z)|}}{|c(z)| - u(z)^2} < \frac{\sqrt{\dfrac{49}{60}\mu\left(\mu-\dfrac{1}{4}\right)}}{\dfrac{49}{60}\left(\mu-\dfrac{1}{4}\right) - \dfrac{4}{75^2}}\, r < \frac{13}{10}r. \quad \square$$

Using Lemma 1 we are now able to prove that the order of convergence of the inclusion method (14) is three.

Theorem 1. *Let the sequence of circular intervals* $\{Z^{(m)}\}_{m=1,2,\ldots}$ *be defined by the iterative formula* (14)*, assuming that the initial disk* $Z^{(0)} = \{a; R\}$ *is chosen so that the condition* (16) *is satisfied. Then, in each iterative step, the following is true:*

(i) $\zeta \in Z^{(m)}$;

(ii) $r^{(m+1)} < \dfrac{17(N-\mu)}{R^2}\left(r^{(m)}\right)^3$.

Proof. The assertion (i) follows from the zero-relation (10) according to the inclusion isotonicity property and the fact that $z^{(m)} \in \{a; R\}$ for each $m = 0, 1, \dots,$ which is obvious because of

$$R - \left|z^{(m)} - a\right| = \rho^{(m)} > 5(N-\mu)r^{(m)} > 0.$$

We now prove that the convergence rate of the iterative method (14) is cubic (the assertion (ii)). Using the inequality (22), which follows from the condition (16), and the bounds (26) and (28), we obtain

$$\begin{aligned} r^{(2)} &= \operatorname{rad} Z^{(2)} = \frac{u^{(1)}}{\left|c^{(1)}\right| - \left(u^{(1)}\right)^2} < \frac{\dfrac{2(N-\mu)r^{(1)}}{3\left(\rho^{(1)}\right)^2}}{\dfrac{3}{5\left(r^{(1)}\right)^2} - \left(\dfrac{2}{75 r^{(1)}}\right)^2} \\ &< \frac{28}{25} \cdot \frac{(N-\mu)\left(r^{(1)}\right)^3}{\left(\rho^{(1)}\right)^2} \end{aligned} \tag{29}$$

and

$$r^{(2)} < \frac{1}{20}\, r^{(1)}. \tag{30}$$

Using the inequality (21) we estimate

$$\rho^{(1)} = R - \left|z^{(1)} - a\right| > R - \frac{16}{25}\, R = \frac{9}{25}\, R \tag{31}$$

and

$$\begin{aligned} \rho^{(2)} &= R - \left|z^{(2)} - a\right| = R - \left| z^{(1)} - a - \sqrt{\mu}\, \frac{\sqrt{\bar{c}^{(1)}}}{\left|c^{(1)}\right| - \left(u^{(1)}\right)^2} \right| \\ &> R - \left|z^{(1)} - a\right| - \sqrt{\mu}\, \frac{\left|\sqrt{c^{(1)}}\right|}{\left|c^{(1)}\right| - \left(u^{(1)}\right)^2} = \rho^{(1)} - \sqrt{\mu}\, \frac{\left|\sqrt{c^{(1)}}\right|}{\left|c^{(1)}\right| - \left(u^{(1)}\right)^2}. \end{aligned}$$

Hence, according to the inequality (25), we find

$$\rho^{(2)} > \rho^{(1)} - \frac{13}{10}\, r^{(1)}.$$

Applying the inequalities (22) and (30), we get

$$\begin{aligned} \rho^{(2)} &> \rho^{(1)} - \frac{13}{10}\, r^{(1)} > 5(N-\mu)r^{(1)} - \frac{13}{10}\, r^{(1)} = \left[5(N-\mu) - \frac{13}{10}\right] r^{(1)} \\ &> 20\left[5(N-\mu) - \frac{13}{10}\right] r^{(2)} > 5(N-\mu) r^{(2)}. \end{aligned}$$

Using the same consideration as for $m = 2$, we prove by induction that the following relations (already proved for $m = 2$) are true for $m \geq 2:$

$$r^{(m+1)} < \frac{28(N-\mu)}{25\left(\rho^{(m)}\right)^2} \left(r^{(m)}\right)^3, \tag{32}$$

$$r^{(m+1)} < \frac{r^{(m)}}{20}, \tag{33}$$

$$\rho^{(m)} > 5(N-\mu) r^{(m)} \tag{34}$$

and

$$\rho^{(m+1)} > \rho^{(m)} - \frac{13}{10}\, r^{(m)}. \tag{35}$$

By successive application of the inequalities (33) and (35), and using the inequalities (23) and (31), we obtain

$$\begin{aligned}\rho^{(m)} &> \rho^{(m-1)} - \frac{13}{10}\, r^{(m-1)} > \cdots > \rho^{(1)} - \frac{13}{10}\, r^{(1)}\Big(1 + \frac{1}{20} + \frac{1}{20^2} + \dots\Big)\\ &\geq \rho^{(1)} - \frac{26}{19}\, r^{(1)} > \frac{9}{25}\, R - \frac{26}{19} \cdot \frac{7}{100}\, R > \frac{13}{50}\, R.\end{aligned}$$

