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Cheap and Tight Bounds: The Recent Result by E. Hansen Can Be Made More Efficient

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Improving the recent result by Eldon Hansen, we give cheap and tight bounds on the solution of a linear interval system as well as on the inverse interval matrix.

Грубая и улучшенная границы: недавний результат Э. Хансена может быть сделан более эффективным

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Улучшая недавний результат Э. Хансена, мы приводим грубую и улучшенную границы для решения линейной интервальной системы и для обратной интервальной матрицы.

1 Introduction

It follows from the general theory [2] that up to 2^n systems of linear equations need to be solved to compute the exact bounds on the solution of a system of linear interval equations in n unknowns. Eldon Hansen has recently published in [1] a remarkable result showing that this number can be reduced to $2n$ for linear interval systems whose midpoint matrix is the unit matrix. In this paper we prove that Hansen's result can be reformulated in such a way that inverting only *one* matrix is needed, and we apply this result to bounding solutions of linear interval systems and inverting interval matrices. The bounds on the inverse interval matrix are shown to be better in general than the classical ones derived via Neumann series.

2 Hansen's result improved

Hansen considered in [1] a linear interval system of the form

$$[I - \Delta, I + \Delta]x = [b_c - \delta, b_c + \delta] \tag{1}$$

where I is the $n \times n$ unit matrix, Δ is a nonnegative $n \times n$ matrix, b_c, $\delta \in R^n$, $\delta \geq 0$ and $[I-\Delta, I+\Delta] = \{A;\ |A-I| \leq \Delta\}$, $[b_c-\delta, b_c+\delta] = \{b;\ |b-b_c| \leq \delta\}$. He used diagonal dominance as a sufficient regularity condition; however, it follows from the assertion (C3) of Theorem 5.1 in [2] that $[I - \Delta, I + \Delta]$ is regular (i.e., all the matrices contained therein are nonsingular) if and only if

$$\varrho(\Delta) < 1 \tag{2}$$

holds (ϱ is the spectral radius). Let us note that this condition implies existence and nonnegativity of the matrix

$$M = (I - \Delta)^{-1} = (m_{ij}).$$

As is well known, the exact bounds on the solution of (1) are defined by

$$\underline{x}_i = \min_{x \in X} x_i$$

$$\overline{x}_i = \max_{x \in X} x_i$$

$(i = 1, \ldots, n)$, where X is the so-called solution set:

$$X = \big\{x;\ Ax = b \quad \text{for some } A \in [I - \Delta, I + \Delta],\ b \in [b_c - \delta, b_c + \delta]\big\}.$$

Hansen showed in [1] that these quantities can be computed by solving only $2n$ systems of linear equations sharing the same coefficient matrix. We shall prove here a reformulation of his result in which only one matrix inversion is needed:

Theorem 1. *Let (2) hold. Then for each $i \in \{1, \ldots, n\}$ we have*

$$\begin{aligned} \underline{x}_i &= \min\{\underset{\sim}{x}_i, \nu_i \underset{\sim}{x}_i\} \\ \overline{x}_i &= \max\{\tilde{x}_i, \nu_i \tilde{x}_i\} \end{aligned}$$

where

$$\begin{aligned} \underset{\sim}{x}_i &= -x_i^* + m_{ii}(b_c + |b_c|)_i \\ \tilde{x}_i &= x_i^* + m_{ii}(b_c - |b_c|)_i \\ x_i^* &= (M(|b_c| + \delta))_i \end{aligned}$$

and

$$\nu_i = \frac{1}{2m_{ii} - 1} \in (0, 1].$$

Proof. Let $i \in \{1, \ldots, n\}$ be fixed. We shall prove: 1) $x_i \le \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}$ for each $x \in X$, 2) $\tilde{x}_i = x'_i$, $\nu_i \tilde{x}_i = x''_i$ for some $x', x'' \in X$; this will prove that $\overline{x}_i = \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}$, 3) the formula for $\underline{x}_i$ using the result for $\overline{x}_i$.

