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On the speed of convergence of the total step iterative method for a class of interval linear algebraic systems

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The equality of the asymptotic convergence factor for the total step method and the spectral radius of the absolute value of its interval matrix is proved for a set of interval linear algebraic systems.

О скорости сходимости полношагового итерационного метода для одного класса интервальных систем линейных алгебраических уравнений

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Для некоторого множества интервальных систем линейных алгебраических уравнений доказано совпадение асимптотического фактора сходимости полношагового итерационного метода со спектральным радиусом модуля матрицы интервальных коэффициентов.

A linear system of algebraic equations

$$x = Ax + b, \quad A = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}, \; b = (b_i)_{i=1}^{n} \in \mathbb{R}^{n}, \quad i, j = \overline{1, n} \tag{1}$$

where $\mathbb{R}^{n\times n}$ is the set of real $n \times n$ matrices and $\mathbb{R}^n$ is the set of real n-dimensional vectors can be transformed into the interval system

$$x = \mathbf{A}x + \mathbf{b} \tag{2}$$

if, instead of real coefficients of (1), we consider the interval coefficients $\mathbf{a}_{ij} = [\underline{\mathbf{a}}_{ij}, \overline{\mathbf{a}}_{ij}]$ and $\mathbf{b}_i = [\underline{\mathbf{b}}_i, \overline{\mathbf{b}}_i]$, $i, j = \overline{1, n}$. The set of all intervals $\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}]$, $\underline{\mathbf{a}}, \overline{\mathbf{a}} \in \mathbb{R}$ will be denoted by $\mathbb{IR}$, the set of all interval $n \times n$ matrices by $\mathbb{IR}^{n\times n}$, and the set of interval n-dimension vectors by $\mathbb{IR}^n$. The set of all solutions of an interval system (2)

$$\{y = (I - A)^{-1}b \mid A \in \mathbf{A}, b \in \mathbf{b}\}$$

may be bound using some iterative method [1] provided that its convergence is guaranteed. This paper is an attempt to estimate the speed of convergence for one of the methods described in [1], namely, the total step method.

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The total step method of successive approximations (also known as the simple iterative method) is defined as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)} + \mathbf{b}, \quad k = 0, 1, \ldots \tag{3}$$

Let us denote by $\rho(|\mathbf{A}|)$ the spectral radius of the real $n \times n$ matrix $|\mathbf{A}|$ consisting of the absolute values of the corresponding coefficients from matrix $\mathbf{A}$. Thus,

$$|\mathbf{a}_{ij}| := \max\{|\underline{\mathbf{a}}_{ij}|, |\overline{\mathbf{a}}_{ij}|\}.$$

It is known [1] that the total step method of successive approximations converges to the single fixed point $\mathbf{x}^*$ of the mapping $\mathbf{A}x + \mathbf{b}$ for any $\mathbf{x}^{(0)} \in \mathbb{IR}^n$ if and only if $\rho(|\mathbf{A}|) < 1$. In this case, the sequence $\mathbf{x}^{(k)}$ converges to the fixed point $\mathbf{x}^* = \lim_{k\to\infty} \mathbf{x}^{(k)}$, i.e., to the interval vector $\mathbf{x}^*$ that satisfies the equation

$$\mathbf{x}^* = \mathbf{A}\mathbf{x}^* + \mathbf{b} \tag{4}$$

that is, after substituting $\mathbf{x}^*$ to the right-hand side of the equation (4) and performing all arithmetical operations according to the rules of (traditional) interval arithmetic, we get the same vector $\mathbf{x}^*$. Also, the following lemma holds:

Lemma 1 [1]. *Let* $\mathbf{A}$ *be an interval matrix such that* $\rho(|\mathbf{A}|) < 1$. *Then for the fixed point* $\mathbf{x}^*$ *of the equation* $\mathbf{x}^* = \mathbf{A}\mathbf{x}^* + \mathbf{b}$ *the following holds:*

$$\{y = (I - A)^{-1}b \mid A \in \mathbf{A}, b \in \mathbf{b}\} \subseteq \{x \mid x \in \mathbf{x}^*\}.$$

