Interval Computations
No 3(5), 1992

AN ALGORITHM OF INTERVAL MATRIX ASYMPTOTIC STABILITY TESTING

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A new sufficient condition of interval matrix asymptotic stability is considered. A simple computer test based on the extended Gershgorin's results is presented.

АЛГОРИТМ ПРОВЕРКИ АСИМПТОТИЧЕСКОЙ УСТОЙЧИВОСТИ ИНТЕРВАЛЬНОЙ МАТРИЦЫ

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Рассматривается новое достаточное условие асимптотической устойчивости интервальной матрицы. Представлен простой компьютерный тест, основанный на расширенных результатах Гершгорина.

1. Introduction

An interval matrix stability problem appears at the stage of the robust control design for n-order interval dynamic system [1]. In recent literature this problem is discussed in [2]–[7]. Here a simple programmed technique for its studying is worked out. The calculating procedure is based on the extended Gershgorin's results and sufficient condition of the asymptotic stability. The proposed algorithm can be used for larger class of interval matrices unlike the other methods from [2]–[7]. Its efficiency is illustrated by numerical examples.

2. The circle approximation method and its interval version

At first consider a matrix A having real elements $a_{ij}, i, j = \overline{1, n}$.

Definition 1. A real matrix A can be called asymptotic stable if all its eigenvalues $\lambda_i, i = \overline{1, n}$ have strictly negative real parts $Re\lambda_i < 0, i = \overline{1, n}$. Let A is asymptotic stable and all its eigenvalues are covered by circular region of radius R. This circle is placed on the left part of the complex λ-plane and has a center in the point $(-R, 0)$ of the real axis. Using linear function

$$\lambda = R(\omega - 1) \tag{1}$$

the circular region can be transformed into a circle of the complex ω-plane. The center of the first circle is also changed to the point $(0, 0)$ which is a center of the unit radius circle. The characteristic equation for the given matrix A $\det(A - \lambda I_n) = 0$ is transformed by (1) to $\det(B - \omega I_n) = 0$ for the following matrix

$$B = A/R + I_n \tag{2}$$

where I_n is the unit matrix of n order. Since all eigenvalues $\lambda_i, i = \overline{1, n}$ belong to the circle of radius R then all eigenvalues $\omega_j, j = \overline{1, n}$ of the matrix B (2) have to be contained in the inner part of the unit circle. Then the matrix B (2) degrees B^k, $k = 1, 2, 3, \ldots$, are built. Simultaneously all eigenvalues of B (2) $\omega_1, \ldots, \omega_n$ have to be raised to the same power k : $\omega_1^k, \ldots, \omega_n^k$ [8]. As the modules of the eigenvalues B (2) are less than the unit $\mid \omega_j \mid < 1, j = \overline{1, n}$, the modules of their degrees $\mid \omega_j^k \mid, j = \overline{1, n}$ will converge to zero as $k \to \infty$. Hence for $k \to \infty$ the consistency of the matrix degrees $\{B^k\} \to \theta_n$ converges to θ_n where θ_n is a null matrix of n order. Instead of the elements B^k reducing one can study matrix B^k norms which also have to be less than the unit. So, if for any real matrix A the converging matrix degrees B^k consistency can be constructed then the given A is asymptotic stable. This is a sufficient criterion known as a "circle approximation" method [9].

Consider an interval version of the given algorithm for a matrix $[A]$ with the interval elements $[a_{ij}] = [\underline{a}_{ij}, \overline{a}_{ij}], i, j = \overline{1, n}$.

Definition 2. An interval matrix $[A]$ is called asymptotic stable if all real matrices $A \in [A]$ are asymptotic stable according to the Definition 1.

Definition 3. A set of the characteristic equations for all real matrices $A \in [A]$ $\{\det(A - \lambda I_n) = 0,\ A \in [A]\}$ may be called a characteristic equation for the interval matrix $[A]$ and formally marked as follows [1]: $\det([A] - \lambda I_n) = 0$.

