Higher Order Methods for the Inclusion of Multiple Zeros of Polynomials*

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In memory of Professor Jürgen Herzberger

Abstract

Starting from a suitable fixed point relation, we derive higher order iterative methods for the simultaneous inclusion of polynomial multiple zeros in circular complex interval arithmetic. Each of the resulting disks contain one and only one zero in every iteration. This convenient inclusion property, together with very fast convergence, ranks these methods among the most powerful iterative methods for the inclusion of polynomial zeros. Using the concept of R-order of convergence of mutually dependent sequences, we present the convergence analysis of the total-step and the single-step methods with Schröder's and Halley's corrections under computationally verifiable initial conditions. The proposed self-validated methods possess a great computational efficiency since the acceleration of the convergence rate from four to five and six is achieved without additional calculations. To demonstrate convergence behavior of the presented methods, two numerical examples are given.

Keywords: Zeros of polynomials; multiple zeros; simultaneous methods; inclusion methods; circular interval arithmetic; convergence.
AMS subject classifications: 65H05, 65G20, 30C15.

1 Introduction

In the application of any iterative root finding method a considerable amount of work must be engaged in obtaining rigorous error bounds of the improved approximations

*Submitted: January 18, 2009; Accepted: January 10, 2010.
†This research was supported by the Serbian Ministry of Science under grant number 174022

to the roots. Iterative methods for the simultaneous determination of complex zeros of a given polynomial, realized in complex interval arithmetic, are very efficient device to error estimates for a given set of approximate zeros. This kind of self-validated methods possess the important inclusion property, meaning that the obtained interval approximations enclose the sought zeros in each iteration. The aim of this paper is continuation of the research concerned with fast iterative methods for the simultaneous inclusion of polynomial zeros, which can be regarded as interval versions of Halley-like iterative method presented in [16] and [26] and discussed later in [15], [17] and [21].

Starting from some initial disks that contain multiple complex zeros of a polynomial and a suitable fixed point relation, we construct interval methods for the refinement of these disks. The improvement of these inclusion disks, in the sense of their contraction, is performed by higher order iterative procedures using circular complex interval arithmetic. The acceleration of convergence is attained by applying the centered inversion of disks and using Schröder's and Halley's corrections. Further improvement can be achieved employing Gauss-Seidel approach dealing with already calculated disks in the current iteration as soon as they are available.

The presentation of the paper is organized as follows. Some basic definitions and operations of circular complex interval arithmetic, necessary for the construction and the convergence analysis of inclusion methods, are given at the end of Introduction. The modified total-step methods with the increased convergence speed are developed in Section 2 using Schröder's and Halley's corrections. We give the convergence analysis in Section 3 and discuss the single-step versions of these methods in Section 4. Section 5 contains numerical examples.

The development and convergence analysis of the proposed inclusion methods need the basic properties of the so-called circular complex arithmetic introduced by Gargantini and Henrici [4]. A circular closed region (disk) $Z := \{z : |z-c| \le r\}$ with center $c :=$ mid Z and radius $r :=$ rad Z will be denoted in parametric notation by $Z := \{c; r\}$. If $Z_k := \{c_k; r_k\}$ $(k = 1, 2)$, then

$$
\begin{aligned}
Z_1 \pm Z_2 &= \{c_1 \pm c_2; r_1 + r_2\},\\
w \cdot \{c; r\} &= \{wc; |w|r\} \quad (w \in \mathcal{C}),\\
Z_1 \cdot Z_2 &= \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\},\\
\{c; r\}^{-1} &= \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (0 \notin \{c; r\}), \quad \text{(exact inversion)}, && (1.1)\\
\{c; r\}^I &= \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \quad (0 \notin \{c; r\}), \quad \text{(centered inversion)}, && (1.2)\\
Z_1 : Z_2 &= Z_1 \cdot Z_2^{-1} \quad \text{or} \quad Z_1 : Z_2 = Z_1 \cdot Z_2^I \quad (0 \notin Z_2).
\end{aligned}
$$

For the basic interval operations $+, -, \cdot, :$ there holds the *inclusion property*, that is,

$$Z_k \subseteq W_k \Rightarrow Z_1 * Z_2 \subseteq W_1 * W_2 \quad (k = 1, 2; * \in \{+, -, \cdot, :\}).$$

An interval function F is called *complex circular extension* of a complex function f if

$$F(z) = f(z), \quad (z \in Z), \quad F(Z) \supseteq \{f(z) :\ z \in Z\}.$$

If f is a rational function and F is its complex circular extension, then

$$Z_k \subseteq W_k \ (k = 1, \dots, q) \Rightarrow F(Z_1, \dots, Z_q) \subseteq F(W_1, \dots, W_q).$$

In particular, we have

$$w_k \in W_k \ (k = 1, \dots, q;\ w_k \in \mathcal{C}) \Rightarrow f(w_1, \dots, w_q) \in F(W_1, \dots, W_q).$$

In this paper we will use the following obvious properties:

$$z \in \{c; r\} \iff |z - c| \le r. \tag{1.3}$$

$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \iff |c_1 - c_2| > r_1 + r_2. \tag{1.4}$$

More details about circular arithmetic can be found in the books [1], [15] and [24]. Throughout this paper disks in the complex plane will be denoted by capital letters.

2 Total-step method with corrections

Let f be a monic polynomial of degree n with simple or multiple complex zeros $\zeta_1, \dots, \zeta_\nu$ $(2 \le \nu \le n)$, with respective multiplicities $\mu_1, \dots, \mu_\nu$ $(\mu_1 + \cdots + \mu_\nu = n)$ and let

$$\begin{aligned} \Delta_{0,i} &= 1, \\ \Delta_{k,i}(z) &= \sum_{\nu=1}^{k} (-1)^{k-\nu} \frac{1}{\mu_i}\Big(\frac{1}{\mu_i} + 1\Big) \dots \Big(\frac{1}{\mu_i} + \nu - 1\Big) \sum \prod_{\lambda=1}^{k} \frac{1}{p_\lambda !}\Big(\frac{f^{(\lambda)}(z)}{\lambda! f(z)}\Big)^{p_\lambda}, \end{aligned}$$

where $k = 1, 2, \dots$ and the second sum on the right-hand side runs over all nonnegative integers $(p_1, \dots, p_k)$ which satisfy $p_1 + 2p_2 + \cdots + kp_k = k$, $p_1 + p_2 + \cdots + p_k = \nu \in \{1, \dots, k\}$. For example, we have

$$\Delta_{1,i}(z) = \frac{1}{\mu_i}\frac{f'(z)}{f(z)}, \quad \Delta_{2,i}(z) = \frac{1}{2\mu_i}\Big(\frac{1}{\mu_i} + 1\Big)\Big(\frac{f'(z)}{f(z)}\Big)^2 - \frac{1}{2\mu_i}\frac{f''(z)}{f(z)}.$$

We observe that the function $N_i(z) = \dfrac{\Delta_{0,i}(z)}{\Delta_{1,i}(z)} = \mu_i \dfrac{f(z)}{f'(z)}$ appears in the Schröder iterative method $\hat{z} = z - N_i(z)$ of the second order and

$$H_i(z) = \frac{\Delta_{1,i}(z)}{\Delta_{2,i}(z)} = \left(\Big(\frac{1 + 1/\mu_i}{2}\Big)\frac{f'(z)}{f(z)} - \frac{f''(z)}{2f'(z)}\right)^{-1}$$

occurs in cubically convergent Halley's iterative formula $\hat{z} = z - H_i(z)$.

