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AN ORDER-THEORETIC APPROACH TO INTERVAL ANALYSIS

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Interval analysis makes heavy use of inclusion order for intervals, and relies on the notion of monotonicity of functions for this order. It turns out that $(\mathbf{IR}, \supseteq)$, where $\mathbf{IR}$ is the set of Moore intervals plus $[-\infty, +\infty]$, is a complete partial order (**cpo**). Then it is appropriate to use the theory of **cpo**'s in order to fully formalize the role of inclusion order. This work develops basic concepts of interval analysis within this order-theoretic framework. There is a natural topology for **cpo**'s, the Scott topology. It is shown that Scott topology on $\mathbf{IR}$ is compatible with both inclusion-monotonicity and the usual topology on the real line.

ТЕОРЕТИКО-ПОРЯДКОВЫЙ ПОДХОД К ИНТЕРВАЛЬНОМУ АНАЛИЗУ

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В интервальном анализе широко используется отношение порядка по включению для интервалов; на нем базируется понятие монотонности функции относительно этого порядка. Можно показать, что множество $(\mathbf{IR}, \supseteq)$, где $\mathbf{IR}$ – множество интервалов Мура плюс $[-\infty, +\infty]$, есть законченное частично упорядоченное множество (**cpo**). В этой ситуации естественно использовать теорию **cpo** с целью полностью формализовать некоторые вещи, связанные с порядком по включению. Настоящая работа развивает

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основные положения интервального анализа в рамках теоретико-порядкового подхода. Существует естественная топология для **cpo** – топология Скотта. Показано, что топология Скотта на **IR** совместима как с монотонностью по включению, так и с обычной топологией вещественной прямой.

1. Introduction

Interval analysis [1] makes heavy use of inclusion order for intervals, and relies on the notion of monotonicity of functions for this order. It turns out that $(\mathbf{IR}, \supseteq)$, where **IR** is the set of Moore intervals plus $[-\infty, +\infty]$, is a complete partial order (**cpo**) [2]. Then it is appropriate to use the theory of **cpo**'s in order to fully formalize the role of inclusion order. This work develops basic concepts of interval analysis within this order-theoretic framework (see also [3–6]), as described below.

Section 2 introduces basic concepts of interval analysis. Section 3 presents the definition of **cpo** and some fundamental notions relative to **cpo**'s. In particular, the notion of continuous functions on **cpo**'s is defined, and continuity for functions on **IR** is related to continuity of functions on **R**. Section 4 introduces a standard topology on **cpo**'s, the Scott's topology, and relates it to the continuity notion for **cpo**'s. Moreover, the Scott's topology on **IR** is related to the usual topology on the real line. Finally, section 5 presents conclusions and discusses further applications of the **cpo**-structure of **IR**.

2. Basic concepts of interval theory

Definition 1.(Intervals). Let **R** be the ordered field of real numbers.

For real numbers $a \leq b$ the closed interval $\{x \in \mathbf{R} | \ a \leq x \leq b\}$ is denoted by $[a, b]$.

In this work, the closed set of all real numbers is also considered as a closed interval, denoted by $[-\infty, +\infty]$.

Definition 2.(IR). The set of all closed intervals is denoted by **IR**.

The letters X, Y, Z are interval variables over **IR**, the letters $\vec{X}, \vec{Y}, \vec{Z}$ are interval variables over $\mathbf{IR}^n$, and F, G, H are interval function variables over $\mathbf{IR}^n \to \mathbf{IR}$.

Real numbers are identified with single-point intervals, so that $\mathbf{R} \subseteq \mathbf{IR}$. Hence for $x \in \mathbf{R}$ we have $x = [x, x]$.

Then it is possible to extend functions $\mathbf{R}^n \to \mathbf{R}$ to functions $\mathbf{IR}^n \to \mathbf{IR}$.