According to this, from the inequality (32) it follows

$$r^{(m+1)} < \frac{17(N-\mu)}{R^2}\big(r^{(m)}\big)^3.$$

We will complete the proof of Theorem 1 providing that the iterative method (14) is defined in each iterative step under the initial condition (16), that is, $0 \notin \{c^{(m)}; r^{(m)}\}$ for each $m = 1, 2, \dots$. Indeed, from the condition (16) the inequality (34) follows for each $m = 1, 2, \dots$ so that Lemma 1 holds in each iteration. □

Remark 1. In the case of simple zero Theorem 1 can be proved under more relaxed condition

$$|\delta_2(a)| > \frac{3(N-1)^2}{2R^2}. \tag{36}$$

In this special case we derive the inequality

$$r^{(m+1)} < \frac{15(N-1)}{R^2}\big(r^{(m)}\big)^3.$$

4 Numerical Examples

The presented algorithm (14) was tested in solving many polynomial equations. To provide the enclosure of the zeros in the presence of rounding errors, we used INTLAB [13], the Matlab toolbox for reliable computing and self-validating algorithms. For the comparison purpose, we also tested the following third order methods for the inclusion of one polynomial zero:

Halley-like method [7]:

$$Z^{(m+1)} = z^{(m)} - \cfrac{1}{f\big(z^{(m)}\big) - \cfrac{P\big(z^{(m)}\big)}{2P'\big(z^{(m)}\big)}\,\cfrac{N(N-\mu)}{\mu}\big(V^{(m)}\big)^2}, \tag{37}$$

where

$$f(z) = \Big(1 + \frac{1}{\mu}\Big)\frac{P'(z)}{2P(z)} - \frac{P''(z)}{2P'(z)}.$$

Euler-like method [9]:

$$Z^{(m+1)} = z^{(m)} - \frac{2\mu}{\delta_1\big(z^{(m)}\big) + \Big[2\mu\delta_2\big(z^{(m)}\big) - \delta_1\big(z^{(m)}\big)^2 + 2N(N-\mu)\big(V^{(m)}\big)^2\Big]_*^{1/2}}. \tag{38}$$

Square-root method [9]:

$$Z^{(m+1)} = z^{(m)} - \frac{3\mu}{\delta_1\big(z^{(m)}\big) + \Big[6\mu\delta_2\big(z^{(m)}\big) - 2\delta_1\big(z^{(m)}\big)^2 + 3(N-\mu)(N-3\mu)\big(V^{(m)}\big)^2\Big]_*^{1/2}}. \tag{39}$$

Third-order method [10]

$$Z^{(m+1)} = z^{(m)} - \mu v\big(z^{(m)}\big) - \frac{\mu v\big(z^{(m)}\big)\Big(A\big(z^{(m)}\big) - v\big(z^{(m)}\big)^2(N-\mu)(N-2\mu)\big(V^{(m)}\big)^2\Big)}{2\big(1 - v\big(z^{(m)}\big)(N-\mu)V^{(m)}\big)^2}, \tag{40}$$

where

$$\delta_1(z) = \frac{P'(z)}{P(z)}, \quad v(z) = \frac{P(z)}{P'(z)} \quad \text{and} \quad A(z) = 1 - \mu + \mu v(z)\,\frac{P''(z)}{P'(z)}.$$

Example 1. To find circular inclusion approximations to a simple zero of the polynomial

$$P(z) = (z-1)(z^2-12z+85)(z^2+12z+100)(z^2-14z+85)(z^2+14z+98) \times (z^4-6561)(z^4-4096),$$

we implemented the interval methods (14), (37), (38), (39) and (40). The zero $\zeta_1 = 1$ of P was isolated in the initial disk $Z_1^{(0)} = \{0.8+0.2i; 6\}$. All zeros of the polynomial P and the initial disk are displayed in the Figure 1.

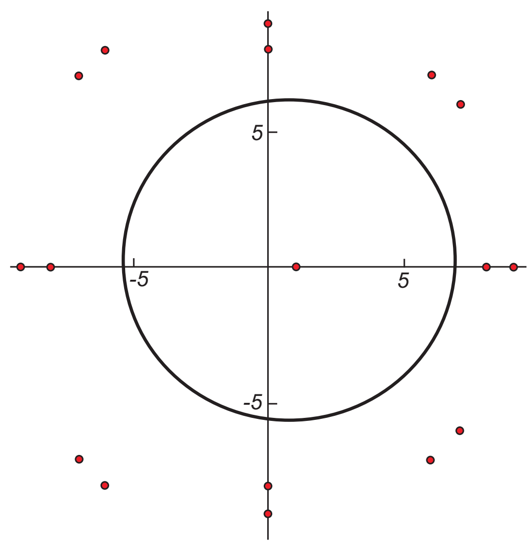

Figure 1: Distribution of zeros of P

The radii of the inclusion disks produced in the first three iterative steps are given in Table 1, where the denotation $A(-q)$ means $A \times 10^{-q}$. In this case the initial condition (36) is satisfied, that is,

$$|\delta_2(a)| = |\delta_2(0.8 + 0.2i)| \approx 12 > \frac{3(N-1)^2}{2R^2} \approx 10.67.$$

