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ON CONTROLLED SOLUTION SET OF INTERVAL ALGEBRAIC SYSTEMS

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For interval system of algebraic equations the concept of *controlled solution set* is introduced and its practical interpretation is given. The main result of the work is a simple sufficient criterion for the controlled solution set of interval linear system to be nonempty.

ОБ УПРАВЛЯЕМОМ МНОЖЕСТВЕ РЕШЕНИЙ ИНТЕРВАЛЬНЫХ АЛГЕБРАИЧЕСКИХ СИСТЕМ

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Для интервальных систем алгебраических уравнений в работе вводится понятие *управляемого множества решений* и указывается его содержательная интерпретация. Основной результат работы — простой достаточный признак непустоты управляемого множества решений для интервальной линейной системы.

Let interval system of linear algebraic equations be given

$$\mathbf{A}x = \mathbf{b} \tag{1}$$

with an interval $m \times n$-matrix $\mathbf{A}$ and interval right hand side m-vector $\mathbf{b}$. It is common knowledge that (1) is only formal symbol, which in itself can mean, for instance, a collection of all point linear algebraic systems with elements from $\mathbf{A}$ and $\mathbf{b}$, respectively. To pose a problem correctly we should at least define what is taken to mean the solution or the solution set to (1). Back in the early 80-th A.A. Vatolin [15], taking the linear programming problem as an example, had demonstrated the variety of concepts of the solution set to the problems with interval data. Up till now, however, in interval analysis the subjects of investigation were the following three solution sets:

- *united solution set*, formed by solutions of all point systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$, i.e., the set

  $$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b)\},$$

  historically first and undoubtedly the most popular of the solution sets; it is called by Western authors simply as *solution set* and usually is denoted by $\Sigma(\mathbf{A}, \mathbf{b})$ (see [1,5,6,8,13] and the extensive references there);
- *tolerable solution set*, formed by all point vectors x such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$, i.e., the set

  $$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b)\}$$

  (see [7,8,9,10,14] et al); A. Neumaier in [7] followed by some other authors call it *restricted solution set*, denoting $\Sigma_0(\mathbf{A}, \mathbf{b})$;
- *interval algebraic solution* [12], i.e., such an interval vector $\mathbf{x}_a$ that substituting it into (1) and executing all interval arithmetic operations results in the valid equality $\mathbf{A}\mathbf{x}_a = \mathbf{b}$.

But in this work we would like to draw the attention of the researchers to the new solution set to interval linear algebraic system, namely the set

$$\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)\}, \qquad (2)$$

formed by all point vectors $x \in \mathbb{R}^n$, such that for *any* desirable $b \in \mathbf{b}$ we can find the corresponding $A \in \mathbf{A}$ satisfying $Ax = b$. Its contentive interpretation is as follows. Let "the black box" be given with the input subjection vector $x \in \mathbb{R}^n$ and the output reply vector $y \in \mathbb{R}^m$, the input–output relationship being linear, i.e., $y = Ax$ with a real $m \times n$-matrix A. Suppose also that the elements of A may be varied by our will within some prescribed intervals $\mathbf{a}_{ij}$, so that A can be made any one from the corresponding interval matrix $\mathbf{A} = (\mathbf{a}_{ij})$. In other words, we have the possibility in some way to control the parameters of the black box in the designated bounds $(\mathbf{a}_{ij})$.

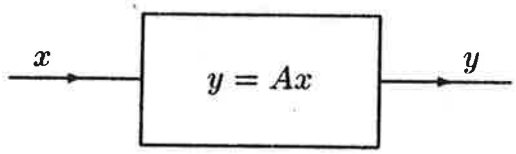

Fig. 1

If some interval vector $\mathbf{y}$ represents the set of output vectors, then the question naturally arises of whether exists input signal x which can be transferred to *any* desired output state $y \in \mathbf{y}$ by appropriate choice of the black box parameters $(\mathbf{a}_{ij})$. The set of all such x (if nonempty) just constitutes $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{y})$. Hence, it makes sence to refer to $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ as *controlled solution set* of interval linear algebraic system $\mathbf{A}x = \mathbf{b}$, since it distinguishes the set of input subjections x with respect to their controllability properties.