Definition 3.(Interval ranges). Let $f : \mathbf{R}^n \to \mathbf{R}$. The *interval range* of f is $F : \mathbf{IR}^n \to \mathbf{IR}$ defined by

$$F(\vec{X}) = \bigcap \{Y \in \mathbf{IR} | Y \subseteq f(\vec{X})\}$$

i.e., the least interval in $\mathbf{IR}$ which contains $f(\vec{X})$.

F is denoted by $\hat{f}$ in order to make explicit its dependence on f.

Note that F is an extension of f. The proof for $n = 1$ is given by

$$\begin{aligned} F(x) =& F([x,x]) = \bigcap \{Y \in \mathbf{IR} | Y \subseteq f([x,x])\} \\ =& \{f(x)\} = [f(x), f(x)] = f(x). \end{aligned}$$

Some of this equalities are due to the identification of real numbers with single-point intervals (cf. definition 2).

Theorem 1. (Characterization of interval ranges). *For all* $f : \mathbf{R}^n \to \mathbf{R}$, *the interval range* $\hat{f} : \mathbf{IR}^n \to \mathbf{IR}$ *of* f *satisfies*

$$\hat{f}(\vec{X}) = [\inf f(\vec{X}), \sup f(\vec{X})].$$

Theorem 2. (Monotonicity of interval ranges). *For all* $f : \mathbf{R}^n \to \mathbf{R}$, $\hat{f}$ *is inclusion-monotonic, i.e.,* $\vec{X} \subseteq \vec{Y}$ *implies* $\hat{f}(\vec{X}) \subseteq \hat{f}(\vec{Y})$, *where* $\vec{X} \subseteq \vec{Y}$ *iff* $X_i \subseteq Y_i$ *for all* $i = 1..n$.

3. Order-theoretic structure of IR

Definition 4.(Partial order). Let D be a set, and $\sqsubseteq$ a relation on D. Then $(D, \sqsubseteq)$ is said to be a *partial order* (**po**) iff $\sqsubseteq$ is reflexive, transitive, and antisymmetric.

Definition 5.(Complete partial order). Let $(D, \sqsubseteq)$ be a **po**.

A non-empty set $X \subseteq D$ is said to be *directed* iff for all $x, y \in X$, $x \sqsubseteq z$ and $y \sqsubseteq z$ for some $z \in X$.

$(D, \sqsubseteq)$ is said to be a *complete partial order* (**cpo**) iff D has a least element $\bot$, and for all $X \subseteq D$ directed, X has a least upper bound, denoted by $\bigsqcup X$.

Definition 6.(ω-continuous cpo). Let $(D, \sqsubseteq)$ be a **cpo**.

The *way-below* order $\ll$ on D is defined by $d \ll e$ iff for all $X \subseteq D$ directed, $e \sqsubseteq \bigsqcup X$ implies $d \sqsubseteq x$ for some $x \in X$.

A set $B \subseteq D$ is said to be **a basis** for D iff for all $x \in D$, the set $\{b \in B | b \ll x\}$ is directed and its least upper bound is x.

$(D, \sqsubseteq)$ is said to be an *ω-continuous* **cpo** iff D has a denumerable basis.

Definition 7.(Domain). Let $(D, \sqsubseteq)$ be an ω-continuous **cpo**.

A set $X \subseteq D$ is said to *be bounded* iff X has an upper bound in D.

D is said to be *bounded-complete* iff every bounded subset of D has a least upper bound.

A *domain* is a bounded-complete, ω-continuous **cpo**.

