ON TWO ALGORITHMS FOR BOUNDING THE INVERSES OF AN INTERVAL MATRIX

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We consider two different algorithms - an iteration method and a finite approximation method - for including the inverses of an interval matrix. Some practical properties are derived and it is shown that in some cases the results of both methods are asymptotically the same.

О ДВУХ АЛГОРИТМАХ ОЦЕНКИ РЕЗУЛЬТАТОВ ОБРАЩЕНИЯ ИНТЕРВАЛЬНЫХ МАТРИЦ

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Рассмотрены два различных алгоритма для оценки результата обращения интервальной матрицы - итерационной и конечной аппроксимации. Получены некоторые практические свойства этих алгоритмов и показано, что в некоторых случаях они асимптотически одинаковы.

1. Introduction. Given an n x n-interval matrix $\mathbf{A} = ([a^1_{ij}, a^2_{ij}])$ which only consists of regular real n x n-matrices $A = (a_{ij})$. We consider the problem of including the set of inverses A^{-1} into a proper interval matrix **X**. The best possible inclusion would be the interval hull of this set of inverses, but solving this problem is extremely expensive (see [8]). So, we are trying to compute only a more or less crude inclusion interval matrix by some simpler methods. Historically, the first method for this problem was proposed by Hansen in [4] . But no results were given concerning the quality of the method. For a special choice of a parameter matrix B in this method we can derive formulas for the resulting inclusion matrix and can prove a monotonicity property. On the other hand, Herzberger has proposed an interval Schulz-method for improving a given inclusion matrix in [5] . We are able to show that, starting with the simplest inclusion matrix according to Hansen's algorithm, this iteration method is monotonic and has asymptotically the same behaviour as Hansen's method. Finally, we prove an estimation for the rate of improvement for both methods compared with the initial inclusion and thus generalize a result in [6] .

2. Notation. The methods under consideration make use of interval operations for interval matrices **A**, **B**, **C**, ... Their definitions and basic properties can be found in the monograph of Alefeld and Herzberger [2] Chapter 10. As usual $d(\mathbf{A}) = (a^2_{ij} - a^1_{ij})$ denote the width matrix of an interval matrix and $|\mathbf{A}| = (\max\{|a^1_{ij}|, |a^2_{ij}|\})$ the matrix of the absolute value. Remember that for a symmetric null-matrix **G** with $\mathbf{G} = -\mathbf{G}$ we have the special rules in [2] Chapter 10 like

$$\mathbf{G} = [-1, 1] \cdot |\mathbf{G}| \quad ,$$
$$d(\mathbf{G}) = 2 \cdot |\mathbf{G}| \quad ,$$
$$H \cdot \mathbf{G} = |H| \cdot \mathbf{G}$$

which will be frequently applied in the following derivations. Furthermore, we are using the easy-to-prove equality

$$|\, H + \mathbf{G} \,| = |H| + |\mathbf{G}| \quad .$$

By $\|\cdot\|$ we denote the row-sum norm and get a norm for interval matrices by $\|\, |\mathbf{A}| \,\|$. In contrast to [2] we denote the midpoint matrix of an interval matrix **A** by $\mathrm{mid}(\mathbf{A}) = \left(\frac{a^1_{ij} + a^2_{ij}}{2}\right)$. Finally, we mention that real matrices $A = (a_{ij})$ with $a_{ij} \equiv a$ are simply written as $A = (a)$ if no confusion arises. In the same manner we write interval matrices with equal entries.