Suppose, a circle of the known radius R cowers all eigenvalues of the matrices $A \in [A]$. For real matrix B (2) the following interval analogy

$$[B] = [A]/R + I_n \tag{3}$$

also can be built. It can be shown that all eigenvalues of every matrix $B \in [B]$ will be located in the unit circle of the ω-plane. Study the interval matrix degrees consistency $\{[B]^k\}$ for $k \to \infty$. If lower and upper bounds of all interval elements $[b_{ij}^{(k)}]$, $i, j = \overline{1, n}$ simultaneously reduce to zero then the given matrix $[A]$ is asymptotic stable.

In practice it is better to build the powers $k = 2^l, l = 0, 1, 2, \ldots$ and to investigate the convergence of the matrix $[B]^k$ norms consistency.

Put into consideration a real matrix $M^{(k)} =\mid [B]^k \mid$ containing the interval elements $[b_{ij}^{(k)}]$ modules:

$$m_{ij}^{(k)} =\mid [b_{ij}^{(k)}] \mid = \max\{\mid \underline{b}_{ij}^{(k)} \mid, \mid \overline{b}_{ij}^{(k)} \mid\}, i, j = \overline{1, n}.$$

Note by

$$\begin{aligned}
\| M^{(k)} \|_I &= \max_{i=\overline{1,n}} \sum_{j=1}^{n} \mid m_{ij}^{(k)} \mid \\
\| M^{(k)} \|_{II} &= \max_{j=\overline{1,n}} \sum_{i=1}^{n} \mid m_{ij}^{(k)} \mid \\
\| M^{(k)} \|_{III} &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \mid m_{ij}^{(k)} \mid \right)^{1/2} \\
\| M^{(k)} \|_{IV} &= n \times \max_{i,j=\overline{1,n}} \mid m_{ij}^{(k)} \mid
\end{aligned} \tag{4}$$

the well-known matrix norms [8],[9]. Let

$$P(k) = \min\left(\| M^{(k)} \|_I, \| M^{(k)} \|_{II}, \| M^{(k)} \|_{III}, \| M^{(k)} \|_{IV}\right) \tag{5}$$

is minimal matrix $M^{(k)}$norm. Then the simple sufficient criterion includes in the following test: if the inequality

$$P(k) < 1 \tag{6}$$

is satisfied for some k then the interval matrix $[A]$ is asymptotic stable. In the other cases $[A]$ can be stable or unstable. Note if the conditions (5),(6) are fulfilled for $m = 0$, i.e. $k = 1$, the building and studying matrix degrees $[B]^k$ are not necessary.

The parameter R was supposed to be known in both real and interval methods. In the next section its calculation will be presented for any matrix.

3. Evaluation of radius R

To determine the radius R covering all eigenvalues of real matrix A one can use any known method which allows to estimate the size of its eigenvalues location region [8],[9]. For example, Gershgorin's results can be used for this purpose. According to his theorem all eigenvalues of real matrix A are included in the union of the discs [9]

$$| \lambda - a_{ii} | \leq P_i,$$

$$P_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} | a_{ij} |$$

placed on the complex λ-plane. Every circle has a center in the point a_{ii} and the radius P_i. These results have been extended to the interval matrix $[A]$ in [1]. The special intervals

$$[G_i] = [\underline{a}_{ii} - R_i, \overline{a}_{ii} + R_i], \; i = \overline{1,n}, \tag{7}$$

$$R_i = \sum_{\substack{i=1 \\ j \neq i}}^{n} | [a_{ij}] | \tag{8}$$

have been introduced there. They are the segments of the real axis since $\underline{a}_{ii}$ and $\overline{a}_{ii}$ are the real bounds of the interval $[a_{ii}]$. The centers of the i Gershgorin's discs for all matrices $A \in [A]$ are included in the interval $[G_i]$

(7), (8) and the parameter R_i may be considered as its maximal radius. Describe an algorithm for the R calculation consisting of following steps.