1) First notice that for $M = (I - \Delta)^{-1}$ we have $M\Delta = \Delta M = M - I$ and $m_{ii} \ge 1$, implying $2m_{ii} - 1 \ge 1$ and $\nu_i \in (0, 1]$. Define a diagonal matrix D by

$$D_{jj} = \begin{cases} 1 & \text{if } j \ne i \text{ and } (b_c)_j \ge 0 \\ -1 & \text{if } j \ne i \text{ and } (b_c)_j < 0 \\ 1 & \text{if } j = i \end{cases}$$

$(j = 1, \ldots, n)$, and let

$$\tilde{b} = Db_c + \delta.$$

Then we have

$$\tilde{x}_i = x_i^* + m_{ii}(b_c - |b_c|)_i = (M\tilde{b})_i.$$

Now, let $x \in X$, so that $Ax = b$ for some $A \in [I - \Delta, I + \Delta]$ and $b \in [b_c - \delta, b_c + \delta]$. Put

$$x' = (|x_1|, \ \ldots, \ |x_{i-1}|, x_i, |x_{i+1}|, \ \ldots, \ |x_n|)^T.$$

We shall prove that x satisfies the inequality

$$M(x' - |x|) + |x| \leq M\tilde{b}. \tag{3}$$

In fact, $x'_i = x_i = b_i + ((I - A)x)_i \leq (b_c + \delta)_i + (\Delta|x|)_i = (\tilde{b} + \Delta|x|)_i$, and if $j \neq i$, then $x'_j = |x_j| \leq |b_j| + |((I-A)x)_j| \leq |b_c|_j + \delta_j + (\Delta|x|)_j = (\tilde{b} + \Delta|x|)_j$, which together gives

$$x' \leq \tilde{b} + \Delta|x|$$

and premultiplying this inequality by the nonnegative matrix M yields $Mx' \leq qM\tilde{b} + M\Delta|x| = M\tilde{b} + (M - I)|x|$, which implies (3). Now, if $x_i \geq 0$, then $x' = |x|$ and from (3) we have

$$x_i = |x_i| \leq (M\tilde{b})_i = \tilde{x}_i$$

and if $x_i < 0$, then from (3) we obtain

$$(2m_{ii} - 1)x_i \leq (M\tilde{b})_i = \tilde{x}_i$$

which implies

$$x_i \leq \nu_i \tilde{x}_i$$

hence in both the cases we have

$$x_i \leq \max\{\tilde{x}_i, \nu_i \tilde{x}_i\}.$$

2) Put

$$x' = DM\tilde{b}$$

$$x'' = DM(\tilde{b} - 2\nu_i \tilde{x}_i \Delta e_i)$$

where e_i is the i-th column of I. We shall prove that x' and x'' belong to X. Since $(I - D\Delta D)x' = DM\tilde{b} - D(M - I)\tilde{b} = D\tilde{b} = b_c + D\delta$, we see that x' satisfies

$$(I - D\Delta D)x' = b_c + D\delta$$

where $I - D\Delta D \in [I - \Delta, I + \Delta]$ and $b_c + D\delta \in [b_c - \delta, b_c + \delta]$, which proves that $x' \in X$. Furthermore, define a diagonal matrix D' by $D'_{ii} = -1$ and

$D'_{jj} = D_{jj}$ otherwise. Then $(I - D\Delta D')DM = DM - D\Delta(I - 2e_ie_i^T)M = DM - D(M-I) + 2D\Delta e_ie_i^T M = D + 2D\Delta e_ie_i^T M$, hence $(I - D\Delta D')x'' = (D + 2D\Delta e_ie_i^T M)(\tilde{b} - 2\nu_i\tilde{x}_i\Delta e_i) = D\tilde{b} + 2\tilde{x}_i D\Delta e_i(-\nu_i + 1 - 2\nu_i(m_{ii} - 1)) = D\tilde{b} = b_c + D\delta$, which again gives that $x'' \in X$. Now, since $e_i^T D = e_i^T$, we have

$$x'_i = e_i^T DM\tilde{b} = \tilde{x}_i$$

$$x''_i = \tilde{x}_i - 2\nu_i\tilde{x}_i(m_{ii} - 1) = \nu_i\tilde{x}_i$$

which in conjunction with 1) proves that

$$\overline{x}_i = \max\{\tilde{x}_i, \nu_i\tilde{x}_i\}.$$

3) To prove the formula for $\underline{x}_i$, consider the linear interval system $[I - \Delta, I + \Delta]x = [-b_c - \delta, -b_c + \delta]$ with the solution set $X_0 = -X$. Then, applying the formula for $\overline{x}_i$ to it, we have