3. Hansen's method. Let $\mathbf{A}^{-1} = \{A^{-1} : A \in \mathbf{A}\}$ denote the set of the inverses of all matrices contained in $\mathbf{A}$, then we consider a real matrix B and define

$$\mathbf{E} = I - \mathbf{A} \cdot B$$

Now, we assume that $\| \, |\mathbf{E}| \, \| < 1$ is fulfilled and compute the inclusion sets for $\mathbf{A}^{-1}$ by

(1)
$$\mathbf{A}^{-1} \subseteq \mathbf{X}^{(k)} = B \cdot \left(\sum_{\nu=0}^{k} \mathbf{E}^{\nu} + \mathbf{R}^{(k)} \right) \; , \quad \mathbf{E}^{0} = I \, ,$$
$$\text{with } \mathbf{R}^{(k)} = ([-r^{(k)}, r^{(k)}]) \, , \; r^{(k)} = \frac{\| \, |\mathbf{E}| \, \|^{k+1}}{1 - \| \, |\mathbf{E}| \, \|}$$

(see Hansen [4]) . For a proper evaluation of the right-hand side expression in interval arithmetic see also Hansen [4] . In practical computations it is advisible to choose $B \approx (\mathrm{mid}(\mathbf{A}))^{-1} = A_c^{-1}$. This is the reason why we are setting $B = A_c^{-1}$ in the sequel in order to derive - at least theoretically - good results.

If we choose $B = A_c^{-1}$, then obviously

$$\mathbf{E} = - \mathbf{E} \text{ and } \mathbf{E} = [-1, 1] \cdot |\mathbf{E}|$$

holds true. Thus we get the explicit formula

$$\sum_{\nu=0}^{k} \mathbf{E}^{\nu} = I + \mathbf{E} \cdot \left(\sum_{\nu=0}^{k-1} \mathbf{E}^{\nu} \right) = I + [-1, 1] \cdot |\mathbf{E}| \, (I - |\mathbf{E}|^{k})(I - |\mathbf{E}|)^{-1}$$

for the power-sum in (1) which is, of course, not of practical importance. In (1) the question of the optimal inclusion arises in the sequence $\{\mathbf{X}^{(k)}\}$. The answer gives the following lemma.

Lemma 1: Let $\mathbf{X}^{(k)}$ be defined as in (1) then

$$\mathbf{X}^{(k+1)} \subseteq \mathbf{X}^{(k)} , \; k \geq 0 \, .$$

Proof: Due to the inclusion monotonicity of the interval matrix operations it is sufficient to show that

$$\sum_{\nu=0}^{k+1} \mathbf{E}^{\nu} + \mathbf{R}^{(k+1)} \subseteq \sum_{\nu=0}^{k} \mathbf{E}^{\nu} + \mathbf{R}^{(k)}$$

holds true. By the same argument it is sufficient to show that

$$\mathbf{E}^{k+1} + \mathbf{R}^{(k+1)} \subseteq \mathbf{R}^{(k)}$$

is valid. But since we have

$$R^{(k)} = [-1, 1] \cdot \left(\| \, |E| \, \|^{k+1} + \frac{\| \, |E| \, \|^{k+2}}{1 - \| \, |E| \, \|} \right)$$

$$= [-1, 1] \cdot (\| \, |E| \, \|^{k+1}) + R^{(k+1)}$$

and using again the inclusion monotonicity of the interval matrix operations it remains to show that the inclusion

$$E^{k+1} = [-1, 1] \cdot |E|^{k+1} \subseteq [-1, 1] \cdot (\| \, |E| \, \|^{k+1})$$

holds. But this is surely the case since we have

$$|E|^{k+1} \leq (\| \, |E| \, \|^{k+1}) \quad . \quad \square$$

From this lemma we can conclude that the optimal result achievable by Hansens's method is

$$X^* = \lim_{k \longrightarrow \infty} X^{(k)} \quad .$$

Lemma 2: $A^{-1} \subseteq X^* = A_c^{-1} \cdot (\sum_{\nu=0}^{\infty} E^\nu) = A_c^{-1} + [-1, 1] \cdot A_c^{-1} \cdot |E| \cdot (I - |E|)^{-1}$.