In our consideration we will use the abbreviations

$$q_{\lambda,i} := \sum_{j=1}^{\nu} \frac{\mu_j}{(z - z_j)^\lambda}, \quad \sigma_{\lambda,i} := \sum_{\substack{j=1 \\ j \ne i}}^{\nu} \frac{\mu_j}{(z - \zeta_j)^\lambda} \quad (\lambda = 1, 2).$$

The following fixed point relation was derived in [26]:

$$\zeta_i = z - \frac{1}{H_i(z)^{-1} - \dfrac{f(z)}{2f'(z)}\Big(\dfrac{1}{\mu_i}\sigma_{1,i}^2 + \sigma_{2,i}\Big)}. \tag{2.1}$$

Let us define the disk

$$S_{\lambda,i}(\boldsymbol{X}, \boldsymbol{Y}) := \sum_{j=1}^{i-1} \mu_j \Big((z - X_j)^I\Big)^\lambda + \sum_{j=i+1}^{\nu} \mu_j \Big((z - Y_j)^I\Big)^\lambda \quad (\lambda = 1, 2),$$

where $\boldsymbol{X} = (X_1, \ldots, X_\nu)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_\nu)$ are vectors whose components are disks. Taking disks $Z_1, \ldots, Z_\nu$ containing the zeros $\zeta_1, \ldots, \zeta_\nu$ instead of these zeros and taking $z = z_i :=$ mid Z_i in (2.1), we obtain the following inclusion,

$$\zeta_i \in z_i - \mathrm{INV}\Big(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i} S_{1,i}^2(\boldsymbol{Z},\boldsymbol{Z}) + S_{2,i}(\boldsymbol{Z},\boldsymbol{Z})\Big]\Big), \tag{2.2}$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_\nu)$ and $\mathrm{INV} \in \{()^{-1}, ()^I\}$. To indicate the type of inversion that we use, denoting some quantities we will add the superscript indices "e" (exact inversion) and "c" (centered inversion).

The convergence speed of iterative interval methods will be estimated in this paper by the concept of the R-order of convergence. Consider the sequences of inclusion disks $\{Z_i^{(m)}\}$ $(i = 1, \ldots, k)$ produced by the iterative interval method IM such that rad $Z_i^{(m)} \to \zeta_i$ $(i = 1, \ldots, k)$. Let $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$ be the vector of polynomial zeros and let $\{r^{(m)}\}$ be the sequence of maximal radii of disks $Z_i^{(m)}$. The concept of R-order of convergence, introduced by Ortega and Rheinboldt [14], is defined using the R factor

$$R_q\{r^{(m)}\} = \begin{cases} \lim\limits_{m\to\infty} \sup \big(r^{(m)}\big)^{1/q}, & q = 1 \\ \lim\limits_{m\to\infty} \sup \big(r^{(m)}\big)^{1/q^m}, & q > 1 \end{cases}$$

in the case of interval methods (see [1, Appendix A]). Then the R-order of convergence of an interval method IM is defined as

$$O_R(IM, \boldsymbol{\zeta}) = \begin{cases} +\infty & \text{if } R_q(IM,\boldsymbol{\zeta}) = 0 \text{ for } q \ge 1 \\ \inf\{q : q \in [1,\infty),\ R_q(IM,\boldsymbol{\zeta}) = 1\} & \text{otherwise.} \end{cases}$$

In particular, if there exists a $q \ge 1$ and a constant γ such that the inequalities

$$r^{(m+1)} \le \gamma\big(r^{(m)}\big)^q \quad (m \ge m_0)$$

hold for all sequences $\{Z_i^{(m)}\}$, then $O_R(IM, \boldsymbol{\zeta}) \ge q$ (see [1, Appendix A]).

Let $(Z_1, \ldots, Z_\nu) := (Z_1^{(0)}, \ldots, Z_\nu^{(0)})$ be initial disjoint disks containing the zeros $\zeta_1, \ldots, \zeta_\nu$, that is, $\zeta_i \in Z_i^{(0)}$ for all i, and let $z_i =$ mid Z_i. The relation (2.2) suggests the following *total-step* method for the simultaneous inclusion of all zeros of f:

$$\hat{Z}_i = z_i - \mathrm{INV}\Big(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i} S_{1,i}^2(\boldsymbol{Z},\boldsymbol{Z}) + S_{2,i}(\boldsymbol{Z},\boldsymbol{Z})\Big]\Big). \tag{2.3}$$

The iterative method (2.3) has the order of convergence equal to *four* (see [15]). The convergence of this method can be accelerated by applying the Gauss-Seidel approach. In this manner we obtain the *single-step* method

$$\hat{Z}_i = z_i - \mathrm{INV}\Big(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i} S_{1,i}^2(\widehat{\boldsymbol{Z}},\boldsymbol{Z}) + S_{2,i}(\widehat{\boldsymbol{Z}},\boldsymbol{Z})\Big]\Big). \tag{2.4}$$

The R-order of convergence of the single-step method (2.4) is at least $3 + x_n$, where $x_n > 1$ is the unique positive root of the equation $x^n - x - 3 = 0$ (for the proof see [15]).

Remark 1 Evidently, the main part in the iteration formulas (2.3) and (2.4) is Halley's correction $H(z)$. For this reason, these methods as well as their modifications which will be considered in this paper are referred to as *Halley-like* methods.

Remark 2 At first sight, it seems unreasonable to use the centered inversion in the computation of sums $S_{\lambda,i}$ since the exact inversion gives smaller disks. In fact, the application of the centered inversion produces centers of the resulting disks $\hat{Z}_i$, which coincide with the very fast iterative methods (in ordinary complex arithmetic). These fast methods significantly force contraction of the disks which leads to the accelerated convergence of interval methods. On the other hand, the exact inversion gives "shifted" centers of the inverted disks. For this reason, the use of exact inversion can accelerate convergence to a certain extent when the Schröder corrections are used and cannot increase the convergence rate applying corrections that appear in iterative methods of the order higher than two (see [17] for a detailed analysis).

Remark 3 The iterative formulas (2.3), (2.4) and those developed later require initial disks that contain the desired zeros and the knowledge of their multiplicities in advance. Both tasks are very important in the theory of iterative interval processes. The problem of obtaining initial disks containing the desired zeros was studied, for instance, in [2], [7] and [19], while efficient procedures for determination of the order of multiplicity can be found in [8], [9], [10], [11], [12], [13] and [22].

Let us introduce the abbreviations

$$\begin{aligned} r^{(m)} &= \max_{1\le i\le \nu} r_i^{(m)}, \quad \eta^{(m)} = \min_{\substack{i,j\\ i\neq j}} \{|z_i^{(m)} - z_j^{(m)}| - r_j^{(m)}\}, \\ \varepsilon_i^{(m)} &= z_i^{(m)} - \zeta_i, \quad \epsilon^{(m)} = \max_{1\le i\le \nu} |\varepsilon_i^{(m)}|, \quad \mu^{(m)} = \min_{1\le i\le \nu} \mu_i^{(m)}, \end{aligned}$$

where $m = 0, 1, 2, \ldots$ is the iteration index. The increase of convergence speed of iteration methods (2.3) and (2.4) can be attained with Schröder's correction $N_i(z_i)$ or Halley's correction $H_i(z_i)$, similarly as in [3] and [18], and later in [20], [21] and [23]. In this construction we assume the choice of initial inclusion disks $Z_1^{(0)}, \ldots, Z_\nu^{(0)}$ containing the zeros $\zeta_1, \ldots, \zeta_\nu$ in such a way that each disk $Z_i^{(0)} - N_i\big(\text{mid } (Z_i^{(0)})\big)$ and $Z_i^{(0)} - H_i\big(\text{mid } (Z_i^{(0)})\big)$ also contains the zero ζ_i $(i = 1, \ldots, \nu)$. This issue is considered in the following assertion where, for simplicity, the iteration indices are omitted.