By the evident means all the above developed ideas are transferred to the general case of interval nonlinear system

$$F(x, \mathbf{a}) = \mathbf{b}\,, \tag{3}$$

with $F(x, a) = (f_1(x, a), f_2(x, a), \ldots, f_m(x, a))$ and $\mathbf{x}$, $\mathbf{a}$ being interval vectors of the same dimension as x, a, respectively. Let us call

$$\Sigma_{\exists\forall}(F; \mathbf{a}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists a \in \mathbf{a})(F(x, a) = b) \,\}$$

the controlled solution set to the interval system (3). If for the above considered "black box" the input–output relationship has the form $y = F(x, a)$, then the points of $\Sigma_{\exists\forall}(F; \mathbf{a}, \mathbf{y})$ (and no one else) can be transformed to any requested output state $y \in \mathbf{y}$ through some control subjection $a \in \mathbf{a}$.

In implicit form the controlled solution set seems to appear even in the work of N.A. Khlebalin and Yu.I. Shokin [3]. But, presumably, for the first time the definition (2) was written out explicitly by A.V. Lakeev and S.I. Noskov [4], who gave no name to this set, but examined some of its properties. Their main result concerning the set (2) is the following

Proposition. *(A.V. Lakeev and S.I. Noskov* [4])

$$\begin{aligned}&\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})\\ &= \{\, x' - x'' \mid x',\, x'' \in \mathbb{R}^n,\ x',\, x'' \geq 0,\ (\,x',\, x'') = 0,\\ &\qquad \underline{\mathbf{A}}x' - \overline{\mathbf{A}}x'' \leq \underline{\mathbf{b}},\ \overline{\mathbf{A}}x' - \underline{\mathbf{A}}x'' \geq \overline{\mathbf{b}}\,\},\end{aligned}$$

where $(\,\cdot\,,\,\cdot\,)$ is the standard scalar product in $\mathbb{R}^n$ (that is, the sum of products of components), and $\underline{\mathbf{a}}$, $\overline{\mathbf{a}}$ stand for lower and upper bounds of an interval (interval vector or matrix), respectively.

It is fairly simple to realize that

$$\begin{aligned}\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) &= \{\, x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)\}\\ &\subseteq \{\, x \in \mathbb{R}^n \mid (\exists b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)\} = \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}),\end{aligned}$$

i.e., the controlled solution set is always a subset of the united solution set. Hence, if $\mathbf{A}$ contains only nonsingular point matrices, then $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$ is bounded coincidentally with $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$.

To make our considerations more vivid and pictorial, turn to the Figure 2, where the controlled solution set to the interval system

$$\begin{pmatrix} [-2;1] & [-1;1] \\ [-1;1] & [-1;2] \end{pmatrix} x = \begin{pmatrix} [-2;1] \\ [-1;2] \end{pmatrix} \qquad (4)$$

is depicted. It is the whole plane with the star around the origin of coordinates removed.

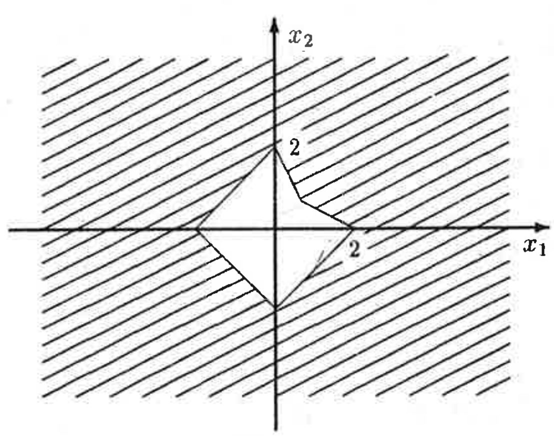

Fig. 2

The configuration of this picture is the typical one in some sence, in so far as the zero vector may belong to $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ only when $\mathbf{b} \subseteq \mathbf{A} \cdot 0 = 0$, i.e., $\mathbf{b} = 0$. Just for this reason the controlled solution set to (4) avoids the origin of coordinates on the Figure 2. Besides, the above Proposition implies that the intersection of $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ with each orthant of $\mathbb{R}^n$ is a polyhedral set, i.e., the intersection of finite number of half-spaces.