Proof: This formula is the (existing) limit of the formula for $X^{(k)}$ given above. $\square$

In [2] Chapter 16 we can find some estimations for E , namely

$$|E| = \frac{1}{2} \, d(A) \cdot |A_c^{-1}|$$

and

$$\| \, |E| \, \| \leq \frac{1}{2} \, \varkappa(A_c) \cdot \frac{\| \, d(A) \, \|}{\| \, A_c \, \|} \quad ,$$

with the condition number $\varkappa(C) = \| C \| \cdot \| C^{-1} \|$.

4. Interval Schulz-method. Let $Y^{(o)} \supseteq A^{-1}$ be given, then we are considering the following iterative process:

$$\tilde{Y}^{(k+1)} = \text{mid}(Y^{(k)}) + Y^{(k)}(I - A \, \text{mid}(Y^{(k)}))$$

(2)

$$Y^{(k+1)} = \tilde{Y}^{(k+1)} \cap Y^{(k)}, \quad k \geq 0 \; .$$

This is a monotonic version of the interval Schulz-method in [2] Chapter 18 generalized for an interval matrix $\mathbf{A}$. In [5] one can find sufficient conditifons for $\mathbf{Y}^{(o)}$ such that the iteration according to (2) improves $\mathbf{Y}^{(o)}$. Since, according to the definition of (2) , the property

$$\mathbf{Y}^{(o)} \supseteq \mathbf{Y}^{(1)} \supseteq \mathbf{Y}^{(2)} \supseteq \dots \supseteq \mathbf{Y}^{(k)} \supseteq \mathbf{A}^{-1}$$

is trivial, we have again that the optimal inclusion achievable by (2) is

$$\mathbf{Y}^{*} = \lim_{k \longrightarrow \infty} \mathbf{Y}^{(k)} .$$

Method (2) requires an initial inclusion matrix $\mathbf{Y}^{(o)} \supseteq \mathbf{A}^{-1}$ and we take the interval matrix $\mathbf{X}^{(o)}$ for this initial inclusion from Hansen's method. Doing this, we can prove following lemma.

Lemma 3: If we take $\mathbf{Y}^{(o)} = \mathbf{X}^{(o)}$, where $\mathbf{X}^{(o)}$ is the interval matrix according to Hansen's method (1), then we have

$$\text{mid}(\mathbf{Y}^{(k)}) = A_c^{-1} , \; k \geq 0 .$$

Proof: This property is obvious for k = 0. For k = 1 we conclude as follows:

$$\tilde{\mathbf{Y}}^{(1)} = \text{mid}(\mathbf{X}^{(o)}) + \mathbf{X}^{(o)}(I - \mathbf{A}\; \text{mid}(\mathbf{X}^{(o)}))$$

$$= A_c^{-1} + \mathbf{X}^{(o)}(I - \mathbf{A} \cdot A_c^{-1}) = A_c^{-1} + \mathbf{X}^{(o)} \cdot \mathbf{E}$$

with $\mathbf{E} = -\mathbf{E}$. Thus we have $\text{mid}(\tilde{\mathbf{Y}}^{(1)}) = A_c^{-1}$ and finally get

$$\text{mid}(\mathbf{Y}^{(1)}) = \text{mid}(\tilde{\mathbf{Y}}^{(1)} \cap \mathbf{X}^{(o)}) = A_c^{-1} .$$

The rest is done by complete induction. □

An immediate consequence of this lemma is the next property.

Corollary 4: If we take $\mathbf{Y}^{(o)} = \mathbf{X}^{(o)}$, where $\mathbf{X}^{(o)}$ is the interval matrix according to Hansen's method (1) , then the method (2) has the special form

$$(2') \quad \begin{aligned} \tilde{\mathbf{Y}}^{(k+1)} &= A_c^{-1} + \mathbf{Y}^{(k)} \cdot \mathbf{E} , \\ \mathbf{Y}^{(k+1)} &= \tilde{\mathbf{Y}}^{(k+1)} \cap \mathbf{Y}^{(k)} , \; k \geq 0 . \end{aligned}$$

But iteration method (2') can be further simplified. This is the content of the next lemma.

