Interval Computations
No 2(4), 1992

ON METRIZATION OF INTERVAL SETS $I(R), I(R^n)$

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Some distances in interval sets $I(R), I(R^n)$ are defined. These sets are shown to be complete metric spaces. Further, estimates for interval matrix norms are given. The matrices are acting from one metric space into another.

О МЕТРИЗАЦИИ ИНТЕРВАЛЬНЫХ МНОЖЕСТВ $I(R), I(R^n)$

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В интервальных множествах $I(R), I(R^n)$ вводятся различные расстояния. Показывается, что указанные множества с введенными расстояниями являются полными метрическими пространствами. Затем даются оценки норм интервальных матриц, действующих из одного метрического пространства в другое.

The use of the tools of functional analysis simplifies considerably solution of a number of interval problems. The works [1]–[7] demonstrate this.

We shall confine ourselves to the following notation. R is the set of real numbers,
R^n is the set of n-dimensional vectors,
$I(R)$ is the set of all real interval numbers, that is

$$I(R) := \{x := [\underline{x}, \overline{x}] \mid \underline{x} \leqslant \overline{x},\ \underline{x}, \overline{x} \in R\}.$$

$I(R^n)$ is the set of all real interval vectors, that is

$$I(R^n) := \{x := (x_1, \ldots, x_n) \mid x_i \in I(R),\ i = \overline{1, n}\}.$$

In this paper, the most commonly used distances are defined for the interval sets $I(R)$ and $I(R^n)$. These sets are shown to be complete metric spaces with any distances considered. Then we give estimates for the norms of interval matrices acting from one metric space into another.

As is known, interval sets considered with traditional interval arithmetic are not linear spaces, and there is no need to metrize the sets in question (see, for example, [3]).

By metrization of a set, we mean the introduction of a distance such that the set would be a metric space.

In [7], it is shown that introducing special arithmetic operations allows us to transform $I(R)$ and $I(R^n)$ into Banach spaces.

1. Metrization of $I(R)$

We shall recall the definition of a metric space.

Definition. A set X is called a metric space if a nonnegative number $\rho(x, y)$, called a distance, is associated with every pair of its elements $x, y \in X$ and satisfies the following conditions:

(1) $\rho(x, y) = 0$ if and only if $x = y$ (axiom of identity);
(2) $\rho(x, y) = \rho(y, x)$ (axiom of symmetry);
(3) for all $x, y, z \in X$, the inequality $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$ holds (triangle axiom).

For X, we shall consider here $I(R)$; that is $X := I(R)$. Consider the most commonly used distances for every pair $x, y \in I(R)$:

$$\rho_1(x, y) := \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}, \tag{1}$$

$$\rho_2(x, y) := \left((\underline{x} - \underline{y})^2 + (\overline{x} - \overline{y})^2\right)^{1/2}, \tag{2}$$

$$\rho_3(x, y) := |\underline{x} - \underline{y}| + |\overline{x} - \overline{y}|, \tag{3}$$

$$\rho_4(x, y) := (|\underline{x} - \underline{y}| + |\overline{x} - \overline{y}|)/(1 + |\underline{x} - \underline{y}| + |\overline{x} - \overline{y}|). \tag{4}$$

Theorem 1. *The set $I(R)$ with any distance of the form $\rho_k(x, y)$, $k := \overline{1, 4}$, is a complete metric space.*

We give the proof for the case $k = 2$. Clearly, the axioms of identity and symmetry hold. We show that the triangle inequality axiom holds.

Using the Cauchy inequality we obtain

$$
\begin{aligned}
\rho_2(x,y) &= \left((\underline{x}-\underline{y})^2+(\overline{x}-\overline{y})^2\right)^{1/2}\\
&= \left((\underline{x}-\underline{z}+\underline{z}-\underline{y})^2+(\overline{x}-\overline{z}+\overline{z}-\overline{y})^2\right)^{1/2}\\
&\leqslant \Big\{(\underline{x}-\underline{z})^2+(\overline{x}-\overline{z})^2+2\left((\underline{x}-\underline{z})^2+(\overline{x}-\overline{z})^2\right)^{1/2}\left((\underline{z}-\underline{y})^2\right.\\
&\quad \left.+(\overline{z}-\overline{y})^2\right)^{1/2}+(\underline{z}-\underline{y})^2+(\overline{z}-\overline{y})^2\Big\}^{1/2}\\
&= \left((\underline{x}-\underline{z})^2+(\overline{x}-\overline{z})^2\right)^{1/2}+\left((\underline{z}-\underline{y})^2+(\overline{z}-\overline{y})^2\right)^{1/2}\\
&=\rho_2(x,y)+\rho_2(z,y).
\end{aligned}
$$