$$\underline{x}_i = \min_X x_i = -\max_{X_0} x_i = -\max\{\tilde{\tilde{x}}_i, \nu_i\tilde{\tilde{x}}_i\}$$

where $\tilde{\tilde{x}}_i = x_i^* + m_{ii}(-b_c - |b_c|)_i = x_i^* - m_{ii}(b_c + |b_c|)_i = -\underset{\sim}{x}_i$, which finally gives

$$\underline{x}_i = -\max\{-\underset{\sim}{x}_i, -\nu_i\underset{\sim}{x}_i\} = \min\{\underset{\sim}{x}_i, \nu_i\underset{\sim}{x}_i\}.$$

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3 Solving linear interval systems

Consider a linear interval system

$$[A_c - \Delta, A_c + \Delta]x = [b_c - \delta, b_c + \delta] \tag{4}$$

and its solution set

$$X_0 = \{x;\ Ax = b \quad \text{for some } A \in [A_c - \Delta, A_c + \Delta], b \in [b_c - \delta, b_c + \delta]\}.$$

Let R be an arbitrary $n \times n$ matrix. If $Ax = b$ for some $A \in [A_c - \Delta, A_c + \Delta]$ and $b \in [b_c - \delta, b_c + \delta]$, then we have $RAx = Rb$ and

$$|RA - I| = |RA_c - I + R(A - A_c)| \leq G_R$$

$$|Rb - Rb_c| \le |R|\delta$$

where we have denoted

$$G_R = |RA_c - I| + |R|\Delta.$$

Thus we can see that the solution set X_0 of (4) is contained in the solution set X of the system

$$[I - G_R, I + G_R]x = [Rb_c - |R|\delta, Rb_c + |R|\delta] \tag{5}$$

which is of the form (1). Now, if the condition

$$\varrho(G_R) < 1 \tag{6}$$

is satisfied, then we can apply Theorem 1 to the system (5) to obtain the exact bounds $\underline{x}_i$, $\overline{x}_i$ ($i = 1, \ldots, n$) on X. Since $X_0 \subset X$, this implies that

$$\underline{x}_i \le x_i \le \overline{x}_i \qquad (i = 1, \ldots, n)$$

holds for each $x \in X_0$. In this way we have obtained an *interval enclosure* of the solution set X_0 of (4). This enclosure is generally not sharp, but can be expected to be very tight if the radii Δ and δ are narrow; cf. Neumaier [3] for a detailed discussion.

The procedure described is performable if we can find a matrix R satisfying (6). It follows from Theorem 4.1.2 in [3] that such a matrix exists if and only if $[A_c - \Delta, A_c + \Delta]$ is strongly regular (i.e., if A_c is nonsingular and $\varrho(|A_c^{-1}|\Delta) < 1$); if this is the case, then $R := A_c^{-1}$ has the required property. Therefore, for practical purposes it is recommendable to set R equal to the computed value of A_c^{-1}.

4 Inverting interval matrices

For an interval matrix $[I - \Delta, I + \Delta]$, consider its interval inverse $[\underline{B}, \overline{B}]$ defined by

$$\underline{B}_{ij} = \min\{(A^{-1})_{ij};\ A \in [I - \Delta, I + \Delta]\}$$
$$\overline{B}_{ij} = \max\{(A^{-1})_{ij};\ A \in [I - \Delta, I + \Delta]\}$$

($i, j = 1, \ldots, n$). Applying Theorem 1 to the systems $[I - \Delta, I + \Delta]x = [e_j, e_j]$, where e_j is the j-th column of I ($j = 1, \ldots, n$), we obtain this explicit form of the inverse (where, as before, $M = (I - \Delta)^{-1}$):

Theorem 2. *Let (2) hold. Then we have*

$$\begin{aligned} \overline{B}_{ij} &= m_{ij} \\ \underline{B}_{ij} &= \begin{cases} -m_{ij} & \text{if } i \neq j \\ \frac{m_{ii}}{2m_{ii}-1} & \text{if } i = j \end{cases} \end{aligned}$$

$(i, j = 1, \ldots, n)$.