We now give the other simple proof that the intersection of the controlled solution set with each orthant is a convex set. If x, y are points from some one orthant of $\mathbb{R}^n$, so that $\lambda x_j(1-\lambda)y_j \geq 0$ for all $j = 1, 2, \ldots, n$ and $\lambda \in (0; 1)$, then the distributivity relationship holds

$$\mathbf{A}(\lambda x + (1-\lambda)y) = \lambda \mathbf{A}x + (1-\lambda)\mathbf{A}y$$

(see, e.g., [1,5,6,8]). In case x, $y \in \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ we have in addition $\mathbf{A}x \supseteq \mathbf{b}$ and $\mathbf{A}y \supseteq \mathbf{b}$. Therefore

$$\mathbf{A}(\lambda x + (1-\lambda)y) \supseteq \lambda \mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b},$$

as required.

If the dimensionality of interval system is considerable, then the direct description of its controlled solution set became laborious and practically useless (its complexity is proportional to $m \cdot 2^n$). For this reason it is expedient to confine ourselves to finding some simple subsets $\Pi \subseteq \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$, since for any $x \in \Pi$ the condition

$$(\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)$$

remains valid. To put this another way, we change $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ for its inner approximation. Most likely interval vectors, i.e., the direct products of segments of the real axis, have the simplest structure and so we formulate the forthcoming interval problem in the following form:

find an interval vector which is contained in the controlled

solution set of the given interval linear algebraic system.

In common with the tolerable solution set, the controlled solution set may turn out to be empty even for "good" interval data, as, for instance, it does in the one-dimensional case $\mathbf{A} = [2; 3]$, $\mathbf{b} = [1; 2]$. The popular model system

$$\begin{pmatrix} [2;4] & [-2;1] \\ [-1;2] & [2;4] \end{pmatrix} x = \begin{pmatrix} [-2;2] \\ [-2;2] \end{pmatrix}$$

from the work of W. Barth and E. Nuding [2] affords more complex example of the empty controlled solution set.

The main result of this work is a simple sufficient criterion for the controlled solution set of interval linear algebraic system to be nonempty. But before proceeding, we have to revise some auxiliary notions and facts of interval analysis.

To begin with, note that if the i-th row of $\mathbf{A}$ contains only zero elements, the controlled solution set being nonempty requires $\mathbf{b}_i = 0$ as a necessary condition. But then the property of $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ to be empty or nonempty depends from the other, not the i-th, rows of $\mathbf{A}$ and components of $\mathbf{b}$. So, we may consider in general that $\mathbf{A}$ has not zero rows.

In what follows the central role will play in our considerations the midpoint (median) of an interval, i.e., the quantity

$$\operatorname{med} \mathbf{p} = (\overline{\mathbf{p}} + \underline{\mathbf{p}})/2.$$

In case $\mathbf{p}$ is an interval vector or matrix, this operation shall be understood component-wise.

To characterize "the relative narrowness" of nonzero intervals H. Ratschek has introduced in [11] the functional

$$\chi(\mathbf{p}) = \begin{cases} \underline{\mathbf{p}}/\overline{\mathbf{p}}\,, & \text{if } |\underline{\mathbf{p}}| \le |\overline{\mathbf{p}}|, \\ \overline{\mathbf{p}}/\underline{\mathbf{p}}\,, & \text{otherwise}\,. \end{cases}$$

Clearly, $-1 \le \chi(\mathbf{p}) \le 1$, and $\chi(\mathbf{p}) = 1$ if and only if $\mathbf{p} \in \mathbb{R}$. Moreover, it turns out that

$$\chi(\mathbf{p}) = \chi(\mathbf{q}) \text{ if and only if } \mathbf{p} = t\,\mathbf{q},\ t \in \mathbb{R},\ t \ne 0, \tag{5}$$

$$\text{if } \mathbf{p} + \mathbf{q} \ne 0, \text{ then } \chi(\mathbf{p} + \mathbf{q}) \le \max\{\, \chi(\mathbf{p}), \chi(\mathbf{q}) \,\}. \tag{6}$$

The siraightforward proofs of these facts can be found in [11,12]. Also, sometimes the following obvious property can prove helpful:

if $\mathbf{p} \supseteq \mathbf{q}$ and $\chi(\mathbf{q}) \ge 0$, then $\chi(\mathbf{p}) \le \chi(\mathbf{q})$.