Lemma 2.1 *Let $Z_1, \ldots, Z_\nu$ be inclusion disks of the zeros $\zeta_1, \ldots, \zeta_\nu$, $\zeta_i \in Z_i$. If the inclusion disks $Z_1, \ldots, Z_\nu$ are chosen so that the inequality*

$$\eta > 3(n - \mu) r \tag{2.5}$$

is satisfied, then

(i) $\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_{N,i} := Z_i - N_i(z_i) \quad (i = 1, \ldots, \nu)$;

(ii) $\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_{H,i} := Z_i - H_i(z_i) \quad (i = 1, \ldots, \nu)$.

Proof. Of (i): By virtue of (1.3) we have to prove the implication

$$|z_i - \zeta_i| \le r_i \Rightarrow |z_i - N_i(z_i) - \zeta_i| \le r_i.$$

Since

$$|z_i - \zeta_j| \ge |z_i - z_j| - |z_j - \zeta_j| \ge |z_i - z_j| - r_j \ge \eta,$$

we find

$$|\sigma_{1,i}| = \left| \sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j}{(z_i - \zeta_j)^k} \right| \le \sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j}{|z_i - \zeta_j|^k} \le \frac{n - \mu_i}{\eta^k}, \quad k = 1, 2.$$

According to this and (2.5) we have

$$r_i < \frac{\eta}{3(n-\mu)} < \frac{1}{3|\sigma_{1,i}|},$$

wherefrom there follows

$$\frac{r_i|\sigma_{1,i}|}{\mu_i - r_i|\sigma_{1,i}|} < \frac{1}{2}.$$

Since $q_{1,i} = 1/\varepsilon_i + \sigma_{1,i}$, using the last inequality we get

$$\begin{aligned} |z_i - N(z_i) - \zeta_i| &= |\varepsilon_i - N(z_i)| = |\varepsilon_i - 1/q_{1,i}| = |\varepsilon_i|^2 \left|\frac{\sigma_{1,i}}{1+\varepsilon_i\sigma_{1,i}}\right| \\ &\le \frac{r_i^2|\sigma_{1,i}|}{1 - r_i|\sigma_{1,i}|} < r_i. \end{aligned}$$

Of (ii) Similarly as in the proof of (i) we should prove the implication

$$|z_i - \zeta_i| \le r_i \Rightarrow |z_i - H_i(z_i) - \zeta_i| \le r_i.$$

First, we find

$$\begin{aligned} H(z_i) &= \frac{2\delta_{1,i}}{\dfrac{1}{\mu_i}\delta_{1,i}^2 + \delta_{2,i}} = \frac{2\displaystyle\sum_{j=1}^{\nu}\frac{\mu_j}{z_i-\zeta_j}}{\dfrac{1}{\mu_i}\Big(\displaystyle\sum_{j=1}^{\nu}\frac{\mu_j}{z_i-\zeta_j}\Big)^2 + \displaystyle\sum_{j=1}^{\nu}\frac{\mu_j}{(z_i-\zeta_j)^2}} \\ &= \frac{2(\mu_i/\varepsilon_i + \sigma_{1,i})}{(\mu_i/\varepsilon_i+\sigma_{1,i})^2/\mu_i + \mu_i/\varepsilon_i^2 + \sigma_{2,i}} = \frac{2\varepsilon_i(\mu_i + \varepsilon_i\sigma_{1,i})}{2\mu_i + 2\varepsilon_i\sigma_{1,i} + \varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \Sigma_{2,i})}. \end{aligned}$$

Hence

$$\begin{aligned} z_i - H_i(z_i) - \zeta_i &= \varepsilon_i - \frac{2\varepsilon_i(\mu_i+\varepsilon_i\sigma_{1,i})}{2\mu_i + 2\varepsilon_i\sigma_{1,i} + \varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})} \\ &= \frac{\varepsilon_i^3(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}{2\mu_i + 2\varepsilon_i\sigma_{1,i} + \varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}, \end{aligned}$$

so that

$$\begin{aligned} |z_i - H_i(z_i) - \zeta_i| &= \left|\frac{\varepsilon_i^3(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}{2\mu_i + 2\varepsilon_i\sigma_{1,i} + \varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}\right| \\ &< \frac{\dfrac{1}{\mu_i}\Big(\dfrac{n-\mu_i}{\eta}\Big)^2 + \dfrac{n-\mu_i}{\eta^2}}{2\mu_i - 2\dfrac{n-\mu_i}{\eta}r_i - \dfrac{1}{\mu_i}\Big(\dfrac{n-\mu_i}{\eta}\Big)^2 r_i^2 - \dfrac{n-\mu_i}{\eta^2}r_i^2}\cdot r_i^3 \\ &= \frac{\dfrac{(n-\mu_i)nr_i^2}{\mu_i\eta^2}}{\dfrac{4}{3} - \dfrac{n(n-\mu_i)r_i^2}{\mu_i\eta^2}}\cdot r_i < \frac{\dfrac{n}{9\mu_i(n-\mu_i)}}{\dfrac{4}{3} - \dfrac{n}{9\mu_i(n-\mu_i)}}\cdot r_i < r_i. \quad \square \end{aligned}$$

Starting from the fixed point relation (2.1) we can construct the total-step Halley-like inclusion method with Schröder's and Halley's corrections. Studying the convergence analysis of these methods, we will consider both methods simultaneously.

For this purpose we indicate these methods with the additional indices $\lambda = 1$ (for Schröder's correction) and $\lambda = 2$ (for Halley's correction) and, in the same manner, we denote the corresponding vectors of approximations as follows:

$$\begin{aligned}\boldsymbol{Z}^{(1)} &= \big(Z_1^{(1)}, \ldots, Z_\nu^{(1)}\big) = \big(Z_{N,1}, \ldots, Z_{N,\nu}\big)\\ \boldsymbol{Z}^{(2)} &= \big(Z_1^{(2)}, \ldots, Z_\nu^{(2)}\big) = \big(Z_{H,1}, \ldots, Z_{H,\nu}\big).\end{aligned}$$

Both corrections $N(z_i)$ and $H(z_i)$ will be also denoted by $C^{(\lambda)}(z_i)$. For simplicity, we will omit the iteration index for all quantities at the m-th iteration, while the quantities at the $(m+1)$-st iteration will be denoted with the additional symbol ˆ ("hat"). Such denotation will eliminate possible confusion about the iteration index and the index of methods with corrections. Now we are able to express both methods in the unique form as

$$\hat{Z}_i = z_i - \mathrm{INV}\bigg(H(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\big[S_{1,i}^2(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)}) + S_{2,i}(\boldsymbol{Z}^{(\lambda)}, \boldsymbol{Z}^{(\lambda)})\big]\bigg) \tag{2.6}$$

for $i = 1, \ldots, \nu$ and $\mathrm{INV} \in \{()^{-1}, ()^I\}$.

