APPLIED NONLINEAR ANALYSIS

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ACADEMIC PRESS
NEW YORK SAN FRANCISCO LONDON 1979
A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS RAPID MANUSCRIPT REPRODUCTION

Proceedings of an International Conference on Applied Nonlinear Analysis, Held at The University of Texas at Arlington, Arlington, Texas, April 20–22, 1978.

A SUMMARY OF RECENT EXPERIMENTS TO COMPUTE THE TOPOLOGICAL DEGREE

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Classically, we obtain numerical solutions of systems of nonlinear algebraic equations by Newton's method or by related gradient methods. In the past decade, however, several new approaches have been explored. In particular, fixed points (or roots) of continuous functions can be approximated by constructive methods of combinatorial topology ([2], [3], [5], [7], [13], [14]). Resulting topological (or combinatorial) algorithms may be employed to locate reliably all roots of a map ([2], [3]).

To ascertain a priori the number of such roots within a given volume of $\mathbb{R}^n$, it is possible to compute the Brouwer degree of the map. Related to other combinatorial methods, new ideas for degree computation have recently been investigated ([8], [9], [11], [15], [16]). Below, we define the Brouwer degree, summarize our method of computation, discuss the algorithm's salient characteristics, and give c.p.u. times for test problems. References are provided where necessary.

ISBN 0-12-434180-2

I. THE CONCEPT AND THE COMPUTATION FORMULA

Consider a continuous map F from a compact n-polygon D into $\mathbb{R}^n$, such that $F(X) \neq 0$ if $X \in b(D)$, where $b(D)$ is the boundary of D. Suppose $\det(F'(X)) \neq 0$ when $F(X) = 0$. Then the degree of F at 0 relative to D, denoted $d(F,D,0)$, is equal to the number of zeros of F where $\det(F'(X)) > 0$, minus the number of zeros of F where $\det(F'(X)) < 0$.

We generalize $d(F,D,0)$ to non-differentiable continuous maps by defining the index of a zero X of F. Given a sufficiently small neighborhood M of X, $f(M)$ covers a neighborhood N of 0 in an m-to-one fashion. Considering the m branches of f/M, the index of X is the number of coverings of N with positive orientation minus the number of coverings of N with negative orientation. Then, $d(F,D,0)$ is the sum of the indices of the zeros of F in D ([4], ch. 1).

The above definitions for $d(F,D,0)$ coincide when F is differentiable with no multiple roots.

To compute $d(F,D,0)$ we consider $F/\|F\|\ |b(D)$. If $Y \in F(b(D))$, it can be shown that $d(F,D,0)$ is equal to the number of times $F/\|F\|\ |b(D)$ covers Y with a positive orientation in $\mathbb{R}^{n-1}$, minus the number of times $F/\|F\|\ |b(D)$ covers Y with a negative orientation ([4], ch. 1). Assume $b(D)$ is polygonal, and triangulate $b(D)$ into simplexes $\{S_i\}_{i=1}^{k}$ such that at least one component of F does not vanish on each S_i. Choose Y to be the intersection of the unit n-sphere with the positive first coordinate axis, and assume appropriate components of $F = (f_1, f_2, \ldots, f_n)$ do not vanish on the $(n-2)$-dimensional boundaries $b(S_i)$ ([8], ch. 3, etc.). We then have:

$$d(F,D,0) = \sum_{j=1}^{m} d(F\ ,S_{i_j}\ ,0).$$

Above, the sum is over all S_i on which $f_1 > 0$, and $F_1 = (f_2, \ldots, f_n)\colon\ S_{i_j} \to \mathbb{R}^{n-1}$ ([8], [11], [15]).

Formulas not involving recursion (in the computer programming sense) have also been presented ([8], [9], [15]). However, the recursion formula has been easiest to implement if we do not allow heuristic determination of the mesh on $b(D)$ ([11]).

II. THE ALGORITHM

We proceed as follows:

(1) Triangulate $b(D)$ to obtain a sufficiently small mesh, as in Section 1, then continue to step (2).

(2) Replace F by F_1 and D by S_{i_j}, and do step (3), for $j = 1,\dots,m$.

(3) (a) If the S_{i_j} are one-dimensional, compute $d(F,S_{i_j},0)$ directly.

(b) If the dimension of the S_{i_j} is greater than 1, repeat steps (1) and (2) with S_{i_j} and F_1 in place of D and F.

In one dimension, $d(f,S,0) = d(f,\langle a,b\rangle,0) = \frac{1}{2}\{\mathrm{sgn}[f(b)] - \mathrm{sgn}[f(a)]\}$. Also, observe that a stack of executions of steps (1), (2), and (3) is generated if $n > 2$.

It is convenient to construct the triangulation by "generalized bisection", which we define as follows: given the $(n-1)$-simplex $S = \langle X_1, X_2, \dots, X_n \rangle$, we define two new simplexes by replacing X_k by $(X_k + X_m)/2$ or X_m by $(X_k + X_m)/2$, where $\langle X_k, X_m \rangle$ is the longest side of S ([8], [10], [15]). The simplexes in such triangulations correspond to nodes in binary trees, and the elements in the final triangulation can be considered with a minimum of computation and storage by a depth-first search of such trees ([11]). The depth of each path is set by examining the moduli of continuity of the components of F ([11]).

A more detailed exposition of the algorithm appears in [11].