1. For the given interval matrix $[A]$ the intervals $[G_i]$ (7), (8) are constructed.

2. The interval $[D] \supseteq \cup_{i=1}^{n}[G_i]$ including their union or coinciding with one is determined.

3. For the transposed matrix $[A]^\top$ the segments (7), (8)

$$[G_i^\top] = [\underline{a}_{ii} - Q_i, \overline{a}_{ii} + Q_i], \ i = \overline{1,n}$$

$$Q_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} | \, [a_{ji}] \, |, \ i = \overline{1,n}$$

are built.

4. The interval $[D^\top] \supseteq \cup_{i=1}^{n}[G_i^\top]$ can also be calculated for $[A]^\top$.

5. The intersection $[Z] = [D] \cap [D^\top]$ is found and its width $R = w[Z]$ can be used as a radius of the circle containing all eigenvalues of real matrices $A \in [A]$.

4. Examples

Example 1. Consider interval matrix [5]

$$[A_1] = \begin{pmatrix} [0, 0.1] & 1 & 0 \\ [-1, -0.9] & -1 & [-0.1, 0] \\ 0 & [0, 0.1] & -1 \end{pmatrix}.$$

In according with the proposed method one can evaluate radius $R = W[-2.1, 1] = 3.2$. The norms (4) for the transformed matrix $[B]$ (3) are equal to $(1.344, 1.344, 1.485, 3.094)$. It is obvious that the minimal norm is greater then the unit. Hence the matrix degrees $[B]^k$ have to be tested. For $m = 3$, i.e. $k = 8$, the inequality (6) $P(8) = \| M^{(8)} \|_{III} = 0.889 < 1$ is fulfilled. So, $[A_1]$ is asymptotic stable as it has been shown in [5].

Example 2. Use the proposed sufficient criterion for another interval matrix from [5]

$$[A_2] = \begin{pmatrix} [-10, -9] & [-7, -6] \\ [20, 25] & [-4, -3] \end{pmatrix}.$$

The radius of the circular region is equal to $R = W[-29, 16] = 45$ and the matrix $[B]$ (3) has the following norms $(1.481, 1.356, 1.358, 1.867)$ the minimal of which is greater than the unit. For $m = 4$, i.e. $k = 16$, one of the mentioned above norms $\| M^{(16)} \|_{III} = 0.901$ satisfies the inequality (6). It means that $[A_2]$ is asymptotic stable as it follows from [5].

Example 3. Take the interval matrices $[A_1]$ and $[A_2]$ and build a block interval matrix of 5-order

$$[A_3] = \begin{pmatrix} [A_1] & 0 \\ 0 & [A_2] \end{pmatrix}.$$

Study its asymptotic stability using the worked out method. The radius $R = 45$ and the norms $(1.489, 1.356, 2.182, 5.011)$ can be evaluated according to the mentioned above procedure. For $m = 7$, i.e. $k = 128$, simultaneously three norms $\| M^{(7)} \|_I = 0.744$, $\| M^{(7)} \|_{II} = 0.74$, $\| M^{(7)} \|_{III} = 0.653$ have become less than the unit hence $[A_3]$ is asymptotic stable. Note that for a large order n longer consistencies of matrix degrees $[B]^k$ have to be tested.

Example 4. Apply the presented algorithm to the studying of an unstable interval matrix [5]

$$[A_4] = \begin{pmatrix} [-8, -7] & [3, 4] \\ [5, 6] & [4, 5] \end{pmatrix}.$$

It has $R = W[-12.9] = 21$ and the norms $(1.524, 1.429, 1.447, 2.476)$. The evaluation of the matrix degrees $[B]^k$ for $m = 1, 2, 3\ldots$ results to the norms $(11.833, 10.636, 9.363, 16.483)$ which rapidly increase for $m > 3$.

5. Conclusion

The test presented in the paper is a sufficient criterion of interval matrix asymptotic stability. This algorithm can be applied not only for the matrices having negative interval elements on the main diagonal or order $n \leq 4$ unlike the methods worked out in [2]–[7]. But for large n longer run time is needed.

Author thanks I.Yu.Naslednikova for the program realization of the proposed method in IBM PC PASCAL.

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