This implies that the set $I(R)$ with the distance $\rho_2(x,y)$ is a metric space.

Now, we show completeness of the metric space in question. Let the fundamental sequence $\{x^p\}^\infty$ be given, where $x^p \in I(R)$. That is, for every $\varepsilon > 0$ there is a natural number $n(\varepsilon)$ such that $\rho_2(x^p,x^q) < \varepsilon$ for all p and $q > n(\varepsilon)$. This implies $((\underline{x}^p-\underline{x}^q)^2+(\overline{x}^p-\overline{x}^q)^2)^{1/2} < \varepsilon$ for $p,q > n(\varepsilon)$. Then $|\underline{x}^p-\underline{x}^q| < \varepsilon$ and $|\overline{x}^p-\overline{x}^q| < \varepsilon$ for p and $q > n(\varepsilon)$. We conclude that the sequences of numbers $\{\underline{x}^p\}_{p=1}^\infty$ and $\{\overline{x}^p\}_{p=1}^\infty$ are fundamental and have limits $\underline{x}^0 \leqslant \overline{x}^0$, respectively. We shall see that the interval $x^0 := [\underline{x}^0,\overline{x}^0]$ is the limit of our sequence $\{x^p\}_{p=1}^\infty$, $\rho_2(x^p,x^0) = \left((\underline{x}^p-\underline{x}^0)^2+(\overline{x}^p-\overline{x}^0)^2\right)^{1/2} \to 0$ as $p\to\infty$, since $\underline{x}^p-\underline{x}^0 \to 0$ and $\overline{x}^p-\overline{x}^0 \to 0$. This implies completeness of the metric space considered.

The other cases are demonstrated in a similar way. ■

We give some properties of the metrics (1)–(4) in $I(R)$ which we will need later.

Theorem 2. *Let a,b,c and $d \in I(R)$. Then*

$$\rho_k(a+b,a+c)=\rho_k(b,c),\quad k=\overline{1,4}, \tag{5}$$

$$\rho_k(a+b,c+d)\leqslant\rho_k(a,c)+\rho_k(b,d),\quad k=\overline{1,4}, \tag{6}$$

$$\rho_k(\alpha b,\alpha c)=|\alpha|\rho_k(b,c),\quad k=\overline{1,3}, \tag{7}$$

$$\rho_k(ab,ac)\leqslant|a|\rho_k(b,c),\quad k=\overline{1,3}. \tag{8}$$

where $|a| := \max\{|\underline{a}|,|\overline{a}|\}$.

Proof. The case $k = 1$ is proven. (See, for example, [6].) Here we shall show the theorem for the case $k = 2$. The proof for the other cases is similar.

(5): The definition of metric implies that

$$\begin{aligned}\rho_2(a+b, a+c) &= \left((\underline{a}+\underline{c}-\underline{a}-\underline{b})^2 + (\overline{a}+\overline{c}-\overline{a}-\overline{b})^2\right)^{1/2} \\ &= \left((\underline{c}-\underline{b})^2 + (\overline{c}-\overline{b})^2\right)^{1/2} = \rho_2(b,c).\end{aligned}$$