Theorem 3. (Order-theoretic structure of IR). *For $X, Y \in \mathbf{IR}$ define $X \sqsubseteq Y$ iff $X \supseteq Y$. Then $(\mathbf{IR}, \sqsubseteq)$ is a domain. Moreover, the least element of* **IR** *is $\bot := [-\infty, +\infty]$; the way-below order for* **IR** *satisfies*

$$X \ll Y \text{ iff } X = \bot \text{ or } x_1 < y_1 \leq y_2 < x_2,$$

where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$; and a basis for **IR** *is $\{[p, q] \in \mathbf{Q}^2 | p < q\}$.*

An interval $[a, b]$ can be seen as a *totally defined* real number if $a = b$, and as a *partially defined* real number if $a < b$; the interval $\bot$ can be seen as a *completely undefined* real number (see [7]). Totality and partiality correspond to respectively maximality and non-maximality for $\sqsubseteq$. The fact that $\bot$ is a completely undefined real number corresponds to the fact that it is the least element for $\sqsubseteq$.

Definition 8.($\bigsqcup$-Continuity) Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be **cpo**'s. A function $f : D \to E$ is said to be $\bigsqcup$*-continuous* iff for all D-directed X, $f(X)$ is E-directed, and $f(\bigsqcup_D X) = \bigsqcup_E f(X)$.

Theorem 4. (Monotonicity of continuous functions). *Every $\bigsqcup$-continuous function $f : D \to E$ is $\sqsubseteq$-monotonic.*

Theorem 4 shows that the notion of $\bigsqcup$-continuity for functions $\mathbf{IR} \to \mathbf{IR}$ is compatible with the notion of inclusion-monotonicity. On the other hand, there are continuous functions $\mathbf{IR} \to \mathbf{IR}$ for Moore topology which are not inclusion monotonic, as it is well known (see [3]).

Theorem 5. (Cartesian products of domains). *Let D_i for $i = 1..n$ be domains. Then $D_1 \times \cdots \times D_n$ coordinatewise ordered by*

$$(x_1, \ldots, x_n) \sqsubseteq (y_1, \ldots y_n) \text{ iff } x_i \sqsubseteq y_i \text{ for } i = 1..n$$

is a domain.

In particular, $\mathbf{IR}^n$ is a domain.

Theorem 6. (Continuity of interval ranges). *For all $f : \mathbf{R}^n \to \mathbf{R}$, $\hat{f} : \mathbf{IR}^n \to \mathbf{IR}$ is $\bigsqcup$-continuous iff f is continuous for the usual Hausdorff topology on $\mathbf{R}$.*

There are functions which are not continuous in the whole of $\mathbf{R}$ but are continuous on a subset of $\mathbf{R}$. A similar theorem holds for these functions. For instance, $x \mapsto 1/x$ is a continuous function on $\mathbf{R} - \{0\}$, which can be extended to a $\bigsqcup$-continuous function F on the whole of $\mathbf{IR}$ such that $F(X) = \perp$ if $X \sqsubseteq 0$, i.e., $0 \in X$. In general, every continuous function on $A \subseteq \mathbf{R}$ can be extended to a $\bigsqcup$-continuous function on the whole of $\mathbf{IR}$ (see theorem 11, section 4).

Theorem 7. (Function spaces). *Let D, E be domains. Then $[D \to E]$, the set of $\bigsqcup$-continuous functions $D \to E$ pointwise ordered by*

$$f \sqsubseteq g \text{ iff } f(x) \sqsubseteq g(x) \text{ for all } x \in D,$$

is a domain.

In particular, $[\mathbf{IR}^n \to \mathbf{IR}]$ and $[[\mathbf{IR}^n \to \mathbf{IR}] \to [\mathbf{IR}^m \to \mathbf{IR}]]$ are domains.

In the same way that an interval is a set of real numbers, a $\bigsqcup$-continuous interval function can be seen as a set of real functions, according to the following theorem.