To estimate the speed of convergence of the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ to $\mathbf{x}^*$, we need to find the distance between the interval vectors. In the set of real intervals $\mathbb{IR}$, the distance $q(\mathbf{a}, \mathbf{b})$ between two intervals $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}]$, $\mathbf{b} = [\underline{\mathbf{b}}, \overline{\mathbf{b}}] \in \mathbb{IR}$, is defined as

$$q(\mathbf{a}, \mathbf{b}) := \max\{|\underline{\mathbf{a}} - \underline{\mathbf{b}}|, |\overline{\mathbf{a}} - \overline{\mathbf{b}}|\},$$

and the distance $q(\mathbf{a}, \mathbf{b})$ between interval vectors $\mathbf{a}, \mathbf{b} \in \mathbb{IR}^n$ is defined [1] as a real n-dimensional vector formed by the distances between the corresponding interval elements of these vectors:

$$q(\mathbf{a}, \mathbf{b}) = \left(q(\mathbf{a}_i, \mathbf{b}_i)\right)_{i=1}^{n}, \quad q(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^n.$$

Suppose that we have a generic iterative procedure on the set of interval vectors $\mathbb{IR}^n$:

$$\mathbf{x}^{(k+1)} = f(\mathbf{x}^{(k)}), \quad k = 0, 1, \ldots \tag{5}$$

To estimate its convergence speed, we need to introduce the notion of *asymptotic convergence factor*.

Definition 1 [1]. *Let* $\mathbf{x}^* = f(\mathbf{x}^*)$ *and let* $\mathcal{G}$ *be the set of all sequences* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *obtained using (5) and satisfying the condition* $\lim_{k\to\infty} \mathbf{x}^{(k)} = \mathbf{x}^*$. *Then the quantity*

$$\alpha = \sup\left\{\limsup_{k\to\infty} \|q(\mathbf{x}^{(k)}, \mathbf{x}^*)\|^{1/k} \mid \{\mathbf{x}^{(k)}\}_{k=0}^{\infty} \in \mathcal{G}\right\}$$

is called the asymptotic convergence factor of the iterations (5) to the point $\mathbf{x}^*$.

It should be noted that in Russian mathematical works, the denotation

$$\overline{\lim_{k\to\infty}} \quad \text{instead of} \quad \limsup_{k\to\infty}$$

is commonly used.

The exact upper bound in Definition 1 is chosen with respect to all sequences from $\mathcal{G}$ and provides a characteristic for the asymptotically worst choice of initial approximation $\mathbf{x}^{(0)}$. The asymptotic convergence factor of the total step method of successive approximations will be denoted by α_T. In [1] it is proved that $\alpha_T \leq \rho(|\mathbf{A}|)$, and the following problem is stated: Find an $\mathbf{x}^{(0)}$ which satisfies the following equation:

$$\limsup_{k\to\infty} \|q(\mathbf{x}^{(k)}, \mathbf{x}^*)\|^{1/k} = \rho(|\mathbf{A}|)$$

i.e., for which $\alpha_T = \rho(|\mathbf{A}|)$. Below, we present a solution to this problem for a special kind of systems.

Theorem. *Let*

1. *a matrix* $\mathbf{A} = (\mathbf{a}_{ij})_{i,j=1}^n$ *in a converging total step method (3) be such that* $0 \notin \mathbf{a}_{ij}$, $i, j = \overline{1, n}$;
2. *the solution* $\mathbf{x}^*$ *of the equation (4) be such that* $0 \in \mathbf{x}_i^*$, $i = \overline{1, n}$.

Then $\alpha_T = \rho(|A|)$.

Corollary. *If the condition 1 of the above theorem is true and the vector* $\mathbf{b}$ *in (3) is such that* $0 \in \mathbf{b}_i$, $i = \overline{1, n}$, *then* $\alpha_T = \rho(|\mathbf{A}|)$.

To prove these statements, we will need to deploy several well known results.

Definition 2 [2]. *A rectangular matrix A with real elements*

$$A = (a_{ij}), \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n$$

is called non-negative (denoted as $A \geq 0$) or positive (denoted as $A > 0$), if all the elements of the matrix A are non-negative (or positive): $a_{ij} \geq 0$ (or $a_{ij} > 0$).