Lemma 5: If we take $Y^{(o)} = X^{(o)}$, where $X^{(o)}$ is the interval matrix according to Hansen's method (1), then the iterates of (2') have the property

$$Y^{(k+1)} \subseteq Y^{(k)}, \quad k \geq 0,$$

i.e. the intersection in (2') is a redundant operation.

Proof: Since (2') is a linear interval iteration method, it follows from the inclusion monotonicity of the interval matrix operations that $\tilde{Y}^{(1)} \subseteq X^{(o)} = Y^{(o)}$ is sufficient for the monotonicity of the whole sequence of iterates. Let us assume that

$$X^{(o)} \subset \tilde{Y}^{(1)} = A_c^{-1} + X^{(o)} \cdot E$$

holds true. From (1) we get with $F = [-1, 1] \cdot \left(\frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|} \right)$

$$A_c^{-1} \cdot F \subset X^{(o)} \cdot E$$

or

$$F \subset [I + F] \cdot E \quad .$$

Applying the width operator d on this inclusion leads to

$$d(F) < d(E) + d(F \cdot E)$$

$$2 \cdot |F| < 2 \cdot |E| + 2 \cdot |F| \cdot |E|$$

$$|F| < |E| \cdot (I - |E|)^{-1}$$

Now, applying the row-sum norm $\| \cdot \|$ to this inequality we finally get

$$n \cdot \frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|} < \| \, |E| \cdot (I - |E|)^{-1} \, \| \leq \frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|}$$

which is a contradiction for $n \geq 1$. Since $\mathrm{mid}(\tilde{Y}^{(1)}) = \mathrm{mid}(X^{(o)}) = A_c^{-1}$ the asserted property $\tilde{Y}^{(1)} \subseteq X^{(o)}$ must hold. □

Corollary 6: If we take $Y^{(o)} = X^{(o)}$, where $X^{(o)}$ is the interval matrix according to Hansen's method (1), then method (2) has the simple form

(2") $$Y^{(k+1)} = A_c^{-1} + Y^{(k)} \cdot E, \quad k \geq 0 .$$

We have considered two different methods for improving the initial interval matrix $X^{(o)} \supseteq A^{-1}$, namely (1) and (2"). Both methods produce a nested sequence of interval matrices $\{X^{(k)}\}$ and $\{Y^{(k)}\}$, respectively. Now the question arises which method finally gives the better result. The answer of that question is the following theorem.

Theorem 7: If we take $Y^{(o)} = X^{(o)}$, where $X^{(o)}$ is the interval matrix according to Hansen's method (1), then we have

$$Y^* = X^* \ .$$

Proof: All we have to do is to prove the same explicit formula for Y^* as it was shown to be valid for X^* in Lemma 2. For this purpose we are going to verify that iteration (2") can be written in the form

$$Y^{(k)} = A_c^{-1} + Z^{(k)} \cdot E \ , \text{ with } Z^{(k)} \geq 0 \ ,$$

$$\text{(2''')} \qquad \text{where } Z^{(1)} = |A_c^{-1}| + |A_c^{-1}| \cdot |F| \quad (\text{for } F \text{ see Lemma 5}) \ ,$$

$$Z^{(k)} = |A_c^{-1}| + Z^{(k-1)} \cdot |E| \quad \text{for } k \geq 2 \ .$$

We are getting the expression for $Z^{(1)}$ by direct substitution of the expression for $X^{(o)}$ according to (1) into the iteration formula (2") . The general formula for $Z^{(k)}$ can be proved by complete induction in the following way.

$$\begin{aligned} Y^{(k+1)} &= A_c^{-1} + Y^{(k)} \cdot E = A_c^{-1} + [A_c^{-1} + Z^{(k)} \cdot E] \cdot E \\ &= A_c^{-1} + |A_c^{-1} + Z^{(k)} \cdot E| \cdot E \\ &= A_c^{-1} + [|A_c^{-1}| + Z^{(k)} \cdot |E|] \cdot E = A_c^{-1} + Z^{(k+1)} \cdot E \ . \end{aligned}$$

