A Note on

Homotopy Methods for Solving Systems of Polynomial Equations Defined on the (n-1)-Sphere and Other Manifolds

by

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ABSTRACT

If $P(x_1,\ldots,x_n) = (p_1(x_1,\ldots,x_n),\ldots,p_{n-1}(x_1,\ldots,x_n)) = 0$ is a system of n-1 polynomial equations in n unknowns, it is explained how to find, with probability 1, all solutions to $P(x_1,\ldots,x_n) = 0$ subject to $\sum_{k=1}^{n} x_k^2 = 1$. The method is a simple extension of existing homotopy techniques for finding all solutions to polynomial systems with n equations in n unknowns, and can be applied to more general constraints than $\sum_{k=1}^{n} x_k^2 = 1$, including non-polynomial expressions.

The method can also be thought of as a hybrid symbolic-numerical method for solving n by n systems of polynomial equations and mixed polynomial-analytic equations. This point of view is most useful when the Jacobi matrix of the system is lower triangular, but may always be applied to reduce considerably the total number of paths required to obtain all solutions. The method can be implemented with existing continuation method software.

The method is described, applications are given, and practical considerations are discussed.

The method is related to Rheinboldt's study of differential equations on manifolds.

Key words: continuation methods, nonlinear algebraic systems, polynomial systems of equations, homotopy method

1. Introduction and Motivation.

Suppose $z \in \mathbb{C}^n$ and suppose $P(z) = (p_1(z), p_2(z), \ldots, p_{n-1}(z))$: $\mathbb{C}^n \to \mathbb{C}^{n-1}$ is a polynomial in the sense that $p_k(z)$ is a polynomial in z_m, $1 \leq k \leq n-1$, $1 \leq m \leq n$, for $z = (z_1, z_2, \ldots, z_n)$. Define a manifold $\mathcal{M}$ by:

(1.1) $\mathcal{M} = \{z \in \mathbb{C}^n\colon f(z)=0\}$,

where $f\colon \mathbb{C}^n \to \mathbb{C}$ is analytic (cf. eg. [3] or [5]), and where we assume $f(z_1, \ldots, z_n)=0$ can be solved for z_i for some i: $1 \leq i \leq n$. (Without loss of generality we will assume $i=n$). The problem we will consider will be to find all $z \in \mathbb{C}^n$ with:

(1.2) $P(z) = 0$ and $z \in \mathcal{M}$.

Several applications originally led us to problem (1.2). In these, $f(z) = \sum_{m=1}^{n} z_m^2 - 1$, and solutions of $P(z)=0$ on the $(n-1)$-sphere $\mathcal{S}^{n-1}$ are desired. For example, in the numerical treatment of bifurcation problems, the tangents to arcs intersecting at a bifurcation point can be represented as solutions to $P(z)=0$, $z \in \mathcal{S}^{n-1}$, where each p_k is a homogeneous quadratic, and the order of the bifurcation point is $n-1$ (cf. eg. [13]).

A second application where $f(z)$ defines $\mathcal{S}^{n-1}$ is in the numerical computation of the Brouwer degree of maps (see [8], [9], or [15]). In particular, suppose $F = (f_1, \ldots, f_n)\colon \mathcal{B}^n \to \mathbb{R}^n$. where $\mathcal{B}^n = \{x \in \mathbb{R}^n, ||x|| \leq 1\}$, and $F(x) \neq 0$ for $x \in \mathcal{S}^{n-1}$. Choose an arbitrary m and fix s $\{-1,1\}$. Then the Brouwer degree of at 0 relative to $\mathcal{B}^n$, written $d(F,\mathcal{B}^n,0)$, can be shown to equal the number of $x \in \mathcal{S}^{n-1}$ with $\mathrm{sgn}(f_m(x))=s$ and $f_k(x)=0$ for $k \neq m$, corresponding to a "positive" orientation, minus the number of such x corresponding to a negative orientation (cf. [1], [8], [9], [15], etc.). Thus, if f_k is a polynomial in the components of x for $k \neq m$, the Brouwer degree $d(F,\mathcal{B}^n,0)$ may be computed by finding all solutions to a problem of the form (1.2), where $\mathcal{M} = \mathcal{S}^{n-1}$.