Definition 3 [2]. *A square matrix $A = (a_{ij})_{i,j=1}^n$ is called decomposable if it is possible to divide all indices $1, 2, \ldots, n$ into two disjoint subsets $\{i_1, i_2, \ldots, i_\mu\}$ and $\{j_1, j_2, \ldots, j_\nu\}$ $(\mu + \nu = n)$ so that*

$$a_{i_\alpha j_\beta} = 0, \quad \alpha = 1, 2, \ldots, \mu; \; \beta = 1, 2, \ldots, \nu.$$

Otherwise, the matrix A is called non-decomposable.

Lemma 2 [2]. *A non-decomposable non-negative matrix $A = (a_{ij})_{i,j=1}^n$ always has a positive eigenvalue r that is equal to the spectral radius of A. All the coordinates of the corresponding eigenvector are positive.*

Proof of the theorem. Since $0 \notin \mathbf{a}_{ij}$, $i, j = \overline{1, n}$, the interval matrix $|\mathbf{A}|$ does not contain any zero elements and is thus positive. Hence, it is a non-decomposable non-negative matrix. Let us find the spectral radius of this matrix: $\lambda := \rho(|\mathbf{A}|)$. According to Lemma 2, an eigenvalue λ corresponds to the eigenvector $x^\lambda = (x_i^\lambda)_{i=1}^n$ with positive coordinates. Let $\|\cdot\|$ be a norm in the n-dimensional space of real vectors $\mathbb{R}^n$. We choose such a vector x^λ that $\|x^\lambda\| = 1$. For x^λ, the equality $|\mathbf{A}|\cdot x^\lambda = \lambda x^\lambda$ holds, or componentwise

$$\sum_{j=1}^{n} |\mathbf{a}_{ij}| x_j^\lambda = \lambda x_i^\lambda, \quad i = \overline{1,n}.$$

Let us denote by $\mathbf{e}_i$ the interval $[-x_i^\lambda, x_i^\lambda]$. As an initial approximation we take an interval vector with components

$$\mathbf{x}_i^{(0)} = \mathbf{x}_i^* + \mathbf{e}_i = [\underline{\mathbf{x}}_i^* - x_i^\lambda, \overline{\mathbf{x}}_i^* + x_i^\lambda], \quad i = \overline{1,n}.$$

In this case

$$q(\mathbf{x}_i^*, \mathbf{x}_i^{(0)}) = \max\{|\underline{\mathbf{x}}_i^* - \underline{\mathbf{x}}_i^{(0)}|, |\overline{\mathbf{x}}_i^* - \overline{\mathbf{x}}_i^{(0)}|\} = x_i^\lambda, \quad i = \overline{1,n}.$$

Then

$$q(\mathbf{x}^*, \mathbf{x}^{(0)}) = \left(q(\mathbf{x}_i^*, \mathbf{x}_i^{(0)})\right)_{i=1}^{n} = (x_i^\lambda)_{i=1}^{n} \quad \text{and} \quad \|q(\mathbf{x}^*, \mathbf{x}^{(0)})\| = 1.$$

Let us find $\mathbf{x}^{(1)} = \mathbf{A}\mathbf{x}^{(0)} + \mathbf{b}$:

$$\mathbf{x}_i^{(1)} = \sum_{j=1}^{n} \mathbf{a}_{ij}\mathbf{x}_j^{(0)} + \mathbf{b}_i = \sum_{j=1}^{n} \mathbf{a}_{ij}(\mathbf{x}_j^* + [-x_j^\lambda, x_j^\lambda]) + \mathbf{b}_i, \quad i = \overline{1,n}. \tag{6}$$

Given the conditions of the theorem, we have $0 \in \mathbf{x}_j^*$, $j = \overline{1,n}$, and thus $0 \in \mathbf{x}_j^* + [-x_j^\lambda, x_j^\lambda]$, $j = \overline{1,n}$. On the other hand, $0 \notin \mathbf{a}_{ij}$, $i,j = \overline{1,n}$, and in the right-hand side of (6), the distributivity law holds:

1. if $\underline{\mathbf{a}}_{ij} > 0$, then

$$\begin{aligned}\mathbf{a}_{ij}(\mathbf{x}_j^* + [-x_j^\lambda, x_j^\lambda]) &= \mathbf{a}_{ij}\left[\underline{\mathbf{x}}_j^* - x_j^\lambda, \overline{\mathbf{x}}_j^* + x_j^\lambda\right] = \left[|\mathbf{a}_{ij}|(\underline{\mathbf{x}}_j^* - x_j^\lambda), |\mathbf{a}_{ij}|(\overline{\mathbf{x}}_j^* + x_j^\lambda)\right] \\ &= \left[|\mathbf{a}_{ij}|\underline{\mathbf{x}}_j^*, |\mathbf{a}_{ij}|\overline{\mathbf{x}}_j^*\right] + \left[-|\mathbf{a}_{ij}|x_j^\lambda, |\mathbf{a}_{ij}|x_j^\lambda\right] = \mathbf{a}_{ij}\mathbf{x}_j^* + \mathbf{a}_{ij}\left[-x_j^\lambda, x_j^\lambda\right];\end{aligned}$$

2. if $\overline{\mathbf{a}}_{ij} < 0$, then

$$\begin{aligned}\mathbf{a}_{ij}(\mathbf{x}_j^* + [-x_j^\lambda, x_j^\lambda]) &= \mathbf{a}_{ij}\left[\underline{\mathbf{x}}_j^* - x_j^\lambda, \overline{\mathbf{x}}_j^* + x_j^\lambda\right] = \left[-|\mathbf{a}_{ij}|(\overline{\mathbf{x}}_j^* + x_j^\lambda), -|\mathbf{a}_{ij}|(\underline{\mathbf{x}}_j^* - x_j^\lambda)\right] \\ &= \left[-|\mathbf{a}_{ij}|\overline{\mathbf{x}}_j^*, -|\mathbf{a}_{ij}|\underline{\mathbf{x}}_j^*\right] + \left[-|\mathbf{a}_{ij}|x_j^\lambda, |\mathbf{a}_{ij}|x_j^\lambda\right] = \mathbf{a}_{ij}\mathbf{x}_j^* + \mathbf{a}_{ij}\left[-x_j^\lambda, x_j^\lambda\right].\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{x}_i^{(1)} &= \left(\sum_{j=1}^{n} \mathbf{a}_{ij}\mathbf{x}_j^* + \mathbf{b}_i\right) + \sum_{j=1}^{n} \mathbf{a}_{ij}\cdot[-x_j^\lambda, x_j^\lambda] = \mathbf{x}_i^* + \left[-\sum_{j=1}^{n}|\mathbf{a}_{ij}|x_j^\lambda, \sum_{j=1}^{n}|\mathbf{a}_{ij}|x_j^\lambda\right] \\ &= \mathbf{x}_i^* + [-\lambda\cdot x_i^\lambda, \lambda\cdot x_i^\lambda] = \mathbf{x}_i^* + \lambda\cdot\mathbf{e}_i \subseteq \mathbf{x}_i^{(0)}, \quad i = \overline{1,n}.\end{aligned}$$

Thus we have $\|q(\mathbf{x}^*, \mathbf{x}^{(1)})\| = \|(\lambda x_i^\lambda)_{i=1}^{n}\| = \lambda\|x^\lambda\| = \lambda$. For similar reasons, we conclude that for $k = 1, 2, \ldots$, we have $\|q(\mathbf{x}^*, \mathbf{x}^{(k)})\| = \lambda^k$, and $\|q(\mathbf{x}^*, \mathbf{x}^{(k)})\|^{1/k} = \lambda = \rho(|\mathbf{A}|)$. Thus $\alpha_T = \rho(|\mathbf{A}|)$. □

Proof of the Corollary. Using Lemma 1, we conclude from the condition $0 \in \mathbf{b}_i$, $i = \overline{1,n}$ that the set

$$\{y = (I - A)^{-1}b \mid A \in \mathbf{A}, b \in \mathbf{b}\}$$

contains the point $y = (0, 0, \dots, 0)$. Consequently, 0 is included in all elements of vector $\mathbf{x}^*$, which contains this set, and condition 2 of this theorem is satisfied. $\square$

The asymptotic convergence factor α is a generalization, for the interval case, of the concept of the R_1-multiplier [6] of a pointwise iterative process.