3 Convergence analysis

Let IM be an iterative numerical method which generates k sequences $\{z_1^{(m)}\}, \ldots, \{z_k^{(m)}\}$ for the approximation of solutions $z_1^*, \ldots, z_k^*$. In order to estimate the order of convergence of the iterative method IM we introduce the error-sequences

$$\varepsilon_i^{(m)} = |z_i^{(m)} - z_i^*| \quad (i = 1, \ldots, k).$$

The order of convergence of inclusion methods with corrections can be suitably determined using the following assertion (see Theorem 3 in [6]:

Theorem 3.1 *Given the error-recursion*

$$\varepsilon_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k} \big(\varepsilon_j^{(m)}\big)^{p_{ij}}, \quad (i = 1, \ldots, k;\ m \ge 0), \tag{3.1}$$

where $p_{ij} \ge 0,\ \alpha_i > 0,\ 1 \le i, j \le k$. Denote the matrix of exponents appearing in (3.1) with P, that is $P = [p_{ij}]_{k\times k}$. If the non-negative matrix P has the spectral radius $\rho(P) > 1$ and a corresponding eigenvector $\boldsymbol{x}_\rho > 0$, then all sequences $\{\varepsilon_i^{(m)}\}$ $(i = 1, \ldots, k)$ have at least the R-order $\rho(P)$.

The matrix $P_k = [p_{ij}]$, concerned with the R-order of convergence, is usually called the *R-matrix.* In the case of interval methods one may take $\varepsilon_j^{(m)} = \mathrm{rad}\ Z_j^{(m)}$ for some indices j from the set $\{1, \ldots, k\}$.

In what follows we will write $w_1 \sim w_2$ or $w_1 = \mathcal{O}_M(w_2)$ (the same order of magnitudes) for two complex numbers w_1 and w_2 that satisfy $|w_1| = \mathcal{O}(|w_2|)$. Let $O_R(IM)$ denote the R-order of convergence of an iteration method IM. For the total-step method (2.6) we can state

Theorem 3.2 *Assume that initial disks $Z_1^{(0)}, \ldots, Z_\nu^{(0)}$ are chosen so that $\zeta_i \in Z_i^{(0)}$ $(i = 1, \ldots, \nu)$ and the inequality*

$$\eta^{(0)} > 3(n-\mu)r^{(0)} \tag{3.2}$$

holds. Then the inclusion method (2.6) *is convergent and the following is true for each* $i = 1, \ldots, \nu$ *and* $m = 1, 2, \ldots$:

1° $\eta^{(m)} > 3(n-\mu)r^{(m)}$;

2° $\zeta_i \in Z_i^{(m)}$ *for each* $i = 1, \ldots, \nu$ *and* $m = 1, 2, \ldots$;

3° *the lower bound of the R-order of convergence of the interval method* (2.6) *is* $O_R(2.6) \geq \lambda + 4$.

Proof. Let us note at the beginning that Theorem 3.2 states that iterative method (2.6) has the order of convergence 5 (the method with Schröder's corrections) and 6 (the method with Halley's corrections). This increase of the convergence rate is forced by the very fast convergence of the sequences $\{z_i^{(m)}\}$ of centers of the disks produced by (2.6). Indeed, taking $Z_i^{(m)} = \{z_i^{(m)}; 0\}$ in (2.6) we obtain Halley-like *fifth* and *sixth* order method in ordinary complex arithmetic (see [25]).

An interval method will be well defined if the inclusion disks are disjoint in every iteration. The condition (3.2) provides that the initial disks $Z_1^{(0)}, \ldots, Z_\nu^{(0)}$ are nonintersecting; indeed, for an arbitrary pair $i, j \in \{1, \ldots, \nu\}$ $(i \neq j)$ we have

$$|z_i^{(0)} - z_j^{(0)}| > \eta^{(0)} > 3(n-\mu)r^{(0)} > 2r^{(0)} \geq r_i^{(0)} + r_j^{(0)},$$

which means that $Z_i^{(0)} \cap Z_j^{(0)} = \emptyset$ (according to (1.4)).

The assertions of Theorem 3.2 will be derived by induction. In our estimation procedures we will often use the inequality (3.2) in the form

$$\frac{r}{\eta} < \frac{1}{3(n-\mu)} \leq \frac{1}{6}, \tag{3.3}$$

sometimes without citing.

Setting $m = 0$ and having in mind the initial condition (3.2), in regard to Lemma 2.1 we immediately obtain the implication

$$\zeta_i \in Z_i \Rightarrow \zeta_i \in Z_i^{(\lambda)} := Z_i - C^{(\lambda)}(z_i) \quad (i = 1, \ldots, \nu).$$

We now prove that the new inclusion disks $Z_1^{(\lambda)}, \ldots, Z_\nu^{(\lambda)}$ are also disjoint. Using some bounds from the proof of Lemma 2.1 we find

$$|N(z_i)| = \left|\frac{\varepsilon_i}{\mu_i + \varepsilon_i \sigma_{1,i}}\right| < \frac{r_i}{\mu_i - r_i|\sigma_{1,i}|} < 2r_i \leq 2r,$$

and

$$|H(z_i)| \leq \frac{1}{\left|1 + \dfrac{\varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}{2(\mu_i + \varepsilon_i \sigma_{1,i})}\right|} \cdot r_i < \frac{1}{1 - \dfrac{1}{8}} \cdot r_i = \frac{8}{7} r_i < 2r_i \leq 2r,$$

because of

$$\left|\frac{\varepsilon_i^2(\sigma_{1,i}^2/\mu_i + \sigma_{2,i})}{2(\mu_i + \varepsilon_i \sigma_{1,i})}\right| < \frac{r^2\Big(\Big(\dfrac{n-\mu_i}{\eta}\Big)^2 + \dfrac{n-\mu_i}{\eta^2}\Big)}{2\Big(\mu_i - r\dfrac{n-\mu_i}{\eta}\Big)} = \frac{\dfrac{(n-\mu_i)nr^2}{\eta^2}}{2\Big(\mu_i - \dfrac{(n-\mu_i)r}{\eta}\Big)} < \frac{1}{8}.$$

According to the bounds of $|N(z_i)|$ and $|H(z_i)|$, we obtain

$$\begin{aligned}\left|\text{mid } Z_i^{(\lambda)} - \text{mid } Z_j^{(\lambda)}\right| &= \left|z_i - C^{(\lambda)}(z_i) - z_j + C^{(\lambda)}(z_j)\right| \\ &\geq |z_i - z_j| - \left|C^{(\lambda)}(z_i)\right| - \left|C^{(\lambda)}(z_j)\right| \\ &> \eta - 4r > 3(n-\mu)r - 4r \geq r_i + r_j.\end{aligned}$$

Thus, due to (1.4) we have $Z_{N,i} \cap Z_{N,j} = \emptyset$ and $Z_{H,i} \cap Z_{H,j} = \emptyset$ $(i \neq j)$.

