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ELLIPSOIDAL ESTIMATES FOR A SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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The total error arising in the numerical solution of a system of differential equations is estimated by the ellipsoid method.

ЭЛЛИПСОИДАЛЬНЫЕ ОЦЕНКИ РЕШЕНИЯ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Общая ошибка, возникающая при численном решении системы дифференциальных уравнений, оценивается методом эллипсоидов.

In the solution of a system of differential equations by difference methods, the interval technique [1] used to obtain strict upper and lower estimates for each component of a solution often leads to exponential growth of the error estimate, even though the error itself does not grow. It occurs, for example, in the case of the system $x' = y$, $y' = -x$; see [1], [2]. A method allowing reduction of the exponential growth to polynomial growth is given in [3]. In [4], a class of systems with right-hand sides nondecreasing in the nondiagonal arguments is described, in which the phenomenon of excessive exponential growth of the error estimate does not occur.

Exponential growth can occur in the componentwise estimate of the size of the accessibility set for controlled systems. In the latter case, the ellipsoid method [5] was proposed to overcome this difficulty. Furthermore, this method is used to find strict estimates for a solution of a system of differential equations, and, therefore, to find a strict error estimate obtained together with approximate computation of a solution. A version of the ellipsoid method is considered in [6].

In the proposed method, the point representative of a solution of the system at arbitrary time is contained in a moving ellipsoid. The evolution equation of this ellipsoid is derived taking into account both the discretization error of the method and the roundoff error both for the solution of the given system and for the solution of the evolution equation of the ellipsoid.

There are other methods for obtaining strict estimates for a solution apart from difference methods; see, for example [2]. We do not consider them in this paper.

1. Notation and auxiliary results

Below, all matrices determining quadratic forms and ellipsoids are assumed to be real and symmetric. A matrix A of dimension $n \times n$ determines a quadratic form $x \cdot Ax$ ($x \in R^n$, the dot denotes the scalar product of vectors). The notation $A > 0$ (or $A \geqslant 0$) denotes $x \cdot Ax > 0$ for all $x \neq 0$ ($x \cdot Ax \geqslant 0$ respectively); I is an identity matrix.

Let $a \in R^n$, and let Q be a matrix with $Q \geq 0$. As in [5], §5, a set of points $x \in R^n$ such that $x = a + Qy$, $y \in R^n$ and $Qy \cdot y \leqslant 1$, is called an ellipsoid, and is denoted $E(a, Q)$. If $Q > 0$, the above set is determined by the inequality $(x - a) \cdot Q^{-1}(x - a) \leqslant 1$. If $rankQ = r < n$, the ellipsoid $E(a, Q)$ is degenerate, that is, the ellipsoid lies in an r-dimensional hyperplane. Examples: the point a (as $Q = 0$); the segment (as $r = 1$); the planar disk in R^3.

Lemma. *If $A \geqslant 0$ and $B \geqslant 0$ are real symmetric matrices, then $E(0, A) \subset E(0, A + B)$.*

Proof. First consider the case $A > 0$. Then $A + B > 0$, and we must demonstrate that the set of points $x \in R^n$ such that $x \cdot A^{-1}x \leqslant 1$ is contained in the set $x \cdot (A+B)^{-1}x \leqslant 1$. To do this, we demonstrate that

$$x \cdot (A+B)^{-1}x \leqslant x \cdot A^{-1}x \qquad (x \in R^n).$$

Denoting $(A+B)^{-1}x = y$ we obtain the equivalent inequlities:

$$x \cdot [A^{-1}x - (A+B)^{-1}x] \geqslant 0, \qquad (A+B)y \cdot [A^{-1}(A+B)y - y] \geqslant 0,$$
$$(A+B)y \cdot A^{-1}By \geqslant 0, \qquad Ay \cdot A^{-1}By + By \cdot A^{-1}By \geqslant 0.$$

The first term of the sum is equal to $y \cdot AA^{-1}By = y \cdot By \geqslant 0$, since the matrix A is symmetric and $B \geqslant 0$; the second term is equal to $Az \cdot z \geqslant 0$, where $z = A^{-1}By$. In the case $A > 0$ the lemma is proven.

Let $A \geqslant 0$, $A_k = A + k^{-1}I, k = 1, 2, \ldots$. Then $A_k > 0$ and by what has been proved, $E(0, A_k) \subset E(0, A_k + B)$. We have $A_k \to A$ as $k \to \infty$, and the statement of the lemma is valid ($E(0, A)$ depends continuously on A; see [5, p.73]).