Definition 4 [6]. *Let $\{x^{(k)}\} \in \mathbb{R}^n$ be an arbitrary sequence converging to x^*. Then the numbers*

$$R_p\{x^{(k)}\} = \begin{cases} \limsup\limits_{k\to\infty} |x^{(k)} - x^*|^{1/k}, & \text{if } p = 1; \\ \limsup\limits_{k\to\infty} |x^{(k)} - x^*|^{1/p^k}, & \text{if } p > 1 \end{cases}$$

are called root convergence multipliers or R-multipliers of this sequence. If $\mathcal{J}$ is an iterative process with the limit point x^* and $C(\mathcal{J}, x^*)$ is a combination of all sequences generated by $\mathcal{J}$ which converge to x^*, then the numbers

$$R_p(\mathcal{J}, x^*) = \sup\left\{R_p\{x^{(k)}\} \mid \{x^{(k)}\} \in C(\mathcal{J}, x^*)\right\}, \quad 1 \le p < \infty$$

are called R-multipliers of the process $\mathcal{J}$ in the point x^*.

Lemma 3 [6]. *Let $B \in L(\mathbb{R}^n)$, $b \in \mathbb{R}^n$. Let us define a mapping $G\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $Gx = Bx + b$, $x \in \mathbb{R}^n$. If $\rho(B) < 1$, then G has the single fixed point x^*, iterations $x^{(k+1)} = Gx^{(k)}$, $k = 0, 1, \dots$ converge to x^* for any $x^{(0)} \in \mathbb{R}^n$, and*

$$R_1(\mathcal{J}, x^*) = \rho(B).$$

Thus, the R_1-multiplier for the pointwise total step iterative method is exactly equal to the spectral radius of the matrix that defines this process. On the other hand, iterative process defined by an interval matrix $\mathbf{A}$ and an interval vector $\mathbf{b}$ (according to the formula (3)), unlike the similar pointwise process, may have asymptotic convergence factor different from $\rho(|\mathbf{A}|)$. For example,

$$\text{for} \quad \mathbf{A} = \begin{pmatrix} [-0.8, 0.5] & 0 \\ 0 & [-0.8, 0.5] \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} [0.06, 0.1] \\ [0.06, 0.1] \end{pmatrix}$$

$$\text{the solution is} \quad \mathbf{x}^* = \begin{pmatrix} [-0.1, 0.2] \\ [-0.1, 0.2] \end{pmatrix}.$$

Actual iterating in the close enough neighborhood of $\mathbf{x}^*$ yields $\alpha_T = 0.5$, whereas $\rho(|\mathbf{A}|) = 0.8$.

Thus, if the matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is such that none of its elements includes 0 and $0 \in \mathbf{b}_i$, $i = \overline{1, n}$, then $\alpha_T = \rho(|\mathbf{A}|)$ and the worst convergence speed of the total step method is observed for $\mathbf{x}_i^{(0)} = \mathbf{x}_i^* + [-x_i^\lambda, x_i^\lambda]$, where x_i^λ is the i-th coordinate of the eigenvalue x^λ which corresponds to the maximum eigenvalue λ equaling the spectral radius of the non-negative non-decomposable matrix $|\mathbf{A}|$.

A detailed report on finding the exact value of α_T and upper and lower estimates of this value for various types of interval linear algebraic systems is presented in [3]. The proof of the facts described in [3] is based on examining the behavior of iterative process in the close enough neighborhood of $\mathbf{x}^*$ and, unlike the present paper, does not make use of a special form of $\mathbf{x}^{(0)}$.

Note that the necessary and sufficient conditions of satisfying the strict inequality $\alpha_T < \rho(|\mathbf{A}|)$ for $\mathbf{A}$ with non-negative interval coefficients (that is, with $\underline{\mathbf{a}}_{ij} \ge 0$, $i, j = \overline{1, n}$) and non-decomposable $|\mathbf{A}|$ are presented in [4] and proved in [5].

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