Since $\| \, |E| \, \| < 1$ is fulfilled, the iteration for $Z^{(k)}$ converges to the unique fixed-point Z^* with

$$Z^* = |A_c^{-1}| + Z^* \cdot |E|$$

or

$$Z^* = |A_c^{-1}| \cdot (I - |E|)^{-1} \ .$$

But then the matrices $Y^{(k)}$ converge to Y^* with

$$Y^* = A_c^{-1} + Z^* \cdot E = A_c^{-1} + [-1, 1] \cdot A_c^{-1} \cdot |E| \cdot (I - |E|)^{-1}$$

which is just the formula for X^* of Lemma 2 . □

After this main result we are now proving an estimation of the rate of improvement for Y^* (= X^*) in comparison to the initial iclusion matrix $Y^{(o)} = X^{(o)}$. This is a generalization of a result given in [6] .

Lemma 8: Let Y^* be the limit of the iterates according to method (2) starting with $Y^{(o)} = X^{(o)}$, where $X^{(o)}$ is the interval matrix according to Hansen's method (1), then we have

$$\frac{\| d(X^*) \|}{\| d(X^{(o)}) \|} = \frac{\| d(Y^*) \|}{\| d(X^{(o)}) \|} \leq \frac{1}{n} .$$

Proof: From Lemma 5 in connection with Lemma 2 we have

$$X^* = Y^* = A_c^{-1} + [-1, 1] \cdot A_c^{-1} \cdot |E| \cdot (I - |E|)^{-1}$$

and after applying the width operator d on this equality we get

$$d(X^*) = d(Y^*) = 2 \cdot |A_c^{-1}| \cdot |E| \cdot (I - |E|)^{-1}$$

on which we apply the row-sum norm $\| \cdot \|$ and derive

$$\| d(X^*) \| = \| d(Y^*) \| \leq 2 \cdot \| A_c^{-1} \| \cdot \frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|} .$$

On the other hand we have

$$d(X^{(o)}) = d(A_c^{-1} \cdot F) = 2 \cdot |A_c^{-1}| \cdot \left(\frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|} \right)$$

on which we are now applying the row-sum norm $\| \cdot \|$ and, observing the special structure of the second matrix, we finally get

$$\| d(X^{(o)}) \| = 2 \cdot \| A_c^{-1} \| \cdot n \cdot \frac{\| \, |E| \, \|}{1 - \| \, |E| \, \|}$$

which proves the assertion. □

5. Numerical Examples

In order to illustrate the estimation of Lemma 8, we are giving some simple numerical examples. Doing this, we consider the interval matrices of the form

$$\mathbf{A} = \mathbf{I} + \mathbf{F}\ ,\ \text{with } \mathbf{F} = ([-f, f])$$

which are regular for a suitable choice of f .

First we fix $f = 5 \cdot 10^{-3}$ and get the following results for several values of the dimension n where $\| \cdot \|$ is the row-sum norm.

Table 1:

n	$\frac{\| d(Y^*) \|}{\| d(X^{(o)}) \|}$
5	1.999999999973E-01
10	9.999999999886E-02
15	6.666666666623E-02

Table 1 clearly shows the predicted dependence of the ratio $\frac{\| d(Y^*) \|}{\| d(X^{(o)}) \|}$ from n .

Next, we fix the dimension $n = 10$ and choose several values for f .

Table 2:

f	$\frac{\| d(Y^*) \|}{\| d(X^{(o)}) \|}$
10^{-2}	6.666806088900
10^{-3}	6.666666311299
10^{-4}	6.666666311113

Table 2 is showing that the ratio of improvement is nearly independent from the choice of the parameter f .

The examples were taken from [3] and are computed on a microcomputer in PASCAL-SC (see [7]).

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