Methods	(14)	(37)	(38)	(39)	(40)
$r^{(1)}$	$5.08(-3)$	$9.33(-2)$	$2.23(-1)$	$4.03(-2)$	diverges
$r^{(2)}$	$2.46(-13)$	$9.17(-5)$	$8.55(-3)$	$3.03(-7)$	

Table 1: The maximal radii of inclusion disks – Example 1, $a = 0.8 + 0.2i$

Let us examine what happen in the case when the initial condition (36) is not satisfied. We shifted the starting point $z^{(0)} = 0.8 + 0.2i$ away from the zero $\zeta_1 = 1$ to the point $z_1^{(0)} = \{0.7 + 0.3i\}$. In this case the initial condition (36) is not satisfied since

$$|\delta_2(a)| = |\delta(0.7 + 0.3i)| \approx 5.56 < \frac{3(N-1)^2}{2R^2} \approx 10.67.$$

In spite of that, the method (14) converges. The radii of the inclusion disks in the first three iterative steps are given in Table 2. We observe that the other methods either diverge or run with efforts.

Methods	(14)	(37)	(38)	(39)	(40)
$r^{(1)}$	$1.74(-2)$	$5.21(-1)$	diverges	$2.06(-1)$	diverges
$r^{(2)}$	$9.10(-11)$	$3.46(-1)$		$1.09(-3)$	

Table 2: The maximal radii of inclusion disks – Example 1, $a = 0.7 + 0.3i$

Example 2. To find circular inclusion approximations to a multiple zero of the polynomial

$$P(z) = (z+1)^3(z+6)^3(z-6)^2(z^2+36)^3$$

we applied the same interval methods from Example 1. The isolated multiple zero of P is $\zeta_1 = -1$ with multiplicity $\mu_1 = 3$. The initial disk was selected to be $Z_1^{(0)} = \{-0.8 - 0.2i; 2\}$. In this example the initial condition (16) is not fulfilled, namely,

$$|\delta_2(-0.8 - 0.2i)| \approx 37.51 < \frac{5(N-\mu)^2\mu}{2R^2} \approx 227.$$

The radii of the inclusion disks, produced in the first three iterative steps, are given in Table 3.

From Tables 1, 2 and 3 we observe that theoretical results, concerning the convergence order of the considered method (14), mainly well coincide with the convergence behavior in practice. We recall that the condition (16) (the condition (36) in the case of simple zeros) is only sufficient. Namely, the interval method (14) can converge in practice although the conditions (16) and (36) are not fulfilled, as can be seen from Examples 1 and 2. On the other hand, the validity of the conditions (16) and (36) always guarantees the convergence of the method (14) and the inclusion of desired zero in each iteration.

Methods	(14)	(37)	(38)	(39)	(40)
$r^{(1)}$	1.06(−2)	4.94(−2)	6.64(−2)	8.91(−3)	diverges
$r^{(2)}$	2.80(−11)	1.66(−6)	9.33(−6)	7.35(−12)	

Table 3: The maximal radii of inclusion disks – Example 2, multiple zero

References

[1] G. Alefeld, J. Herzberger, *Introduction to Interval Computations*, Academic Press, New York, 1983.

[2] I. Gargantini, Parallel Laguerre iterations: The complex case, *Numer. Math.*, 26:317–323, 1976.

[3] I. Gargantini, P. Henrici, Circular arithmetic and the determination of polynomial zeros, *Numer. Math.* 18:305–320, 1972.

[4] R. E. Moore, *Interval Analysis*, Prentice Hall, Englewood Cliffs, 1966.

[5] R. E. Moore, R. Baker Kearfott, M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

[6] M. S. Petković, On a generalization of the root iterations for polynomial complex zeros in circular interval arithmetic, *Computing*, 27:37–55, 1981.

[7] M. S. Petković, Some interval iterations for finding a zero of a polynomial with error bounds, *Comput. Math. Appl.* 14:479–495, 1987.

[8] M. S. Petković, *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.

[9] M. Milošević, Iterative methods for approximating polynomial zeros, Doctoral dissertation, University of Niš, 2011.

[10] M. S. Petković, M. R. Milošević, D. M. Milošević, Efficient methods for the inclusion of polynomial zeros, *Appl. Math. Comput.* 217:7636–7652, 2011.

[11] M. S. Petković, D. M. Milošević, L. D. Petković, High order Euler-like method for the inclusion of polynomial zeros, *Appl. Math. Comput.* 196:762–773, 2011.

[12] M. S. Petković, L. D. Petković, *Complex Interval Arithmetic and its Applications*, Wiley-VCH, Berlin-Weinheim-New York, 1998.

[13] S. Rump, INTLAB - Interval laboratory, in: *Developments in Reliable Computing* (Tibor Scendes, ed.), Kluwer Academic Publishers, 1999, pp .77–104.