It is not difficult to show that

$$z_i - Z_j + C^{(\lambda)}(z_j) = \left\{z_i - \zeta_j + \xi_j^{(\lambda)} \varepsilon_j^{\lambda+1}; r_j\right\},$$

where

$$\begin{aligned}\xi_j^{(1)} &= -\frac{\sigma_{1,j}}{\mu_j + \varepsilon_j \sigma_{1,j}} = \mathcal{I}_n \mathcal{O}_M(1), \\ \xi_j^{(2)} &= -\frac{\sigma_{1,j}^2/\mu_j + \sigma_{2,j}}{2\mu_j + 2\varepsilon_j\sigma_{1,j} + \varepsilon_j^2(\sigma_{1,j}^2/\mu_j + \sigma_{2,j})} = \mathcal{O}_M(1).\end{aligned}$$

Let $h_{ij} := \text{mid } \left(z_i - Z_j + C^{(\lambda)}(z_j)\right) = z_i - \zeta_j + \xi_j^{(\lambda)} \varepsilon_j^{\lambda+1}$. In the proof of Lemma 2.1 we have already estimated

$$|z_i - \zeta_j| > \eta \quad \text{and} \quad |\sigma_{1,j}| < \frac{n-\mu}{\eta},$$

so that

$$\left|\xi_j^{(1)}\right| < \frac{|\sigma_{1,j}|}{\mu_j - |\varepsilon_j||\sigma_{1,j}|} < \frac{\dfrac{n-\mu_j}{\eta}}{\mu_j - \dfrac{(n-\mu_j)r}{\eta}} < \frac{3(n-\mu_j)}{2\eta}$$

and

$$\begin{aligned}\left|\xi_j^{(2)}\right| &\leq \frac{\dfrac{|\sigma_{1,j}|^2}{\mu_j} + |\sigma_{2,j}|}{2\mu_j - 2|\varepsilon_j||\sigma_{1,j}| - |\varepsilon_j|^2\Big(\dfrac{|\sigma_{1,j}|^2}{\mu_j} + |\sigma_{2,j}|\Big)} \\ &\leq \frac{\dfrac{(n-\mu_j)n}{\mu_j\eta^2}}{2\mu_j - 2r\dfrac{n-\mu_j}{\eta} - r^2\dfrac{(n-\mu_j)n}{\mu_j\eta^2}} < \frac{6}{7}\frac{(n-\mu_j)n}{\mu_j\eta^2},\end{aligned}$$

for all $j = 1, \ldots, \nu$.

According to this we bound

$$\left|h_{ij}^{(1)}\right| = \left|z_i - \zeta_j + \xi_j^{(1)}\varepsilon_j^2\right| > |z_i - \zeta_j| - \left|\xi_j^{(1)}\right||\varepsilon_j|^2 > \eta - \frac{3(n-\mu_j)r^2}{2\eta} > \eta - \frac{r}{2} \quad (3.4)$$

and

$$\left|h_{ij}^{(2)}\right| = \left|z_i - \zeta_j + \xi_j^{(2)}\varepsilon_j^3\right| > |z_i - \zeta_j| - \left|\xi_j^{(2)}\right||\varepsilon_j|^3 > \eta - \frac{6(n-\mu_j)r^3}{7\mu_j\eta^2} > \eta - \frac{r}{2}. \quad (3.5)$$

Hence

$$\left|h_{ij}^{(\lambda)}\right|\left(\left|h_{ij}^{(\lambda)}\right| - r_j\right) > \left(\eta - \frac{r}{2}\right)\left(\eta - \frac{3r}{2}\right) = \eta^2\left(1 - \frac{r}{2\eta}\right)\left(1 - \frac{3r}{2\eta}\right) > \frac{11}{16}\eta^2 > \frac{3}{5}\eta^2,$$

which yields

$$\frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)} < \frac{5r}{3\eta^2}. \tag{3.6}$$

Using (3.6) we estimate

$$\sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)} < \frac{5(n-\mu_i)r}{3\eta^2}. \tag{3.7}$$

Let us introduce $u_{ij}^{(\lambda)} = \dfrac{1}{h_{ij}^{(\lambda)}}$, $\;b_i^{(\lambda)} = \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j u_{ij}^{(\lambda)}$. By (3.2), (3.4) and (3.5) we get

$$|u_{ij}^{(\lambda)}| = \frac{1}{|h_{ij}^{(\lambda)}|} < \frac{1}{\eta - \frac{r}{2}} < \frac{12}{11\eta} \tag{3.8}$$

and

$$|b_i^{(\lambda)}| \le \sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j |u_{ij}^{(\lambda)}| < \frac{12(n-\mu_i)}{11\eta}. \tag{3.9}$$

Then, by (3.7), we obtain the inclusion

$$\begin{aligned} S_{1,i} &= \sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j}{z_i - Z_j + C^{(\lambda)}(z_j)} = \sum_{\substack{j=1\\ j\neq i}}^{\nu} \frac{\mu_j}{\{h_{ij}^{(\lambda)}; r_j\}} \\ &= \sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j \Big\{ u_{ij}^{(\lambda)}; \frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)} \Big\} \subset \Big\{ b_i^{(\lambda)}; \frac{5(n-\mu_i)r}{3\eta^2} \Big\}. \end{aligned}$$

Now, using circular interval operations, we find

$$\begin{aligned} S_{1,i}^2 &\subset \Big\{ \big(b_i^{(\lambda)}\big)^2; 2|b_i^{(\lambda)}| \frac{5(n-\mu)r}{3\eta^2} + \Big(\frac{5(n-\mu)r}{3\eta^2}\Big)^2 \Big\} \\ &\subset \big\{ \big(b_i^{(\lambda)}\big)^2; \gamma_1 r \big\}, \quad \gamma_1 = \frac{41(n-\mu)^2}{10\eta^3}. \end{aligned}$$

Applying (3.7) and (3.8), we get

$$\begin{aligned} S_{2,i} &= \sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j \Big(\frac{1}{z_i - Z_j + C^{(\lambda)}(z_j)} \Big)^2 = \sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j \Big\{ u_{ij}^{(\lambda)}; \frac{r_j}{|h_{ij}^{(\lambda)}|\big(|h_{ij}^{(\lambda)}| - r_j\big)} \Big\}^2 \\ &\subset \Big\{ \sum_{\substack{j=1\\ j\neq i}}^{\nu} \mu_j \big(u_{ij}^{(\lambda)}\big)^2; \gamma_2 r \Big\}, \quad \gamma_2 = \frac{41(n-\mu)}{10\eta^3}. \end{aligned}$$

Using the introduced abbreviations, the iterative method (2.6) can be written as

$$\hat{Z}_i = z_i - 2q_{1,i}\mathrm{INV}\Big(q_{1,i}^2 - S_{1,i}^2 + q_{2,i} - S_{2,i} \Big) \quad (i = 1, \ldots, \nu). \tag{3.10}$$

Now, we find some necessary enclosures. Since $q_{1,i} = \mu_i/\varepsilon_i + \sigma_{1,i}$ and $q_{2,i} = \mu_i^2/\varepsilon_i^2 + \sigma_{2,i}$, we have

$$q_{1,i}^2 - S_{1,i}^2 \subset \Big\{ \frac{\mu_i^2}{\varepsilon_i^2} + \frac{2\mu_i}{\varepsilon_i}\sigma_{1,i} + \sigma_{1,i}^2 - \big(b_i^{(\lambda)}\big)^2; \gamma_1 r \Big\} =: A_{1,i}^{(\lambda)}$$

and

$$q_{2,i} - S_{2,i} \subset \Big\{ \frac{\mu_i^2}{\varepsilon_i^2} + \sigma_{2,i} - \sum_{\substack{j=1 \\ j\neq i}}^{\nu} \mu_j \big(u_{ij}^{(\lambda)}\big)^2; \gamma_2 r \Big\} =: A_{2,i}^{(\lambda)}.$$

Then, for INV $= ()^I$ we obtain from (3.10)

$$Z_i \subset \hat{D}_i^{(c)} = z_i - \frac{2(\sigma_{1,i} + \mu_i/\varepsilon_i)}{A_{1,i}^{(\lambda)} + A_{2,i}^{(\lambda)}}.$$

Hence

$$Z_i = z_i - d_i \varepsilon_i \left\{ \frac{1}{d_i + w_i^{(\lambda)}}; \frac{|\varepsilon_i|^2 \gamma r}{\big|d_i + w_i^{(\lambda)}\big| \big(\big|d_i + w_i^{(\lambda)}\big| - \gamma |\varepsilon_i|^2 r\big)\} } \right\}, \tag{3.11}$$

where

$$d_i = 2(\sigma_{1,i}\varepsilon_i + \mu_i), \quad \gamma = \gamma_1 + \gamma_2 = \frac{41n(n-\mu)}{10\eta^3}$$

and

$$w_i^{(\lambda)} = \varepsilon_i^2 \big(\sigma_{1,i}^2 - \big(b_i^{(\lambda)}\big)^2\big) + \varepsilon_i^2 \Big(\sigma_{2,i} - \sum_{\substack{j=1 \\ j\neq i}}^{\nu} \mu_j \big(u_{ij}^{(\lambda)}\big)^2 \Big). \tag{3.12}$$

The superscript index "c" points to the application of the centered inversion, INV $= ()^I$.

