A SUMMARY OF RECENT EXPERIMENTS
TO COMPUTE THE TOPOLOGICAL DEGREE

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Classically, we obtain numerical solutions of systems of nonlinear algebraic equations by Newton's method or by related gradient methods. In the past decade, however, several new approaches have been explored. In particular, fixed points (or roots) of continuous functions can be approximated by constructive methods of combinatorial topology ([2], [3], [5], [7], [13], [14]). Resulting topological (or combinatorial) algorithms may be employed to locate reliably all roots of a map ([2], [3]).

To ascertain a priori the number of such roots within a given volume of $\mathbb{R}^n$, it is possible to compute the Brouwer degree of the map. Related to other combinatorial methods, new ideas for degree computation have recently been investigated ([8], [9], [11], [15], [16]). Below, we define the Brouwer degree, summarize our method of computation, discuss the algorithm's salient characteristics, and give c.p.u. times for test problems.

References are provided where necessary.
I. THE CONCEPT AND THE COMPUTATION FORMULA

Consider a continuous map $F$ from a compact $n$-polygon $D$ into $\mathbb{R}^n$, such that $F(X) \neq 0$ if $X \in \partial(D)$, where $\partial(D)$ is the boundary of $D$. Suppose $\det(F'(X)) \neq 0$ when $F(X) = 0$. Then the degree of $F$ at $0$ relative to $D$, denoted $d(F,D,0)$, is equal to the number of zeros of $F$ where $\det(F'(X)) > 0$, minus the number of zeros of $F$ where $\det(F'(X)) < 0$.

We generalize $d(F,D,0)$ to non-differentiable continuous maps by defining the index of a zero $X$ of $F$. Given a sufficiently small neighborhood $M$ of $X$, $f(M)$ covers a neighborhood $N$ of $0$ in an $m$-to-one fashion. Considering the $m$ branches of $f/M$, the index of $X$ is the number of coverings of $N$ with positive orientation minus the number of coverings of $N$ with negative orientation. Then, $d(F,D,0)$ is the sum of the indices of the zeros of $F$ in $D$ ([4], ch. 1).

The above definitions for $d(F,D,0)$ coincide when $F$ is differentiable with no multiple roots.

To compute $d(F,D,0)$ we consider $F/\|F\| \cdot \partial(D)$. If $Y \in F(\partial(D))$, it can be shown that $d(F,D,0)$ is equal to the number of times $F/\|F\| \cdot \partial(D)$ covers $Y$ with a positive orientation in $\mathbb{R}^{n-1}$, minus the number of times $F/\|F\| \cdot \partial(D)$ covers $Y$ with a negative orientation ([4], ch. 1). Assume $\partial(D)$ is polygonal, and triangulate $\partial(D)$ into simplexes $\{S_{ij}\}_{i=1}^k$ such that at least one component of $F$ does not vanish on each $S_{ij}$. Choose $Y$ to be the intersection of the unit $n$-sphere with the positive first coordinate axis, and assume appropriate components of $F = (f_1,f_2,...,f_n)$ do not vanish on the $(n-2)$-dimensional boundaries $\partial(S_{ij})$ ([8], ch. 3, etc.). We then have:

$$d(F,D,0) = \sum_{j=1}^m d(F,S_{ij},0).$$

Above, the sum is over all $S_{ij}$ on which $f_1 > 0$, and $F_1 = (f_2,...,f_n): S_{ij} \rightarrow \mathbb{R}^{n-1}$ ([8], [11], [15]).
Formulas not involving recursion (in the computer programming sense) have also been presented ([8], [9], [15]). However, the recursion formula has been easiest to implement if we do not allow heuristic determination of the mesh on $b(D)$ ([11]).

II. THE ALGORITHM

We proceed as follows:

(1) Triangulate $b(D)$ to obtain a sufficiently small mesh, as in Section 1, then continue to step (2).

(2) Replace $F$ by $F_1$ and $D$ by $S_{i,j}$, and do step (3), for $j = 1, \ldots, m$.

(3) (a) If the $S_{i,j}$ are one-dimensional, compute $d(F, S_{i,j}, 0)$ directly.

(b) If the dimension of the $S_{i,j}$ is greater than 1, repeat steps (1) and (2) with $S_{i,j}$ and $F_1$ in place of $D$ and $F$.

In one dimension, $d(f, S, 0) = d(f, \langle a, b \rangle, 0) = \frac{1}{2} \left[ \text{sgn}[f(b)] - \text{sgn}[f(a)] \right]$. Also, observe that a stack of executions of steps (1), (2), and (3) is generated if $n > 2$.