The sum S of two ellipsoids $E(a_1, Q_1)$ and $E(a_2, Q_2)$ is the set of all points of the form $x = x_1 + x_2$, where $x_1 \in E(a_1, Q_1)$, $x_2 \in E(a_2, Q_2)$. According to [5], §6, $S \subset E(a, Q)$, where

$$a = a_1 + a_2, \qquad Q = (p^{-1}+1)Q_1 + (p+1)Q_2, \tag{1}$$

the number $p > 0$ being arbitrary ([5], p.107). Formula (6.39) from [5] determines the optimal value of p such that the corresponding ellipsoid $E(a, Q)$ has minimum volume.

In the case where the size of the ellipsoid $E(a_1, Q_1)$ is considerably less than the other, the optimal value of p is close to

$$p = \sqrt{n^{-1}Tr(Q_2^{-1}Q_1)}, \tag{2}$$

See [5], p.106; here Tr is the trace of a matrix.

2. Presentation of the method

Let $z(t_i)$ be approximate values of the solution of the following problem at the points of the net $t_i = t_0 + ih$, $i = 1, 2, \ldots$:

$$x' = f(t, x), \quad x(t_0) = x^0 \quad (x \in R^n) \tag{3}$$

(The case where the approximate solution $z(t)$ is found on the interval $t_0 \leqslant t \leqslant t^*$, is considered in Section 3.) A difference $\Delta_i = z(t_i) - x_{i-1}(t_i)$, where $x_{i-1}(t)$ is an exact solution of equation (3) with the initial condition $x_{i-1}(t_{i-1}) = z(t_{i-1})$, is called a local error. This local error is formed of a local error of the approximate method used and of a roundoff error. Let an estimate for Δ_i be known, for example, $| \Delta_i | \leqslant \delta_i$ or $\Delta_i \in E(d_i, M_i)$, $i = 1, 2, \ldots$. We must estimate the difference $x(t_i) - z(t_i) = w(t_i)$, where $x(t)$ is the exact solution of problem (3). We have

$$w(t_i) = x(t_i) - x_{i-1}(t_i) - \Delta_i.$$

For the function $y(t) = x(t) - x_{i-1}(t)$ we have

$$y'(t) = f\left(t, x_{i-1}(t) + y(t)\right) - f\left(t, x_{i-1}(t)\right) = C(t)y(t) + \psi(t), \qquad (4)$$

where $C(t)$ is the matrix $(\partial f_k / \partial x_j)_{k,j=1,\ldots,n}$, the values of derivatives are taken for $x = x_{i-1}(t)$, and $\psi(t)$ is a remainder term of the Taylor formula. Let the estimate of $\psi(t) \in E\left(u(t), B(t)\right)$ be known. We obtain the estimate $w(t_i) \in E(a_i, Q_i)$ for the difference $w(t_i) = x(t_i) - z(t_i)$ of the exact and approximate solutions. Let $z(t_0) = x_0$, and for the exact solution, let be known that either $x(t_0) \in E(x_0, Q_0)$, or, more strictly, $x(t_0) = x_0$ (the latter case reduces to the preceding for $Q_0 = 0$).

We show how to pass the estimate $w(t_{i-1}) \in E(a_{i-1}, Q_{i-1})$ to a similar estimate for $w(t_i)$. Equation (4) is of the form (8.8) of [5]. Therefore,

$$y(t) \in E\left(a(t), Q(t)\right) \quad \left(t_{i-1} \leqslant t \leqslant t_i\right), \qquad (5)$$

where the vector $a(t)$ and the matrix $Q(t)$ satisfy the equations†

$$a' = C(t)a + u(t), \quad a(t_{i-1}) = a_{i-1}, \qquad (6)$$

$$Q' = CQ + QC^T + qQ + q^{-1}B(t), \quad Q(t_{i-1}) = Q_{i-1} \qquad (7)$$

Here ^{T} is the matrix transposition sign and $q = q(t) > 0$ is any function. Its optimal value (in the sense of [5], p. 127) is $q = \sqrt{n^{-1}Tr(Q^{-1}B)}$. The system (7) has the high dimension $n(n+1)/2$, since the matrix Q

†The matrix is known only at the points of the net t_{i-1}. That enough to solve the equations (6) and (7) by the approximate methods presented in the Section 4.

is symmetric. Thus, it is worthwhile to solve this system by one of less cumbersome one-step methods; see Section 4. To avoid computing the matrix Q^{-1} at each step, we can consider the function q in (2.6) to be constant on the intervals $t_i < t < t_{i+s}$ to several tens of steps of length, since Q does not change greatly over this distance. The discontinuity of the right-hand side of equation (2.6) caused by this for $t = t_i, t_{i+s}, \dots,$ does not change the estimate of the local error of the one-step methods.