Using the estimates (3.8), (3.9) and the bound $|\sigma_{k,i}| \le (n-\mu)/\eta^k$ $(k = 1, 2)$, we find

$$\begin{aligned} \big|d_i + w_i^{(\lambda)}\big| &> 2\mu_i - 2|\sigma_{1,i}||\varepsilon_i| - |\varepsilon_i|^2 \Big(|\sigma_{1,i}|^2 + \big|b_i^{(\lambda)}\big|^2 + |\sigma_{2,i}| + \sum_{\substack{j=1 \\ j\neq i}}^{\nu} \mu_j \big|u_{ij}^{(\lambda)}\big|^2 \Big) \\ &> 2 - \frac{2}{3} - \frac{265(n-\mu+1)(n-\mu)}{121} \frac{r^2}{\eta^2}, \end{aligned}$$

wherefrom, in view of (3.2),

$$\big|d_i + w_i^{(\lambda)}\big| > \frac{9}{10}. \tag{3.13}$$

Besides, by (3.2) we get

$$\gamma |\varepsilon_i|^2 r \le \gamma r^3 = \frac{41n(n-\mu)r^3}{10\eta^3} < \frac{3}{25}. \tag{3.14}$$

In regard to (3.13) and (3.14) there follows

$$\big|d_i + w_i^{(\lambda)}\big| \big(\big|d_i + w_i^{(\lambda)}\big| - \gamma |\varepsilon_i|^2 r \big) > \frac{7}{10}. \tag{3.15}$$

This means that disk $\hat{D}_i^{(c)}$ given by (3.11) and, consequently, disk $\hat{Z}_i$, are closed circular regions so that the sums $S_{1,i}$ and $S_{2,i}$ are closed disks.

The following upper bound is valid for $|d_i|$:

$$|d_i| = 2|\sigma_{1,i}\varepsilon_i + \mu_i| < 2\Big(\frac{(n-\mu)r}{\eta} + 1 \Big) < \frac{8}{3}.$$

Using this bound and (3.15), we find from (3.11)

$$\hat{r}_i = \text{rad}\ \ \hat{Z}_i \le \text{rad}\ \ \hat{D}_i \le \frac{10\gamma|\varepsilon_i|^3 r|d_i|}{7} < \frac{328n(n-\mu)}{21\eta^3}\,|\varepsilon_i|^3 r. \tag{3.16}$$

From (3.16) we conclude that

$$\hat{r} = \mathcal{O}(|\varepsilon|^3 r) \tag{3.17}$$

and also, by (3.2),

$$\hat{r} < \frac{r}{2}. \tag{3.18}$$

¿From (11) we find

$$\hat{z}_i = \text{mid}\ \ \hat{Z}_i = z_i - \frac{\varepsilon_i d_i}{d_i + w_i^{(\lambda)}}, \tag{3.19}$$

whence

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \varepsilon_i\left(1 - \frac{d_i}{d_i + w_i^{(\lambda)}}\right) = \frac{\varepsilon_i w_i^{(\lambda)}}{d_i + w_i^{(\lambda)}},$$

that is

$$\hat{\varepsilon}_i = \mathcal{O}_M\big(\varepsilon_i w_i^{(\lambda)}\big), \tag{3.20}$$

since $\big|d_i + w_i^{(\lambda)}\big|$ is bounded.

Evidently, from (3.2) there follows $w_i^{(\lambda)} = \mathcal{O}_M(\varepsilon_i^2)$. Furthermore, we have

$$\sigma_{1,i} - b_i^{(\lambda)} = \sum_{\substack{j=1\\ j\ne i}}^{\nu} \mu_j\Big(\frac{1}{z_i-\zeta_j} - u_{ij}^{(\lambda)}\Big) = \sum_{\substack{j=1\\ j\ne i}}^{\nu} \frac{\mu_j \xi_j^{(\lambda)} \varepsilon_j^{\lambda+1}}{(z_i-\zeta_j)h_{ij}^{(\lambda)}} = \mathcal{O}_M(\alpha\varepsilon_i^{\lambda+1}),$$

where α is a constant. According to this we get

$$\sigma_{1,i}^2 - \big(b_i^{(\lambda)}\big)^2 = \big(\sigma_{1,i} - b_i^{(\lambda)}\big)\big(\sigma_{1,i} + b_i^{(\lambda)}\big) = \mathcal{O}_M\big(\alpha'\varepsilon^{\lambda+1}\big)$$

and

$$\begin{aligned}
\sigma_{2,i} - \sum_{\substack{j=1\\ j\ne i}}^{\nu} \mu_j\big(u_{ij}^{(\lambda)}\big)^2 \quad &= \quad \sum_{\substack{j=1\\ j\ne i}}^{\nu} \mu_j\bigg(\frac{1}{(z_i-\zeta_j)^2} - \big(u_{ij}^{(\lambda)}\big)^2\bigg)\\
&= \quad \sum_{\substack{j=1\\ j\ne i}}^{\nu} \mu_j\Big(\frac{1}{z_i-\zeta_j} - u_{ij}^{(\lambda)}\Big)\Big(\frac{1}{z_i-\zeta_j} + u_{ij}^{(\lambda)}\Big)\\
&= \quad \mathcal{O}_M\big(\alpha''\epsilon^{\lambda+1}\big).
\end{aligned}$$

Returning to (3.20) we conclude that

$$\hat{\varepsilon} = \varepsilon^3 \mathcal{O}_M\big(\alpha^*\varepsilon^{\lambda+1}\big), \tag{3.21}$$

where α', α'' and α^* are constants.

Starting from the expressions for d_i and $w_i^{(\lambda)}$ and using the already derived bounds, we estimate

$$|d_i| > 2\mu_i - 2|\sigma_{1,i}||\varepsilon_i| > 2 - \frac{2(n-\mu)r}{\eta} > \frac{4}{3}$$

and

$$\begin{aligned}\left|w_i^{(\lambda)}\right| &< |\varepsilon_i|^2\Big(|\sigma_{1,i}|^2+\left|b_i^{(\lambda)}\right|^2+|\sigma_{2,i}|+\sum_{\substack{j=1\\ j\neq i}}^{\nu}\mu_j\left|u_{ij}^{(\lambda)}\right|^2\Big)\\ &< \frac{265(n-\mu+1)(n-\mu)}{121}\frac{r^2}{\eta^2}<\frac{2}{5}.\end{aligned}$$

According to the last two inequalities, we obtain from (3.19)

$$|\hat{z}_i-z_i|=\frac{|\varepsilon_i d_i|}{\left|d_i+w_i^{(\lambda)}\right|}<\frac{r_i}{1-\left|w_i^{(\lambda)}\right|/|d_i|}<\frac{3r_i}{2}. \tag{3.22}$$