It may seem surprising that $\underline{B}_{ii} > \frac{1}{2}$ (since $m_{ii} \geq 1$). This, however, is a consequence of a more general result ([2], Thm. 5.1, (C5)).

For a general square interval matrix $[A_c - \Delta, A_c + \Delta]$, employing Theorem 2 to the preconditioned matrix as in Section 3, we obtain the following result (where we employ matrices R and K to avoid the use of exact inverses A_c^{-1} and $(I - G_R)^{-1}$):

Theorem 3. *For a given interval matrix $[A_c - \Delta, A_c + \Delta]$, let $K = (k_{ij}) \geq 0$ and R be any matrices satisfying*

$$KG_R + I \leq K \tag{7}$$

where

$$G_R = |RA_c - I| + |R|\Delta.$$

Then for each $A \in [A_c - \Delta, A_c + \Delta]$ we have

$$|A^{-1} - TR| \leq (K - T)|R| \tag{8}$$

where T is the diagonal matrix with diagonal entries

$$T_{ii} = \frac{k_{ii}^2}{2k_{ii} - 1} \qquad (i = 1, \ldots, n).$$

Proof. Premultiplying (7) by the nonnegative matrix G_R, we obtain $G_R + I \leq KG_R + I \leq K$ and by induction

$$\sum_{j=0}^{\ell} G_R^j \leq K$$

for each $\ell \geq 0$, hence $\varrho(G_R) < 1$ and $(I - G_R)^{-1} = \sum_{j=0}^{\infty} G_R^j \leq K$. Now, for each $A \in [A_c - \Delta, A_c + \Delta]$ we have $|RA - I| = |RA_c - I + R(A - A_c)| \leq G_R$, hence $RA \in [I - G_R, I + G_R]$, which implies that A is nonsingular and

$$\underline{B} \leq A^{-1}R^{-1} \leq \overline{B}$$

where $\underline{B}$ and $\overline{B}$ are as in Theorem 2, with $M := (I - G_R)^{-1}$. Define matrices $\underset{\sim}{B}$ and $\tilde{B}$ by $\tilde{B} = K$ and

$$\underset{\sim}{B}_{ij} = \begin{cases} -k_{ij} & \text{if } i \neq j \\ \frac{k_{ii}}{2k_{ii}-1} & \text{if } i = j \end{cases}$$

$(i, j = 1, \ldots, n)$, then from $M \leq K$ we obtain $\underset{\sim}{B} \leq \underline{B}$ and $\overline{B} \leq \tilde{B}$, hence

$$\underset{\sim}{B} \leq A^{-1}R^{-1} \leq \tilde{B}$$

which implies that

$$|A^{-1}R^{-1} - T| = |A^{-1}R^{-1} - \frac{1}{2}(\underset{\sim}{B} + \tilde{B})| \leq \frac{1}{2}(\tilde{B} - \underset{\sim}{B}) = K - T$$

and consequently

$$|A^{-1} - TR| = |(A^{-1}R^{-1} - T)R| \leq |A^{-1}R^{-1} - T| \cdot |R| \leq (K - T)|R|.$$

■

To explain what is new in this result, let us notice that the classical argument using Neumann series (see e.g. [2], proof of Thm. 4.4) yields the estimate

$$|A^{-1} - R| \leq (K - I)|R|. \tag{9}$$

However, since

$$T_{ii} - 1 = \frac{(k_{ii} - 1)^2}{2k_{ii} - 1} \geq 0$$

for each i, we have $T \geq I$ and hence

$$(K - T)|R| \leq (K - I)|R|.$$

This shows that (8) is at least as good as (9), but for each i, j with $k_{ii} > 1$ and $R_{ij} \neq 0$ the estimate (8) gives a result which is better than (9) by the amount of

$$\frac{(k_{ii} - 1)^2}{2k_{ii} - 1}|R_{ij}|.$$

Thus for the particular choice $R := A_c^{-1}$ and $K := (I - |A_c^{-1}|\Delta)^{-1}$, the estimate (8) gives a better result than (9) for each i, j such that $(|A_c^{-1}|\Delta)_{ii} > 0$ and $(A_c^{-1})_{ij} \neq 0$. Let us note that Herzberger and Bethke proved in [4] that two well-known methods for bounding the inverse interval matrix cannot improve on the bound (9).

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