Equations (6) and (7) are differential equations for the evolution of the ellipsoid (5) that give an estimate for the difference $w = x - z$ of the exact and approximate solutions of problem (3). From (5), it follows that $x(t) \in E(z(t) + a(t), Q(t))$.

Since we are led to solve equations (6) and (7) approximately, we must ensure that the ellipsoid constructed from the approximate $a(t)$ and $Q(t)$ contains the ellipsoid (5) constructed from the exact $a(t)$ and $Q(t)$.

Assume that we have found the approximate values $a^* = a^*(t_i)$ and $Q^* = Q^*(t_i)$ of the solutions $a(t)$ and $Q(t)$ for the equations (6) and (7) at the point $t = t_i$. Then

$$a(t_i) = a^* - \eta_i, \quad Q(t_i) = Q^* - K_i, \tag{9}$$

where the vector η_i and the symmetric matrix K_i are the local errors that take into account the method's truncation errors for the solution of equations (6) and (7) at one step and roundoff errors. Let the following estimates be known:

$$\eta_i \in E(\alpha, L), \quad \|K_i\| \leqslant \rho \text{ or } |K_i x| \leqslant \rho |x|, \quad x \in R^n;$$

α, L, and ρ can depend on i. In particular, if the absolute value of each entry of the matrix K_i does not exceed μ_0, we can take $\rho = n\mu_0$. Since $w(t_i) = y(t_i) - \Delta_i$, then, denoting $y(t_i) + \eta_i = y^*$, we use (5) and (9) to obtain

$$w(t_i) = y^* - \eta_i - \Delta_i, \quad y^* \in E(a^*, Q(t_i)).$$

Here we pass from $Q(t_i)$ to the known matrix Q^*. Since

$$Q^* + \rho I = Q(t_i) + (K_i + \rho I), \quad K_i + \rho I \geqslant 0,$$

the lemma implies $E(a^*, Q(t_i)) \subset E(a^*, Q^* + \rho I)$,

$$y^* \in E(a^*, Q_\rho), \quad Q_\rho = Q^* + \rho I. \tag{10}$$

Here we must always round up after an addition.

Let us estimate the order of magnitude of the values η_i, Δ_i, and $a(t)$ with respect to h. The approximate method of order k for equation (3) gives the local error $\Delta_i = O(h^{k+1})$, while the cumulative error is $\eta_i = O(h^{k+2})$. Therefore, the estimate for η_i is small in comparison with the estimate for Δ_i. According to (1) and (2),

$$\eta_i + \Delta_i \in E(\alpha, L) + E(d_i, M_i) \subset E(\alpha + d_i, N), \tag{11}$$

$$N = (1 + p_1)M_i + (p_1^{-1} + 1)L + n\mu_1 I, \; p_1 = \sqrt{n^{-1}Tr(M_i^{-1}L)}. \tag{12}$$

According to (10) $w(t_i) \in E(a^*, Q_\rho) + E(-\alpha - d_i, N)$. Since $\eta_i + \Delta_i = O(h^{k+1})$, $y^* = O(h^k)$, (1) and (2) give us

$$w(t_i) \in E(a_i, Q_i), \quad a_i = a^* - \alpha - d_i, \tag{13}$$

$$Q_i = (1 + p_2)Q_\rho + (p_2^{-1} + 1)N + n\mu_2 I, \; p_2 = \sqrt{n^{-1}Tr(Q_\rho^{-1}N)}. \tag{14}$$

To compensate for roundoff errors, the terms $n\mu_1 I$ and $n\mu_2 I$ are introduced just as ρ was in (10). We assume that the roundoff error in computing $a_i = a^* - \alpha - d_i$, is a vector whose length does not exceed ξ, and roundoff errors in the additions and and multiplications in (12) and (13) give errors in each entry of the matrices N and Q_i that do not exceed μ_1 and μ_2, respectively. Here, errors in computing p_1 and p_2 in (12) and (14) must not be taken into account (any p_1 and p_2 are suitable), and the values of p_1^{-1} and p_2^{-1} must be taken with excess whose amount can be neglected. Then, adding the terms $n\mu_1 I$ and $n\mu_2 I$ can only augment the ellipsoids in (11) and (13), in spite of the roundoff errors.