Further we will need, however, a kind of the converse statement:

Lemma. *If* $\operatorname{med} \mathbf{p} = \operatorname{med} \mathbf{q}$ *and* $-1 < \chi(\mathbf{p}) \le \chi(\mathbf{q})$ *, then* $\mathbf{p} \supseteq \mathbf{q}$.

Proof. Denote $\mu = \operatorname{med} \mathbf{p} = \operatorname{med} \mathbf{q}$. If $-1 < \chi(\mathbf{p}) \le \chi(\mathbf{q})$, then $\mu \ne 0$, i.e., $\mu < 0$ or $\mu > 0$. Without loss in generality we may allow the second opportunity, since the case of negative μ is treated in a similar manner. Under these conditions $|\underline{\mathbf{p}}| < |\overline{\mathbf{p}}|$ and $|\underline{\mathbf{q}}| < |\overline{\mathbf{q}}|$, and so $\chi(\mathbf{p}) \le \chi(\mathbf{q})$ implies

$$\underline{\mathbf{p}}/\overline{\mathbf{p}} \le \underline{\mathbf{q}}/\overline{\mathbf{q}},$$

or

$$\frac{\mu - \operatorname{rad}\mathbf{p}}{\mu + \operatorname{rad}\mathbf{p}} \le \frac{\mu - \operatorname{rad}\mathbf{q}}{\mu + \operatorname{rad}\mathbf{q}},$$

where $\operatorname{rad}\mathbf{p} = (\overline{\mathbf{p}} - \underline{\mathbf{p}})/2$ is the radius of an interval. We obtain from it after simple transformations:

$$\mu \cdot \operatorname{rad}\mathbf{p} \ge \mu \cdot \operatorname{rad}\mathbf{q},$$

which is equivalent (in view of $\mu > 0$) to $\operatorname{rad}\mathbf{p} \ge \operatorname{rad}\mathbf{q}$, i.e., $\mathbf{p} \supseteq \mathbf{q}$.

Now we are able to formulate and to prove the

Theorem. *Let interval $m \times n$-matrix $\mathbf{A}$ and interval m-vector $\mathbf{b}$ be such that for each $i \in \{1, 2, \dots, m\}$ the conditions*

(i) $\mathbf{b}_i \ne 0$,

(ii) $-1 < \max\{\, \chi(\mathbf{a}_{ij}) \mid 1 \le j \le n,\ \mathbf{a}_{ij} \ne 0 \,\} \le \chi(\mathbf{b}_i)$

are valid. If "the middle system" $\operatorname{med}\mathbf{A} \cdot x = \operatorname{med}\mathbf{b}$ is compatible, then its solution belongs to the controlled solution set $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ (which is accordingly nonempty).

Proof. If $\tilde{x}$ is a solution to the "middle" point system, then [8]

$$\operatorname{med}(\mathbf{A}\tilde{x}) = (\operatorname{med}\mathbf{A}) \cdot \tilde{x} = \operatorname{med}\mathbf{b}.$$