It is convenient to construct the triangulation by "generalized bisection", which we define as follows: given the $(n-1)$-simplex $S = \langle X_1, X_2, \ldots, X_n \rangle$, we define two new simplexes by replacing $X_k$ by $(X_k + X_m)/2$ or $X_m$ by $(X_k + X_m)/2$, where $\langle X_k, X_m \rangle$ is the longest side of $S$ ([8], [10], [15]). The simplexes in such triangulations correspond to nodes in binary trees, and the elements in the final triangulation can be considered with a minimum of computation and storage by a depth-first search of such trees ([11]). The depth of each path is set by examining the moduli of continuity of the components of $F$ ([11]).

A more detailed exposition of the algorithm appears in [11].
III. PERFORMANCE OF THE ALGORITHM AND SCOPE OF APPLICATION

To date, few other methods for computing $d(F,D,0)$ have appeared. Erdelsky ([6]) described an efficient method, equivalent to ours except for the triangulation, for $n = 2$. O'Neil and Thomas ([12]) computed the degree for arbitrary $n$ by quadrature involving the Kronecker integral ([1], p. 465). These computations involved probabilistic estimates for the accuracy of the result.

Our approach lends itself naturally to root-finding ([8], [9], [11]). Assume $D = < X_0, X_1, \ldots, X_n >$ is an $n$-simplex, and compute $(F,D,0)$, triangulating $b(D)$ by bisection. If $d(F,D,0) \neq 0$, bisect the $n$-simplex $D$, forming $S_1$ and $S_2$, and compute $d(F,S_1,0)$ and $d(F,S_2,0) = d(F,D,0) - d(F,S_1,0)$. This computation is expedited with information retained from computation of $d(F,S,0)$ ([11]). We repeat the process, bisecting the first $S_i$ over which $F$ has non-zero degree and storing the other $S_i$ in a list if $F$ also has non-zero degree on it ([11]). The procedure continues until a simplex with diameter less than a specified tolerance is found. We then repeat the bisection-degree computation process on the stored simplexes until the list is empty (ibid).

Our root-finding algorithm shares properties with other combinatorial fixed point algorithms. Function values only are required, and only rough accuracy is needed. Moreover, all roots, including ones difficult to obtain with gradient methods, may often be located. Our degree-computation method, however, gives lower bounds on the number of roots within the search region, while other methods may find approximate zeros which are not near true roots ([2], [3], etc.).

Degree computation-bisection has several disadvantages. The diameters of the resulting simplexes decrease linearly as bisection proceeds, and the rate of decrease increases with $n$ ([10]). Furthermore, due to the recursive nature, execution time for
functions of comparable smoothness increases exponentially with \( n \). Lastly, we must assume that there is no root of \( F \) on the boundary of any \( n \)-simplex produced by bisection; also, there must be no roots of the truncated functions on the boundaries of any of the lower-dimensional simplices (when \( n > 2 \)). When such roots exist, they are found in the process of degree computation, preventing the algorithm from proceeding further.

In practice, it is possible to avoid roots on boundaries by changing the vertices of \( D \) slightly. Also, the algorithm is not necessarily too costly in small dimensions.

IV. NUMERICAL RESULTS

We present results for several test examples in 2 and 3 dimensions.

The experimental program involved root-finding by bisection. It contained a parameter controlling information storage between successive degree computations ([11]), but we present c.p.u. times for optimal values of that parameter. In all cases, the stopping diameter (tolerance) was .1, and all roots and corresponding indices within \( D \) were found.

The PL/I program was run interactively on a Multics 68/80 system. The results in 2 dimensions appear in Table 1. The trial function in 3 dimensions was: \( f_1 = x_1^2 - x_2, \ f_2 = x_2^2 - x_3, \ f_3 = x_3^2 - x_1, \) and \( D = <(0.9, 1.1, -0.1), (0.1, 0), (1.1, 0, 1.1), (-0.9, -0.8, -0.7)> \). The c.p.u. time for that example was 94.1 seconds.
TABLE I. Two Dimensional Examples
\[ D = \{(-4.1, -3.9), (4, -4), (-0.15, 4)\} \]

<table>
<thead>
<tr>
<th>function</th>
<th>c.p.u. time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z^2 )</td>
<td>3.5</td>
</tr>
<tr>
<td>( z^2 + 1 )</td>
<td>3.7</td>
</tr>
<tr>
<td>( z^3 )</td>
<td>13.6</td>
</tr>
<tr>
<td>( z^3 + 1 )</td>
<td>11.9</td>
</tr>
<tr>
<td>( z^4 )</td>
<td>37.3</td>
</tr>
<tr>
<td>( z^4 + 1 )</td>
<td>37.2</td>
</tr>
</tbody>
</table>

REFERENCES