The roundoff error in the difference $a_i = a^* - \alpha - d_i$ is equivalent to the variation of $\alpha + d_i$ by the amount of the error, and is compensated by adding the ellipsoid (11) to the ball $E(O, \xi^2 I)$. This leads to replacing the matrix N in (14) by the matrix

$$N_1 = (1 + p_3\xi)N + (p_3^{-1}\xi + \xi^2 + n\mu_3)I, \quad p_3 + \sqrt{n^{-1}TrN^{-1}}. \tag{15}$$

The term of the sum $n\mu_3 I$ compensates roundoff errors in the addition and multiplication in (15) if they do not exceed μ_3 for any entry of the matrix N_1.

Therefore, to pass from a_{i-1}, Q_{i-1} to a_i, Q_i, one must find approximate values a^*, Q^* for $t = t_i$ of the solutions of equations (6) and (7). Furthermore, given estimates for the local errors and roundoff errors, one finds Q_ρ from (10), then one finds N and N_1 from (12) and (15). Then, using formulas (13) and (14) with N replaced by N_1, one obtains a_i and Q_i.

The preceding implies that for any exact solution $x(t)$ of the equation $x' = f(t, x)$ satisfying the condition $x(t_i) \in E\left(z(t_i) + a_i, Q_i\right)$, there is an $i = i_0 \geqslant 0$ such that this condition will hold for all $i > i_0$, as long as this solution exists and the estimates used for the local errors hold.

3. Modification of the method

A. Let the approximate solution $z(t)$ of equation (3) be found not only at the points t_i, but on the interval $[t_0, t^*]$, and let it satisfy the equation

$$z' = f(t, z) + \varphi(t), \quad z(t_0) = x_0, \tag{16}$$

and assume function $\varphi(t)$ is small. Then the function $w = x(t) - z(t)$, where $x(t)$ is a solution for problem (3), satisfies the equation

$$w' = C(t)w + \psi(t) - \varphi(t), \quad w(t_0) \in E(0, Q_0), \tag{17}$$

where $C(t)$ is now a matrix of derivatives $\partial f_k / \partial x_j$, taken for $x = z(t)$, and $\psi(t)$ is the remainder term of the Taylor formula

$$f(t, z + w) = f(t, z) + C(t)w + \psi(t).$$

Given the estimate of $\psi(t) - \varphi(t) \in E\left(u(t), B(t)\right)$, equation (17) is of the form (8.8) of [5]. Thus for $t \geqslant t_0$ we have

$$\begin{aligned} &w(t) \in E\left(a(t), Q(t)\right), \quad a' = C(t)a + u(t), \quad a(t_0) = 0, \\ &Q' = CQ + QC^T + qQ + q^{-1}B(t), \quad Q(t_0) = Q_0, \end{aligned} \tag{18}$$

where the function q is determined as in (7). Errors in the solution of this system are taken into account in the same manner as in Section 2, only Δ_i, d_i, M_i are missing.

Remark. The case of Section 2 where the approximate solution is found only at the points t_i can be reduced to the case considered above. To do this, we must interpolate the function $z(t_i)$ on each of intervals $[t_{i-1}, t_i]$ and estimate the residual $\varphi(t)$ in (16) for the function $z(t)$ so constructed. More easily, take

$$z(t) = x_{i-1}(t) + (t - t_{i-1})\Delta_i/h \quad (t_{i-1} \leqslant t \leqslant t_i),$$

where $x_{i-1}(t)$ and Δ_i are the same as in the beginning of Section 2. Then

$$\varphi(t) = z'(t) - f(t, z(t)) = h^{-1}\Delta_i + f(t, x_{i-1}(t)) - f(t, z(t));$$

the latter difference can be estimated from the estimates for $\partial f_k/\partial x_j$. For example, if the norm of the matrix $\|(\partial f_k/\partial x_j)_{k,j=1,\dots,n}\| \leqslant m_1$ then

$$|\varphi(t)| \leqslant h^{-1}|\Delta_i|(1 + (t - t_{i-1})m_1) \quad (t_{i-1} \leqslant t < t_i).$$

Similarly, the case of Section 3A reduces to the case of Section 2, since the difference $\Delta_i(t) = z(t) - x_{i-1}(t)$, can be estimate using Gronwall's lemma, provided the estimate $|\varphi(t)| \leqslant \varphi_0$ is known. This gives the estimate

$$|\Delta_i| = |\Delta_i(t_i)| \leqslant \varphi_0 h e^{m_1 h},$$

where m_1 is defined above.