(6): Using the Cauchy inequality we obtain

$$\begin{aligned}\rho_2(a+b, c+d) =& \left\{\left((\underline{c}-\underline{a})+(\underline{d}-\underline{b})\right)^2 + \left((\overline{c}-\overline{a})+(\overline{d}-\overline{b})\right)^2\right\}^{1/2} \\ \leqslant& \left((\underline{c}-\underline{a})^2 + (\overline{c}-\overline{a})^2\right)^{1/2} + \left((\underline{d}-\underline{b})^2 + (\overline{d}-\overline{b})^2\right)^{1/2} \\ =& \rho_2(a,c) + \rho_2(b,d).\end{aligned}$$

$$\begin{aligned}(7): \quad \rho_2(\alpha b, \alpha c) &= \left((\underline{\alpha c}-\underline{\alpha b})^2 + (\overline{\alpha c}-\overline{\alpha b})^2\right)^{1/2} \\ &= |\alpha|\left((\underline{c}-\underline{b})^2 + (\overline{c}-\overline{b})^2\right)^{1/2} = |\alpha|\rho_2(b,c).\end{aligned}$$

(8): Since $\rho_2(ab, ac) = \left((\underline{ab}-\underline{ac})^2 + (\overline{ab}-\overline{ac})^2\right)^{1/2}$, we suppose that $\underline{ab} \geqslant \underline{ac}, \overline{ab} \geqslant \overline{ac}$. (The other cases are considered in a similar way.) Then constants $\alpha, \beta \in a$ exist such that $\underline{ac} = \underline{\alpha c}$, $\overline{ab} = \overline{\beta b}$. By the property of inclusion monotonicity, it follows that $\alpha b \in ab$, $\beta c \in ac$. This implies

$$0 \leqslant \underline{ab} - \underline{ac} \leqslant \underline{\alpha b} - \underline{\alpha c},$$

$$0 \leqslant \overline{ab} - \overline{ac} \leqslant \overline{\beta b} - \overline{\beta c},$$

$$\begin{aligned}\rho_2(ab, ac) &= \left((\underline{ab}-\underline{ac})^2 + (\overline{ab}-\overline{ac})^2\right)^{1/2} \\ &\leqslant \left((\underline{\alpha b}-\underline{\alpha c})^2 + (\overline{\beta b}-\overline{\beta c})^2\right)^{1/2} \\ &\leqslant \left(|a|^2(\underline{b}-\underline{c})^2 + |a|^2(b-c)^2\right)^{1/2} \\ &= |a|\left((\underline{b}-\underline{c})^2 + (\overline{b}-\overline{c})^2\right)^{1/2} = |a|\rho_2(b,c). \quad \blacksquare\end{aligned}$$

2. Metrization of $I(R^n)$

We introduce the most commonly used distances in $I(R^n)$, in which $x, y \in I(R^n)$, $k = \overline{1,4}$.

$$\rho_{1k}(x,y) := \max_{1\leqslant j\leqslant n} \rho_k(x_j, y_j). \tag{9}$$

$$\rho_{2k}(x,y) := \left(\sum_{j=1}^{n} \rho_k^2(x_j, y_j)\right)^{1/2}. \tag{10}$$

$$\rho_{3k}(x,y) := \sum_{j=1}^{n} \rho_k(x_j, y_j). \tag{11}$$

We can, for example, also introduce the following distances:

$$\rho_{q+3,k}(x,y) := \rho_{qk}(x,y)/(1+\rho_{qk}(x,y)), \tag{12}$$

where $q = \overline{1,3}$.

In what follows, we shall consider only the distances (9)–(11).

Theorem 3. *The set $I(R^n)$ with any distance of the form (9)–(11) is a complete metric space.*

Proof. Consider the case of $I(R^n)$ with the distance (10). Clearly, $\rho_{2k}(x,y) \geqslant 0$. If $\rho_{2k}(x,y) = 0$, it follows that all $\rho_k(x_j, y_j) = 0$, $j = \overline{1,n}$, and $x_j = y_j$. This implies that $x = y$. Conversely, $x = y$ implies that $\rho_{2k}(x,y) = 0$. Therefore, the axiom of identity holds. Clearly, $\rho_k(x_j, y_j) = \rho_k(y_j, x_j)$ implies that the axiom of symmetry holds: $\rho_{2k}(x,y) = \rho_{2k}(y,x)$.