More generally, suppose we wish to solve the system:

$$(1.3) \quad F(z) = (p_1(z), \ldots, p_\ell(z), f_{\ell+1}(z), \ldots, f_n(z)) = 0,$$

where p_k is a polynomial for $1 \leq k \leq \ell$ $(\ell \leq n)$, where f_k is analytic for $\ell+1 \leq k \leq n$, and where the equation $f_k(z)=f_k(z_1,\ldots,z_k)$ can be solved symbolically for z_k. (The solution z_k is allowed to have several branches). We further require that $f_k(z)=0$ and $|z_k| \to \infty$ implies $|z_i| \to \infty$ for some $i<k$. (This certainly is true if f_k is a polynomial, and in many other cases). Then the techniques in this paper may be used to: (1) find all solutions, in the case the f's are generally analytic; and (2) reduce the number of homotopy paths required to solve (1.3) in the case the f's are polynomials. We give details below.

2. The Methods.

The general method upon which our techniques are based is treated fully in [6] and elsewhere, so we give only a brief summary here.

In the general method, we find all solutions to a polynomial system $P(z)=0$, $P: \mathbb{C}^n \to \mathbb{C}^n$, by following the homotopy paths:

$$(2.1) \qquad H(z,t) = tP(z) + (1-t)Q(z)$$

from $t=0$ to $t=1$. Here, $P(z) = (p_1(z), p_2(z),\ldots,p_n(z))$ is such that the degree of p_k is n_k for $1 \le k \le n$; $Q(z) = (q_1(z),\ldots,q_n(z))$ is chosen so that $q_k(z) = z_k^{n_k+1} - 1$. (See [6]; other choices of Q are possible, but the degree of q_k and the dependence of q_k on z_k are important).

It has been shown ([4], [6], [7], etc.) that, with probability 1 over the space of coefficients of P, all $np = \prod_{k=1}^{n} n_k$ roots of P(z) are obtained at t=1 by following paths of H from roots of Q at t=0. However, there are $nq = \prod_{k=1}^{n} (n_k+1) >> np$ roots of Q, so numerous paths beginning at t=0 never reach t=1. (In fact, $||z|| \to \infty$ as $t \to 1$; see [6]).

The convergence proof in [6] is based on the fact that the roots of Q are bounded, the fact that, if $q_k \neq 0$ then:

$$(2.2) \qquad \frac{p_k(z)}{q_k(z)} = 1 - \frac{1}{t};$$

the fact that $||z|| \to \infty$ implies $z_{k_0} \to \infty$ for some k_0, and the fact that $q_{k_0}(z)$ is of degree $n_{k_0}+1$ in z_{k_0}. In our modified method, these conditions will still hold; when implemented to its fullest, though (for systems with triangular Jacobi matrices) it will only require $ns = (n_1+1)\prod_{k=2}^{n} n_k = np + \prod_{k=2}^{n} n_k$ paths be followed, and in no case will require more than $\tilde{n} = n_n \prod_{k=1}^{n-1} (n_k+1)$

paths. (This is a factor of 2 savings when P is a system of quadratics).

Suppose now the system is as in (1.3). We then define the homotopy:

$$(2.3) \quad \tilde{H}(z,t) = (h_1(z,t), h_2(z,t), \ldots, h_n(z,t))$$

where $h_k(z,t) = t\, p_k(z) + (1-t)\, q_k(z)$ with q_k as above for $1 \le k \le \ell$, and $h_k(z) = f_k$ for $\ell+1 \le k \le n$. Because of the assumptions on f_k, we may find all roots of $\tilde{H}(z,0)$ by selecting roots of q_k, $1 \le k \le \ell$, and then using a forward substitution process. Furthermore, if $||z|| \to \infty$, (2.2) remains valid for some $k \le \ell$. Since the other portions of the proof of convergence in [6] depended only on regularity and analyticity, we may conclude that all roots of F may be found by following paths of $\tilde{H} = 0$ from t=0 to t=1.

3. Summary, Conclusions, and Practical Considerations

We have pointed out a simple technique for improving the efficiency of a general method for finding all roots to a system of polynomial equations. Besides making the method more efficient, the technique fits naturally with certain applications, and also extends the method to cases not previously covered.

The general method has been shown to be reliable in practice in many cases (cf. eg. [12]). Furthermore, good software for it is readily available ([11], [14], etc.).

The only modifications necessary to implement the improvements are: (1) programming of the appropriate function $\tilde{H}$; and (2) the initial computing of roots of complex numbers required to get the coordinates z_k, $k>\ell$, of roots of $\tilde{H}(z,0)$. It is hard to imagine that these modifications will cause problems.

It is unclear without further investigation precisely how favorably the technique will compare with other techniques for handling the applications mentioned in section 1.

Finally, we mention that Rheinboldt has also used the idea of including the constraints as part of the function in order to solve equations defined on manifolds (cf. [13]). There, however, more general differential equations, as opposed to algebraic systems, are being solved, and more involved considerations come into play.

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