Using the inequalities (3.2), (3.18) and (3.22) we obtain

$$\begin{aligned}|\hat{z}_i-\hat{z}_j| &\geq |z_i-z_j|-|\hat{z}_i-z_i|-|\hat{z}_j-z_j|>\eta+r_j-\frac{3}{2}r_i-\frac{3}{2}r_j\\ &> 3(n-\mu)r-2r>2\hat{r}\Big[3(n-\mu)-2\Big].\end{aligned}$$

In regard to the last inequality we obtain for any pair i,j $(i\neq j)$

$$|\hat{z}_i-\hat{z}_j|>2\hat{r}\geq\hat{r}_i+\hat{r}_j \quad (i\neq j),$$

which implies that the disks $\hat{Z}_1,\ldots,\hat{Z}_\nu$ are mutually disjoint. Besides, for arbitrary pair i,j $(i\neq j)$ we have

$$|\hat{z}_i-\hat{z}_j|-\hat{r}_j>2\hat{r}\big[3(n-\mu)-2\big]-\hat{r}>3(n-\mu)\hat{r}.$$

Hence

$$\hat{\eta}>3(n-\mu)\hat{r}.$$

In this way we have proved that the initial condition (3.2) implies the inequality of the same form but for the index $m=1$. It is worth noting that the inequality (3.18) points to the contraction of the new disks $Z_1^{(1)},\ldots,Z_\nu^{(1)}$.

Repeating the above procedure and the argumentation for arbitrary index $m\geq 0$ we can derive all above relations for the index $m+1$. Since these relations have been already proved for $m=0$, according to induction it follows that, under the condition (3.2), they are valid for all $m\geq 1$. In particular, we have

$$\eta^{(m)}>3(n-\mu)r^{(m)} \tag{3.23}$$

(the assertion 1°) and

$$r^{(m+1)}<\frac{r^{(m)}}{2}. \tag{3.24}$$

According to the inequality (3.24) we conclude that the sequence $\{r^{(m)}\}$ tends to 0; therefore, the inclusion method (2.6) is convergent. Since (3.23) holds, the assertion of Lemma 2.1 is valid for arbitrary m, which means that the Halley-like methods (2.6) with Schröder's and Halley's correction are well defined at each iterative step.

Suppose that $\zeta_i\in Z_i^{(m)}$ for each $i=1,\ldots,\nu$ and $\mu\leq 1$. With regard to (2.1) and (2.6) we obtain that $\zeta_i\in Z_i^{(m+1)}$ (due to the inclusion isotonicity). Since $\zeta_i\in Z_i^{(0)}$ (the assumption of the theorem) it follows by induction that $\zeta_i\in Z_i^{(m)}$ for each $i=1,\ldots,\nu$ and $m=0,1,\ldots$ (the assertion 2°).

To determine the lower bound of the R-order of convergence of the method (2.6) (the assertion 3°) we use Theorem 3.1 working with sequences $\{z_i^{(m)}\}$ and $\{r_i^{(m)}\}$ of the centers and the radii of disks $Z_i^{(m)}$ produced by the algorithm (2.6). For simplicity, as usual in this type of analysis, we adopt $1 > |\varepsilon^{(0)}| = r^{(0)} > 0$ dealing with the "worst case" model. Such a model does not influence the lower bound of the R-order of convergence since it is obtained in the limit process. By virtue of (3.17) and (3.21) we have

$$\varepsilon^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^{\lambda+4}, \quad r^{(m+1)} \sim \left(\varepsilon^{(m)}\right)^3 r^{(m)}.$$

These relations yield the R-matrix $P_2 = \begin{bmatrix} \lambda+4 & 0 \\ 3 & 1 \end{bmatrix}$ with the spectral radius $\rho(P_2) = \lambda+4$ and the corresponding eigenvector $\boldsymbol{x}_\rho = ((\lambda+3)/3, 1) > 0$. Hence, according to Theorem 3.1, we obtain

$$O_R((2.6)) \geq \rho(P_2) = \lambda + 4 \quad (\lambda = 1, 2).$$

Now we will consider the case INV $= ()^{-1}$, that is, when the exact inversion (1.1) is applied in the final step. Then, from (3.10) we obtain

$$\hat{Z}_i \subset \hat{D}_i^{(e)} := z_i - d_i\varepsilon_i \frac{\left\{\overline{d_i + w_i^{(\lambda)}}; \gamma|\varepsilon_i|^2 r\right\}}{\left|d_i + w_i^{(\lambda)}\right|^2 - \left(\gamma|\varepsilon_i|^2 r\right)^2} \tag{3.25}$$

and

$$\hat{r}_i = \text{rad } \hat{Z}_i < \frac{d_i\gamma|\varepsilon_i|^3 r}{\left|d_i + w_i^{(\lambda)}\right|^2 - \left(\gamma|\varepsilon_i|^2 r\right)^2} < \frac{14n(n-\mu)|\varepsilon_i|^3 r}{\eta^3}. \tag{3.26}$$

The center $\hat{z}_i = \text{mid } \hat{Z}_i$ is slightly shifted in reference to the center mid $\hat{D}_i^{(e)}$ so that from (3.25) we get

$$\hat{z}_i \cong \text{mid } \hat{D}_i^{(e)} = z_i - \frac{d_i\varepsilon_i}{(d_i + w_i^{(\lambda)})\left(1 - \left(\dfrac{\gamma|\varepsilon_i|^2 r}{\left|d_i + w_i^{(\lambda)}\right|}\right)^2\right)}.$$

Using previously derived estimations we obtain $d_i + w_i^{(\lambda)} = \mathcal{O}_M(1)$, $d_i = \mathcal{O}_M(1)$, $\gamma = \mathcal{O}(1)$ and $\gamma|\varepsilon_i|^2 r/\left|d_i + w_i^{(\lambda)}\right| = \mathcal{O}(r\epsilon^2)$. Applying the development into geometric series, from the last relation we find

$$\hat{z}_i \cong z_i - \frac{d_i\varepsilon_i}{d_i + w_i^{(\lambda)}}\left(1 + \left(\frac{\gamma|\varepsilon_i|^2 r}{\left|d_i + w_i^{(\lambda)}\right|}\right)^2 + \ldots\right) = z_i - \frac{d_i\varepsilon_i}{d_i + w_i^{(\lambda)}} + \mathcal{O}_M\left(r^2\epsilon^5\right).$$

Hence

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{d_i\varepsilon_i}{d_i + w_i^{(\lambda)}} + \mathcal{O}_M\left(r^2\epsilon^5\right) = \varepsilon_i^3\mathcal{O}_M\left(\alpha_1\epsilon^{\lambda+1} + \alpha_2 r^2\epsilon^2\right), \tag{3.27}$$

where α_1 and α_2 are some complex quantities such that $|\alpha_1| = \mathcal{O}(1)$ and $|\alpha_2| = \mathcal{O}(1)$. ¿From the last relation we conclude that

$$\hat{\varepsilon}_i = \varepsilon_i^3\mathcal{O}_M\left(\epsilon^{\lambda+1}\right) \quad (\lambda = 1, 2), \tag{3.28}$$

in other words, the relations (3.26) and (3.28) coincide with (3.16) and (3.21). Therefore, the lower bound of the R-order of convergence of the inclusion methods (2.6) when INV $= ()^{-1}$ is the same as in the case when INV $= ()^I$. □

In practice, the application of the centered inversion (INV $= ()^I$) in the final step produces smaller resulting inclusion disks. This is a consequence of shifting the centers which influences the radii of disks, see (3.21) and (3.27).