B. Assume that in Section 2 we have obtained symmetric estimates for Δ_i and $\psi(t)$ (or for $\psi(t) - \varphi(t)$ in Section 3A), that is, $d_i = 0$, $u(t) \equiv 0$.

Then (6) or (18) implies that $a(t) \equiv 0$, the quantities a^*, η_i, α, L, a_i, and ξ are equal to zero, we have no need of formulas (11), (12), and (15), It also implies that, in (14), $N = M_i$ in the case of Section 2 and $N = 0$, $Q_i = Q_\rho$ in the case of Section 3A.

4. Simplest estimates of errors

If we have the estimate $|\Delta_i| \leqslant \delta_i$ for the local error Δ_i (Section 2), then $d_i = 0$, $M_i = \delta_i^2 I$.

In the equality (4), $\psi(t)$ is the remainder term of the Taylor formula. Each of its components $\psi_j(t)$ is estimated by the quadratic form

$$|\psi_j(t)| \leqslant \frac{1}{2}\sum_{k,l=1}^{n} m_{jkl}|y_k y_l|, \quad \left|\frac{\partial^2 f_j}{\partial x_k \partial x_l}\right| \leqslant m_{jlk}. \tag{19}$$

Therefore, $|\psi_j(t)| \leqslant m_j |y|^2/2$, and

$$m_j^2 = \|(m_{jkl})_{k,l=1,\dots,n}\|^2 = \sum_{k,l=1}^{n} m_{jkl}^2. \tag{20}$$

Inequalities (19) for $j,\ k,\ l = 1,\dots,n$ must hold in a convex domain (in x) containing the fragment $t_{i-1} \leqslant t \leqslant t_i$ of the solutions $x(t)$ and $x_{i-1}(t)$. Since these solutions are not known in advance, the domain is taken with excess, and after obtaining the final estimate of a solution one must check that $x(t)$, $z(t_{i-1})$, and the solution $x_{i-1}(t)$ near $z(t_{i-1})$ are contained in this domain. (If not, one must augment the domain and repeat the computation and checking.)

The estimates $|\psi_j(t)|$ imply $u(t) \equiv 0$,

$$\psi(t) \in E\left(0, B(t)\right), \quad B(t) = \frac{n}{4}|y(t)|^4 \operatorname{diag}(m_1^2, \dots, m_n^2),$$

where $\operatorname{diag}(b_1, \dots, b_n)$ is a diagonal matrix with entries $b_1, \dots, b_n$ on the main diagonal. According to Section 3B, we have $a(t) \equiv 0,\ N = N_1 = M_i$, and by (5), $|y(t)|^2 \leqslant \|Q(t)\|$. Indeed, $E(0, Q)$ is an ellipsoid with the major semiaxis equal to $\sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of the matrix Q (see [5], p. 72), and $\lambda_{\max} \leqslant \|Q\|$. For nonnegative definite symmetric matrices $Q = (q_{ij})_{i,j=1,\dots,n}$, the following estimates hold:

$$\|Q\| \leqslant TrQ = \sum_{i=1}^{n} q_{ii}, \quad \lambda_{\max} \leqslant \max_i \sum_{j=1}^{n} |q_{ij}|.$$

Then, we can take in equation (7)

$$B(t) = \frac{n}{4}(TrQ)^2 \operatorname{diag}(m_1^2, \dots, m_n^2). \tag{21}$$

To achieve an acceptable accuracy in the solution of equation (7), it is customary to take a reasonably small step h.

The Euler method gives the approximate value

$$Q^*(t_i) = Q_{i-1} + hQ'(t_{i-1}),$$

where $Q'(t_{i-1})$ is determined from (7). The method's error in one step does not exceed $h^2 s_2/2$ if $\|Q''(t)\| \leqslant s_2$. For a solution of equation (7),

$$Q'' = CQ' + Q'C^T + qQ' + C'Q + Q(C^T)' + q^{-1}B' \quad (t_{i-1} < t < t_i).$$

Here Q' is estimated using (7), and it is assumed that the function q is constant for $t_{i-1} < t < t_i$.