III. PERFORMANCE OF THE ALGORITHM AND SCOPE OF APPLICATION

To date, few other methods for computing $d(F,D,0)$ have appeared. Erdelsky ([6]) described an efficient method, equivalent to ours except for the triangulation, for $n = 2$. O'Neil and Thomas ([12]) computed the degree for arbitrary n by quadrature involving the Kronecker integral ([1], p. 465). These computations involved probabilistic estimates for the accuracy of the result.

Our approach lends itself naturally to root-finding ([8], [9], [11]). Assume $D = < X_0, X_1, \ldots, X_n >$ is an n-simplex, and compute $(F,D,0)$, triangulating $b(D)$ by bisection. If $d(F,D,0) \neq 0$, bisect the n-simplex D, forming S_1 and S_2, and compute $d(F,S_1,0)$ and $d(F,S_2,0) = d(F,D,0) - d(F,S_1,0)$. This computation is expedited with information retained from computation of $d(F,S,0)$ ([11]). We repeat the process, bisecting the first S_i over which F has non-zero degree and storing the other S_i in a list if F also has non-zero degree on it ([11]). The procedure continues until a simplex with diameter less than a specified tolerance is found. We then repeat the bisection-degree computation process on the stored simplexes until the list is empty (ibid).

Our root-finding algorithm shares properties with other combinatorial fixed point algorithms. Function values only are required, and only rough accuracy is needed. Moreover, all roots, including ones difficult to obtain with gradient methods, may often be located. Our degree-computation method, however, gives lower bounds on the number of roots within the search region, while other methods may find approximate zeros which are not near true roots ([2], [3], etc.).

Degree computation-bisection has several disadvantages. The diameters of the resulting simplexes decrease linearly as bisection proceeds, and the rate of decrease increases with n ([10]). Furthermore, due to the recursive nature, execution time for functions of comparable smoothness increases exponentially with n. Lastly, we must assume that there is no root of F on the boundary of any n-simplex produced by bisection; also, there must be no roots of the truncated functions on the boundaries of any of the lower-dimensional simplexes (when $n > 2$). When such roots exist, they are found in the process of degree computation, preventing the algorithm from proceeding further.

In practice, it is possible to avoid roots on boundaries by changing the vertices of D slightly. Also, the algorithm is not necessarily too costly in small dimensions.

IV. NUMERICAL RESULTS

We present results for several test examples in 2 and 3 dimensions.

The experimental program involved root-finding by bisection. It contained a parameter controlling information storage between successive degree computations ([11]), but we present c.p.u. times for optimal values of that parameter. In all cases, the stopping diameter (tolerance) was .1, and all roots and corresponding indices within D were found.

The PL/I program was run interactively on a Multics 68/80 system. The results in 2 dimensions appear in Table 1. The trial function in 3 dimensions was: $f_1 = x_1^2 - x_2$, $f_2 = x_2^2 - x_3$, $f_3 = x_3^2 - x_1$, and $D = < (.9,.1,-.1), (0,1,0), (.1,0,1.1), (-.9,-.8,-.7) >$. The c.p.u. time for that example was 94.1 seconds.

TABLE I. Two Dimensional Examples
$D = \langle(-4.1,-3.9),(4,-4),(-.15,4)\rangle$

function	*c.p.u. time*
z^2	3.5
z^2+1	3.7
z^3	13.6
z^3+1	11.9
z^4	37.3
z^4+1	37.2

REFERENCES

[1] Alexandroff, P., and Hopf, H. (1973, orig. 1935). "Topologie", Chelsea, New York.

[2] Allgower, E. L. (1974). "Numerische Approximation von Lösungen nichtlinearer Randuerts-aufgaben mit meheren Lösungen", *ZAMM, Tagung 54*, München.

[3] Allgower, E. L., and Keller, K. L. (1971). "A search routine for a Sperner simplex", *Computing 8*, 157-165.

[4] Cronin, J. (1964). "Fixed points and topological degree in nonlinear analysis", *A.M.S. Surveys No. 11*.

[5] Eaves, B. C., and Saigal, R. (1972). "Homotopies for the computation of fixed points", Mathematical Programming 13, Nos. 1 and 2.

[6] Erdelsky, P. J. (1973). "Computing the Brouwer degree in $\mathbb{R}^2$", *Math. Comp. 27, #121*.

[7] Jeppson, M. (1972). "A search for the fixed points of a continuous mapping", Mathematical Topics in Economic Theory and Computation, *SIAM*, Philadelphia, 122-125.

[8] Kearfott, R. B. (1977). "Computing the degree of maps and a generalized method of bisection", Ph.D. dissertation, University of Utah.

[9] Kearfott, R. B. "An efficient degree-computation method for a generalized method of bisection", submitted to *Numer. Math.*

[10] Kearfott, R. B. "A proof of convergence and an error bound for the method of bisection in $\mathbb{R}^n$", to appear in *Math. Comp.*

[11] Kearfott, R. B. "Root-finding experiments using direct computation of the topological degree", submitted to *Math. Comp.*

[12] O'Neil, T., and Thomas, J. (1975). "The calculation of the topological degree by quadrature", *SIAM J. Numer. Anal. 12*, 637-689.

[13] Saigal, R. (1976). "Fixed point computing methods", pre-print, the Center for Mathematical Studies in Economics and Management Science, Northwestern University.

[14] Scarf, H. (1967). "The approximation of fixed points of a continuous mapping", *SIAM J. Appl. Math. 15, No. 5.*

[15] Stenger, F. (1976). "An algorithm for the topological degree of a mapping in $\mathbb{R}^n$", *Numer. Math. 25,* 23-28.

[16] Stynes, M. (1977). "An algorithm for the numerical computation of the degree of a mapping", Ph.D. dissertation, University of Oregon.