4 Single-step methods with corrections

Further acceleration of the convergence of the method (2.6) can be attained using the Gauss-Seidel procedure which uses already calculated disks in the current iteration. Starting from (2.4) we can state the following single-step inclusion method with Schröder's or Halley's corrections:

$$\hat{Z}_i = z_i - \mathrm{INV}\Big(H_i(z_i)^{-1} - \frac{f(z_i)}{2f'(z_i)}\Big[\frac{1}{\mu_i}S_{1,i}^2(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)}) + S_{2,i}(\widehat{\boldsymbol{Z}}, \boldsymbol{Z}^{(\lambda)})\Big]\Big) \tag{4.1}$$

for $i = 1, \dots, \nu$ and $\mathrm{INV} \in \{()^{-1}, ()^I\}$.

It is very difficult to find the R-order of convergence of this method since 2ν sequences of centers and radii and the number of zeros ν are involved in the convergence analysis. However, we can estimate easily the bounds of the R-order taking the limit cases $\nu = 2$ and very large ν.

First, since the convergence rate of a single-step method becomes almost the same as the one of the corresponding total-step method when the polynomial degree is very large, according to Theorem 3.2 we have $O_R((4.1), \nu) \geq \lambda + 4$ for very large ν.

Consider now the single-step method (4.1) for $\nu = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)}$ (the "worst case" model). After an extensive calculation we derive the following estimates

$$\hat{\varepsilon}_1 \sim \varepsilon_1^3 \varepsilon_2^{\lambda+1}, \quad \hat{\varepsilon}_2 \sim \varepsilon_1^3 \varepsilon_2^{\lambda+4} \ (\lambda = 1, 2), \quad \hat{r}_1 \sim |\varepsilon_1|^3 r_2, \quad \hat{r}_2 \sim |\varepsilon_1|^3 |\varepsilon_2|^3 r_2.$$

The corresponding R-matrix and their spectral radii and eigenvectors are:

$$P_4 = \begin{bmatrix} 3 & \lambda+1 & 0 & 0 \\ 3 & \lambda+4 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}, \ \rho(P_4) = \begin{cases} 6.64575, & \lambda = 1, \\ 7.8541, & \lambda = 2, \end{cases}$$

$$\boldsymbol{x}_\rho = \begin{cases} (1, 1.8229, 0.6771, 1.5) > 0, & \lambda = 1, \\ (1, 1.6180, 0.5279, 1.1459) > 0, & \lambda = 2. \end{cases}$$

Let $\Omega^{(\lambda)}$ $(\lambda = 1, 2)$ be the ranges of the lower bounds of the R-order of convergence concerning the single-step methods (4.1). Taking into account the previous results, we obtain

$$\Omega^{(1)} = (5, 6.646), \quad \Omega^{(2)} = (6, 7.855).$$

Since the increased convergence is attained without any additional calculations we conclude that the inclusion methods (4.1) possess a great computational efficiency.

5 Numerical examples

The presented algorithms (2.3), (2.4), (2.6) and (4.1) have been tested in solving many polynomial equations. To provide the enclosure of the zeros in the second and third iteration that produce very small disks, we have used the programming package *Mathematica* with multiple precision arithmetic. All methods are realized using only centered inversion, that is $\mathrm{INV} = ()^I$.

Example 1 Circular inclusion approximations to the multiple zeros of the polynomial

$$f(z) = z^9 - 8z^8 + 25z^7 - 34z^6 + 64z^4 - 76z^3 + 8z^2 + 48z - 32$$

are estimated by implementing interval methods (2.3), (2.6) (for $\lambda = 1, 2$) and (4.1) (for $\lambda = 1, 2$). The exact zeros of f are $\zeta_1 = -1$, $\zeta_2 = 2$, $\zeta_3 = 1 + i$, $\zeta_4 = 1 - i$, of the respective multiplicities $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = \mu_4 = 2$. The initial disks were selected to be $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$, with the centers

$$z_1^{(0)} = -1.1 + 0.2i, \quad z_2^{(0)} = 2.1 - 0.2i, \quad z_3^{(0)} = 0.8 + 1.2i, \quad z_4^{(0)} = 0.9 - 1.2i.$$

The maximal radii of the inclusion disks produced in the first three iterative steps are given in Table 5.2, where the denotation $A(-q)$ means $A \times 10^{-q}$.

Methods	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.3)	1.89(−2)	2.48(−9)	9.34(−39)
(2.4)	6.03(−3)	3.38(−12)	7.57(−50)
(2.6), $\lambda = 1$	2.69(−2)	3.18(−11)	1.81(−60)
(4.1), $\lambda = 1$	8.43(−3)	3.27(−14)	1.34(−69)
(2.6), $\lambda = 2$	2.77(−2)	3.41(−14)	1.05(−86)
(4.1), $\lambda = 2$	9.55(−3)	3.48(−16)	4.64(−96)

Table 1: The maximal radii of inclusion disks

Example 2 We implemented the same interval methods as in Example 1 to find inclusion disks of multiple zeros of the polynomial

$$\begin{aligned} f(z) &= z^{16} - 2z^{15} - 9z^{14} + 24z^{13} + 11z^{12} - 90z^{11} + 89z^{10} + 60z^9 - 200z^8 \\ &\quad +240z^7 - 124z^6 - 192z^5 + 336z^4 - 256z^3 - 64z^2 - 256. \end{aligned}$$

The exact zeros of f are $\zeta_1 = -1$, $\zeta_2 = -2$, $\zeta_3 = 1 + i$, $\zeta_4 = 1 - i$, $\zeta_5 = i$, $\zeta_6 = -i$, $\zeta_7 = 2$ of the multiplicity $\mu_1 = 2$, $\mu_2 = 3$, $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 2$, $\mu_7 = 3$, respectively. We have taken the following initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.5\}$, with the centers

$$\begin{aligned} z_1^{(0)} &= -1.1 + 0.1i, \quad z_2^{(0)} = -2.2 - 0.1i, \quad z_3^{(0)} = 1.1 + 1.2i, \quad z_4^{(0)} = 0.9 - 1.1i, \\ z_5^{(0)} &= -0.1 + 0.9i, \quad z_6^{(0)} = 0.1 - 1.1i, \quad z_7^{(0)} = 2.2 - 0.1i. \end{aligned}$$

The maximal radii of the inclusion disks are given in Table 2.

Methods	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
(2.3)	5.83(−2)	6.15(−9)	1.09(−38)
(2.4)	5.08(−2)	2.50(−10)	4.41(−44)
(2.6), $\lambda = 1$	6.74(−2)	3.11(−10)	1.89(−55)
(4.1), $\lambda = 1$	5.80(−2)	9.01(−11)	5.58(−60)
(2.6), $\lambda = 2$	6.43(−2)	1.02(−12)	3.70(−80)
(4.1), $\lambda = 2$	5.50(−2)	2.95(−13)	1.20(−86)

Table 2: The maximal radii of inclusion disks

From Tables 1 and 2 and a lot of numerical experiments we can conclude that the convergence rate of the considered methods, given in Theorem 3.2, mainly well coincides to the convergence speed of these methods in practice, especially in latter iterations. The convergence behavior of the methods tested in Example 2 is considerably good although the initial disks $Z_4^{(0)} = \{0.9 - 1.1i; 0.5\}$ and $Z_6^{(0)} = \{0.1 - 1.1i; 0.5\}$ are even overlapping. Enormously small disks obtained in the third iteration are unnecessary in practice, but we have presented them in both tables to stress the property of inclusion methods with corrections occurring in the growing accuracy as the number of iteration steps increases.

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