A more exact method is the second order Taylor formula

$$Q^*(t_i) = Q_{i-1} + hQ'(t_{i-1}) + h^2 Q''(t_{i-1})/2. \tag{22}$$

The method's error is no more than $h^3 s_3/6$, if $\|Q'''(t)\| \leqslant s_3$; Q''' is obtained by the differentiation of the formula for Q''.

We must take into account the roundoff errors for both methods. We often find more exact estimates if we do the estimates separately for each entry of the matrix Q.

5. Estimate of total error

In Section 2, we obtained the following estimate for the difference $w = x - z$ of the exact and approximate solutions: $w(t_i) \in E(a_i, Q_i)$. Therefore, $|w(t_i) - a_i| \leqslant \sqrt{\|Q_i\|}$. $\|w(t_i)\| \leqslant |a_i| + \sqrt{\|Q_i\|}$.

We estimate w by other means, without using a_i and Q_i, on the basis of an estimate of the local error Δ_i. The remark in Section 3A shows that an approximate solution of $z(t)$ extended from the points t_i to intervals (t_{i-1}, t_i), satisfies equation (16), where

$$|\varphi(t)| \leqslant h^{-1}|\Delta_i|(1 + m_1 h) \sim h^{-1}|\Delta_i| \quad (h \to 0).$$

Therefore, the difference between solutions of equations (3) and (16) should be estimated. The principal part of this difference satisfies the variation equation

$$w' = C(t)w - \varphi(t), \quad w(t_0) = 0 \tag{23}$$

(or $w(t_0) = x^0 - x_0$, see the beginning of Section 2), where the matrix $C(t)$ is the same as in (17). A solution $w(t)$ of equation (23) can be estimated by known methods, for example,

$$\left(|w|^2\right)' = 2w \cdot w' = 2w \cdot C(t)w - 2w \cdot \varphi(t),$$

and if $w \cdot C(t) \leqslant k|w|^2$ ($w \in R^n$) and $|\varphi(t)| \leqslant h^{-1}\delta$, then

$$|w(t)| \leqslant h^{-1}\delta k^{-1}\left(e^{k(t-t_0)} - 1\right) + |w(t_0)|e^{k(t-t_0)} \quad (t \geqslant t_0).$$

The estimate can be made more precise if one takes $k = k(t)$, $\delta = \delta(t)$ and the estimates $|w(t)|$ on the basis of the linear differential inequality obtained, or if one chooses the function $v(t,w)$ of Lyapunov function type and estimates $\frac{d}{dt}v\left(t, w(t)\right)$, and then $v\left(t, w(t)\right)$.

6. Example

One must estimate the error in a solution of the system

$$x' = x - y \quad y' = 2x - y^3; \quad x(0) = 0,25\ , \quad y(0) = 0$$

by the Adams method.

The solution passes inside the domain D_0 bounded by the limiting cycle and presumably is contained in the hexagon $D(|x| \leqslant 1,\ |y| \leqslant 1.25,\ |x-y| \leqslant 1)$. This was found by a preliminary computation. To compute the solution more precisely, the order 5 Adams interpolation method with step $h = 1/256$ was used. The local error of the method is ρ_0,where $|\rho_0| < 0.000054\gamma$ and $\gamma = 2^{-24}$. (The estimates of derivatives $|x^{(6)}| < 3100$, $|y^{(6)}| < 48000$ in D were used.) With a computer with 24 bit mantissas in floating point numbers, and recording the solution x, y in two cells for each value (the other numbers were recorded in one cell each), the roundoff error at each step is less than $0.0778\ \gamma$, and the total local error is less $\Delta = 0.086\gamma$ for x and y separately.

Simultaneously with computing the solution, the error was estimated by the ellipsoid method using formulas (7) and (22).

The following table gives a guaranteed bound for the error obtained by this method, that is, the value b_1 of the major semiaxis of the ellipse that contains the solution.

$t =$	2	4	6	8	10	12	14	16
$10^6 b_1 =$	8.6	20.1	91	98.1	80.5	232	91	437

The major axis reduces near the points of the limit cycle that are nearest to the singular points $x = y = \pm\sqrt{2}$ of the system (6.1). It can be verified that an approximate solution for $0 \leqslant t \leqslant 16$ along with the ellipse enclosing it is contained in D. This justifies the estimates made.

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