Towards the Possibility of Objective Interval Uncertainty in Physics *

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Abstract

In Newtonian physics, each physical variable has some well-defined value. We may not know this value exactly due to measurement uncertainty, but such a value is assumed to exist. In this case, the more accurate our measurements, the closer we get to this actual (unknown) value. In quantum physics, when we have a sequence of objects in the same state, and we measure the value of the same physical quantity for all these objects, we may get different result; the only thing that the quantum physics predicts is the probability of different outcomes. In quantum physics, these probabilities have well-defined values. Physicists have considered a natural next step: when even probability distributions are not uniquely determined, e.g., when only bounds on frequencies are known. In principle, under this assumption, one could expect situations when for the actual sequence of observations, frequencies fluctuate between the given bounds and do not converge to any probabilities at all. In this paper, under some simplifying assumptions, we prove that such fluctuations are not possible: each sequence of observations has a limit.

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Predictions in Newtonian physics: ideal case. In Newtonian physics, once we know the current state of the system, we can predict (at least in principle) all the future states of this system.

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Predictions in Newtonian physics: practical situations. In real life, measurements are never absolutely accurate, so we do not have the exact knowledge of the current state.

Here, the more accurate our measurements of the current state, the more accurate predictions we can make.

Interval uncertainty. The inaccuracy of the existing knowledge and of the resulting predictions can often be described in terms of interval uncertainty.

Predictions in quantum physics. In quantum physics, we cannot predict the *exact* future state of a system; we can only predict the *probabilities* of different future states.

According to the modern quantum physics, if we know the *exact* initial state of the world we can *uniquely* predict these probabilities. This means that the more accurate our measurements of the current state, the more accurate predictions of probabilities we can make.

Comment. In practice, we can often predict the *intervals* of possible values of the probability.

Possibility of objective interval-valued probabilities. It is reasonable to conjecture that: for some real-life processes, there is no objective probability.

In other words, for different subsequences, the corresponding frequencies can indeed take different values from a given interval.

Comment. The analysis of such processes is given by Gorban; see, e.g., [1, 2] and references therein.

How to formalize this: a natural idea. A natural common sense idea is that if an event has probability 0, then it cannot happen.

Limitations of a natural idea. The above statements cannot be literally true since every number has probability 0, and thus, no number is random.

Kolmorogov's definition of randomness. The idea of Kolmorogov and Martin-Löf is that we only require that *definable* events of probability 0 do not happen. As a result, we get a consistent definition of randomness; see, e.g. [3].

The reason why this definition is consistent is that:

- there are only countably many defining texts;
- thus countably many definable events;
- the union of countably many events of probability 0 has probability 0;
- thus, we indeed have a consistent definition of a random object.

How can we define when an object is random: case of exact probabilities. As we have just mentioned, randomness under a known probability distribution P can be defined as follows:

An object x is random *if its does not belong to any definable event E for which $P(E) = 0$.*

The meaning of this definition is straightforward: if a (definable) event E has probability 0, then this event cannot happen.

How can we define when an object is random: case of imprecise probabilities. As we have argued earlier, now we encounter a new situation: we do not know the probability distribution; we only know a class $\mathcal{P}$ of possible probability distributions.

A natural idea is as follows: if a definable event E is guaranteed to have probability 0 (i.e., $P(E) = 0$ for all possible P), then this event cannot happen. This idea leads to the following definition:

An object x is random *if it does not belong to any definable event E for which $P(E) = 0$ for all $P \in \mathcal{P}$.*

Observation. Let us start with the following observation: if an object x is random w.r.t. some $P_0 \in \mathcal{P}$, then it is also random w.r.t. $\mathcal{P}$.

Proof. The proof of this statement is straightforward:

- let E be a definable event for which $P(E) = 0$ for all $P \in \mathcal{P}$;
- we want to prove that $x \notin E$;
- since $P(E) = 0$ for all $P \in \mathcal{P}$ and $P_0 \in \mathcal{P}$, in particular, $P_0(E) = 0$;
- since x is P_0-random, we have $x \notin E$;
- the observation is proven.

Our main result. Let us consider the simplest case when the class $\mathcal{P}$ is finite, i.e., when $\mathcal{P} = \{P_1, \ldots, P_n\}$. In this case, according to the above observation, for every i, every P_i-random object is $\mathcal{P}$-random.

A natural expectation is that there are $\mathcal{P}$-random objects which are not P_i-random.

However, a *surprising result* is that every $\mathcal{P}$-random object is random with respect to one of the probability measures P_i.

Proof of the main result. The proof is by contradiction:

- let x be $\mathcal{P}$-random and not random with respect to all P_i;
- by definition, P_i-random means that $x \notin E$ for all definable E with $P_i(E) = 0$;
- thus, the fact that x is not P_i-random means that there exists an event E_i with $P_i(E_i) = 0$ for which $x \in E_i$;
- since $x \in E_i$ for all i, the object x belongs to the intersection $E \stackrel{\text{def}}{=} \bigcap\limits_{i=1}^{n} E_i$, i.e., $x \in E$;
- since $P_i(E_i) = 0$ and $E \subseteq E_i$, we have $P_i(E) = 0$;

- thus, x belongs to the event E for which $P_i(E) = 0$ for all i;
- this contradicts to our assumption that x is $\mathcal{P}$-random;
- the statement is proven.

Comments.

- The main idea of the proof comes from [3].
- We hope that this problem does not appear in the more physical interval-valued class $\mathcal{P}$.

References

[1] I. I. Gorban, *Theory of Hyper-Random Phenomena*, Kyiv, Ukrainian National Academy of Sciences Publ., 2007 (in Russian).

[2] I. I. Gorban, "Hyper-random phenomena: definition and description", *Information Theories and Applications*, vol. 15, No. 3, pp. 203–211, 2008.

[3] V. Kreinovich and L. Longpré, "Pure quantum states are fundamental, mixtures (composite states) are mathematical constructions: an argument using algorithmic information theory", *International Journal on Theoretical Physics*, vol. 36, no. 1, pp. 167–176, 1997.

[4] M. Li and P. Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications*, Springer, Berlin-Heidelberg, 2008.