We show that the triangle inequality holds for any $x, y, z \in I(R^n)$.

$$\begin{aligned}
\rho_{2k}(x,y) &= \left(\sum_{j=1}^{n} \rho_k^2(x_j, y_j)\right)^{1/2} \\
&\leqslant \left(\sum_{j=1}^{n} (\rho_k(x_j, y_j) + \rho_k(z_j, y_j))^2\right)^{1/2} \\
&\leqslant \left(\sum_{j=1}^{n} \rho_k^2(x_j, z_j)\right)^{1/2} + \left(\sum_{j=1}^{n} \rho_k^2(z_j, y_j)\right)^{1/2} \\
&= \rho_{2k}(x,z) + \rho_{2k}(z,y).
\end{aligned}$$

Now we shall demonstrate completeness of the given metric space.

Let the fundamental sequence $\{x^p\}_{p=1}^{\infty}$ be given, that is, for any $\varepsilon > 0$ a natural number $n(\varepsilon)$ exists such that $p_{2k}(x^p, x^q) < \varepsilon$ for all p and $q > n(\varepsilon)$. Therefore, $\left(\sum_{j=1}^{n} \rho_k^2(x_j^p, x_J^q)\right)^{1/2} < \varepsilon$. Clearly, $\rho_k(x_j^p, x_j^q) < \varepsilon$ for $p, q > n(\varepsilon)$. By Theorem 1, there is an element $x_j^0 \in I(R^n)$ such that $\{x_j^p\}_{p=1}^{\infty}$ converges to x_j^0 . This implies that $x^0 := (x_1^0, x_2^0, \dots x_n^0) \in I(R^n)$ and $\lim_{p\to\infty} x^p = x^0$. ■

3. Estimates for the norms of interval matrices

In this Section we give estimates for the norms of interval matrices acting from one metric space into another. We shall mean by interval matrix a matrix A of dimension $n \times m$ of the form

$$A := \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \tag{13}$$

where $a_{ij} \in I(R), i = \overline{1,m}, j = \overline{1,n}$. This will be denoted as $A \in I(R^{n\times m})$.

Following Collatz [8], page 92, we introduce a concept of the norm of an operator without the requirement that the metric spaces be linear.

Definition. Let X and Y be metric spaces, and let the operator A act from $D(A) \subseteq X$ into $R(A) \subseteq Y$. Then the least real number α in the inequality

$$\rho_y(Ax_1, Ax_2) \leqslant \alpha \rho_x(x_1, x_2), \tag{14}$$

where $x_1, x_2 \in D(A)$ will be called a norm of the operator A, and is denoted $\|A\|_{X\to Y}$ or, simply, $\|A\|$.

Remark. The set of operators from X into Y does not in general form a normed space.

Denote the metric space $I(R^n)$ with the distance $\rho_{qk}(x,y)$ by $I(R^n_q k)$.

We estimate a norm of the matrix acting from $I(R^n_{qk})$ into $I(R^m_{pl})$, where $q, k, p, l = 1, 2, 3$.

3.1. The case $A : I(R^n_{1k}) \to I(R^m_{1k}),\ k = 1, 2, 3$

$$\begin{aligned}
\rho_{1k}(Ax, Ay) &= \max_{1\leqslant i\leqslant m} \rho_k((Ax)_i, (Ay)_i) \\
&= \max_{1\leqslant i\leqslant m} \rho_k\left(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} a_{ij}y_j\right) \\
&\overset{(6)}{\leqslant} \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} \rho_k(a_{ij}x_j, a_{ij}y_j) \\
&\overset{(8)}{\leqslant} \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij}| \max_{1\leqslant j\leqslant n} \rho_k(x_j, y_j) \\
&\overset{(9)}{=} \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij}| \rho_{1k}(x, y).
\end{aligned}$$

We obtain in this case

$$\|A\| \leqslant \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij}|. \tag{15}$$

3.2. The case $A : I(R^n_{2k}) \to I(R^m_{2k}),\ k = 1, 2, 3$

$$\rho_{2k}(Ax, Ay) = \left(\sum_{i-1}^{m} \rho_k^2((Ax)_i, (Ay)_i)\right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \rho_k^2 \left(\sum_{j=1}^{n} a_{ij} x_j, \sum_{j=1}^{n} a_{ij} y_j \right) \right)^{1/2}$$

$$\leqslant \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \rho_k (a_{ij} x_j, a_{ij} y_j) \right)^2 \right)^{1/2}$$

$$\leqslant \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}| \rho_k (x_j, y_j) \right)^2 \right)^{1/2}$$