Furthermore, since $-1 < \chi(\mathbf{b}_i)$, $i = 1, 2, \dots, m$, we have $\operatorname{med}(\mathbf{A}\tilde{x})_i \ne 0$. So, the following calculations are legitimate for each $i \in \{1, 2, \dots, m\}$:

$$\begin{aligned}
\chi((\mathbf{A}\tilde{x})_i) &= \chi\left(\sum_{j=1}^{n} \mathbf{a}_{ij}\tilde{x}_j\right) \\
&\le \max\{\, \chi(\mathbf{a}_{ij}\tilde{x}_j) \mid 1 \le j \le n,\ \mathbf{a}_{ij}\tilde{x}_j \ne 0 \,\} && \text{by (5)} \\
&= \max\{\, \chi(\mathbf{a}_{ij}) \mid 1 \le j \le n,\ \mathbf{a}_{ij}\tilde{x}_j \ne 0 \,\} && \text{by (6)} \\
&\le \max\{\, \chi(\mathbf{a}_{ij}) \mid 1 \le j \le n,\ \mathbf{a}_{ij} \ne 0 \,\}.
\end{aligned}$$

Thus, $-1 < \chi(\,(\mathbf{A}\tilde{x})_i) \le \chi(\mathbf{b}_i)$ holds true for $i \in \{1, 2, \dots, m\}$ and by Lemma

$$(\mathbf{A}\tilde{x})_i \supseteq \mathbf{b}_i\,, \quad i \in \{1, 2, \dots, m\},$$

which proves the Theorem.

For example, both conditions of the Theorem are satisfied for the above considered interval system (4). Solution of "the middle system" is the vector $(1,\,1)^{\top}$ and it is seen (on Fig. 2) to belong to the controlled solution set of (4). On the other hand, the condition (ii) of the Theorem does not hold for the interval linear system

$$\begin{pmatrix} [2;4] & [-2;2] \\ [-2;2] & [2;4] \end{pmatrix} x = \begin{pmatrix} [1;5] \\ [1;5] \end{pmatrix}, \tag{7}$$

while its "midpoint" system is compatible and has the solution $(1,\,1)^{\top}$ obviously contained in the nonempty controlled solution set of (7).

It may appear from the statement of the Theorem that the solution of the "middle system" is the most probable representative of the controlled solution set to interval linear algebraic system. However, the following counterexample shows that this is not so in general. For the system

$$\begin{pmatrix} 3 & -[1;2] \\ [1;2] & 3 \end{pmatrix} x = \begin{pmatrix} [5;7] \\ [7;9] \end{pmatrix}$$

we have $\Sigma_{\exists\forall} = \{(1,\,2)^{\top}\}$, but the "middle system" solution is $(8/9,\,20/9)^{\top}$.

In spite of the apparent unwieldiness of this criterion, its implementation requires as low as $O(mn)$ of arithmetical and logical operations. The question of compatibility of "the middle system" is resolved trivially, if $\mathbf{A}$ is nonsingular interval matrix (i.e., contains only nonsingular point matrices). In its turn there are developed numerical algorithms for testing whether interval matrix is nonsingular [13], though on the whole this problem is not quite trivial.

To summarize, one may assert that the above stated criterion is quite practical, but not sufficiently sensitive. It is intended for the preliminary rough examination of a given problem.

When solving practical problems apart from the solution on its own one not infrequently needs some characteristics of its stability that characterizes the solvability margin or the measure of the compatible state stability. In our opinion, the magnitude

$$\varrho = \min_{1 \le i \le m} \left\{ \chi(\mathbf{b}_i) - \max\{\chi(\mathbf{a}_{ij}) \mid 1 \le j \le n,\ \mathbf{a}_{ij} \ne 0\} \right\}$$

can be applied for this purpose as a crude quantitive measure of compatibility factor in the case $\varrho > 0$.

In conclusion it is useful to correlate the main result of this work with that of the article [14]. Recall that there sufficient criterion is obtained for the tolerable solution set to be empty. It is formulated as follows:

Let interval $m \times n$*-matrix* $\mathbf{A}$ *and interval* m*-vector* $\mathbf{b}$ *be such that for some* $k \in \{1, 2, \dots, m\}$ *the conditions hold*

(i) $0 \notin \mathbf{b}_k$,

(ii) $\max\{\ \chi(\mathbf{a}_{kj}) \mid 1 \le j \le n,\ \mathbf{a}_{kj} \ne 0\,\} < \chi(\mathbf{b}_k)$.

Then the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *is empty.*

As is seen, this statement is in remarkable duality to the Theorem, and the sets $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ and $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ are in a certain antagonism to each other.

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