Theorem 8. (Interval functions as sets of real functions). *For all $\bigsqcup$-continuous $F : \mathbf{IR}^n \to \mathbf{IR}$ define*

$$|F| = \{f : \mathbf{R}^n \to \mathbf{R} \text{ continuous } | F \text{ is an extension of } f\}$$

Then

- $F \sqsubseteq G$ *iff* $|F| \supseteq |G|$
- F *is maximal iff* $|F|$ *is a singleton*
- $\hat{f} = \{f\}$

Corollary 9. *There is a $1-1$ correspondence between continuous real functions and $\bigsqcup$-continuous maximal interval functions, given by $f \mapsto \hat{f}$. Moreover, for each continuous real function f, the set*

$$\{F | F \text{ is a continuous interval extension of } f\}$$

is directed and its least upper bound is $\hat{f}$.

Then, in the same way a real number x was identified with the interval $[x, x]$, we identify a continuous real function f with its interval range $\hat{f}$.

4. Topological structure of IR

Definition 9.(Scott topology). Let $(D, \sqsubseteq)$ be a **cpo**. The *Scott topology* S_D on D is defined by:

$O \subseteq D$ is S_D*-open* iff

- $x \in O$ and $x \sqsubseteq y$ implies $y \in O$,
- $\bigsqcup X \in O$, with $X \subseteq D$ directed implies $X \cap O \neq \emptyset$.

Theorem 10. ($\bigsqcup$-continuity=Scott-continuity). *Let $(D, \sqsubseteq)$ and $(E, \sqsubseteq)$ be* **cpo***'s. Then $f : D \to E$ is $\bigsqcup$-continuous iff f is Scott-continuous.*

Theorem 11. *Let H be the usual Hausdorff topology on the real line and S be the Scott topology on* **IR***. Then $(\mathbf{R}, H)$ is a topological subspace of $(\mathbf{IR}, S)$, i.e.:*

- $\mathbf{R} \subseteq \mathbf{IR}$,
- $H = \{O \cup \mathbf{R} | O \in S\}$.

Corollary 12. *A function* $f : A \to \mathbf{R}$, *for* $A \subseteq \mathbf{R}$, *is* H*-continuous iff* f *has a* S*-continuous extension* $F : \mathbf{IR} \to \mathbf{IR}$. *Conversely, if a function* $F : \mathbf{IR} \to \mathbf{IR}$ *is* S*-continuous, then the function* $f : A \to \mathbf{R}$ *defined by*

- $A = \{x \in \mathbf{R} | F(x) \in \mathbf{R}\}$,
- $f(x) = F(x)$ *for all* $x \in A$

is H*-continuous.*

Note that the Scott topology on **IR** is T_0 and is not T_2 (i.e., Hausdorff). Hence this topology is not metrizable.

5. Conclusion

The Moore topology on **IR** is not compatible with inclusion monotonicity, in the sense that there are continuous functions for this topology which are not monotonic. But the notion of inclusion monotonicity is fundamental for interval theory. This work presents an order-theoretic approach to interval analysis, emphasizing the role of the inclusion order. Section 3 develops this approach, and shows that it is an appropriate foundation for interval analysis. Section 4. introduces Scott topology for **cpo**'s. It turns out that the Scott topology for the **cpo IR** extends the usual Hausdorff topology on the real line (see section 4, theorem 11). Moreover, every continuous function for this topology is monotonic. Hence Scott continuity is compatible with monotonicity.

Section 3 shows that in the same way elements of **IR**, can be seen as approximations of real numbers, elements of the function space $[\mathbf{IR}^n \to \mathbf{IR}]$ can be seen as approximations of real functions. In the same way real numbers are identified with single-point intervals, real functions are identified with interval ranges. Moreover, in the same way approximations of real numbers are seen as sets of real numbers, approximations of real functions can be seen as sets of real functions.

This order-theoretic approach to interval analysis has some immediate applications. First, it is possible to give denotational semantics [8] to programs which allow interval computations. Note that denotational semantics of programming languages usually omit numerical features, except for integer numbers. Second, there is a theory of computability for domains (see [9]), so that we have the notion of computable interval, com

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(i.e., an operator like integration or differentiation). Note that Moore interval theory does not allow us to classify objects as computable or non-computable.

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