(The Cauchy inequality)

$$\leqslant \left(\sum_{i=1}^{m} \left(\left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \rho_k^2 (x_j, y_j) \right)^{1/2} \right)^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \rho_{2k}(x, y).$$

We obtain that in this case

$$\|A\| \leqslant \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{16}$$

3.3. The case $A : I(R^n_{3k}) \to I(R^m_{3k}),\ k = 1, 2, 3$

$$
\begin{aligned}
\rho_{3k}(Ax, Ay) &= \sum_{i=1}^{m} \rho_k((Ax)_i, (Ay)_i) \\
&= \sum_{i=1}^{m} \rho_k \left(\sum_{j=1}^{n} a_{ij} x_j, \sum_{j=1}^{n} a_{ij} y_j \right) \\
&\overset{(6)}{\leqslant} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho_k(a_{ij} x_j, a_{ij} y_j) \\
&\overset{(8)}{\leqslant} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \rho_k(x_j, y_j) \\
&\leqslant \sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |a_{ij}| \sum_{j=1}^{n} \rho_k(x_j, y_j) \\
&= \sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |a_{ij}| \rho_{3k}(x, y).
\end{aligned}
$$

This implies

$$
\|A\| \leqslant \sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |a_{ij}|. \tag{17}
$$

3.4. The case $A : I(R^n_{3k}) \to I(R^m_{2k}), k = 1, 2, 3$

$$\rho_{2k}(Ax, Ay) = \left(\sum_{i=1}^{m} \rho_k^2((Ax)_i, (Ay)_i) \right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \rho_k^2 \left(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} a_{ij}y_j \right) \right)^{1/2}$$

$$\leqslant \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \rho_k(a_{ij}x_j, a_{ij}y_j) \right)^2 \right)^{1/2}$$

$$\overset{(8)}{\leqslant} \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}| \rho_k(x_j, y_j) \right)^2 \right)^{1/2}$$

(The Cauchy inequality)

$$\leqslant \left(\sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |a_{ij}| \left(\sum_{j=1}^{n} \rho_k(x_j, y_j) \right)^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{m} \max_{1 \leqslant i \leqslant n} |a_{ij}| \right)^{1/2} \rho_{3k}(x, y).$$

This implies that

$$\|A\| \leqslant \left(\sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |a_{ij}| \right)^{1/2} \tag{18}$$

We give the estimates without proof for other cases. The proofs are similar. Below, everywhere $k = 1, 2, 3$.

3.5. The case $A : I(R^n_{1k}) \to I(R^m_{2k})$

$$\|A\| \leqslant \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}| \right)^2 \right)^{1/2} \tag{19}$$

3.6. The case $A : I(R^n_{1k}) \to I(R^m_{3k})$

$$\|A\| \leqslant \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|. \tag{20}$$

3.7. The case $A : I(R^n_{2k}) \to I(R^m_{1k})$

$$\|A\| \leqslant \max_{1 \leqslant i \leqslant n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{21}$$

3.8. The case $A : I(R^n_{2k}) \to I(R^m_{3k})$

$$\|A\| \leqslant \sum_{i=1}^{m} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}. \tag{22}$$

3.9. The case $A : I(R^n_{3k}) \to I(R^m_{1k})$

$$\|A\| \leqslant \max_{1 \leqslant i \leqslant m} \max_{1 \leqslant j \leqslant n} |a_{ij}| \tag{23}$$

We recall that $|a_{i,j}| := \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$.

Clearly, the estimates (15)–(23) will hold for real matrices, which are a special case of interval matrices.

Remark. The estimates for the norms of interval matrices can be useful for solving some problems. For example, when solving systems of first order algebraic equations of the form

$$x = Ax + b,$$

where $b \in I(R^n)$ is the given interval vector, x is the required interval vector, and the interval matrix $A : I(R^n_{pk}) \to I(R^n_{pk})$, $p, k = 1, 2, 3$; the Banach theorem on contraction mapping applies (see, for example, [9]). One can use the estimates (15)–(17) in this case.

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