Numerical Analysis Comprehensive Examination
Tuesday, August 13, 2013, 9:00am to 1:00pm, MDD 208.

This exam is closed book, but calculators are permitted. Please use your own paper, and put you name on each sheet.

Q1: Define $F(z) = \int_0^1 x^z e^{-x} dx$. Integration by parts leads to $F(z) = zF(z-1) - e^{-1}$.
(a) Given $F(0.5) \approx 0.378944691641$, use the recursion formula to evaluate $F(16.5)$.
(b) Evaluate $F(26.5)$ by midpoint quadrature rule, and use the recursion to evaluate $F(16.5)$.
(c) Compare the two values, and explain which one is accurate and why.

Q2: Let $g(x) = \ln(2x+1)$. Consider the solution of the equation $x = g(x)$.
(a) Use the contraction mapping theorem to show that there exists a solution $x^* \in [1, 1.5]$.
(b) Give an upper bound of the convergence factor of fixed point iteration. Suppose the bound gives an accurate estimate of the convergence rate. To achieve faster convergence, do you prefer bisection method or fixed point iteration? Perform three steps of the method you prefer.

Q3: Consider approximating $f(x) = e^{-\frac{x}{2}} - 1$ on $[-1, 1]$.
(a) Find the quadratic Lagrange interpolation polynomial $p_2(x)$ based on equidistant nodes $\{-1, 0, 1\}$ and Chebyshev nodes $\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$, respectively, and give a bound for $\|f - p_2\|_\infty$.
(b) Find the quadratic least squares approximation $p_2(x)$ of $f(x)$ with respect to $\langle f, g \rangle = \int_{-1}^1 fg$.
(c) As the order n of the approximation polynomials $p_n(x)$ increases, can we guarantee, in each case, a *monotonically* decreasing $\|f - p_n\|_2$ or $\|f - p_n\|_\infty$? Why?

Q4: Assume that $f(x) \in \mathbf{C}^2[a, b]$, and let $M = \|f''(x)\|_\infty$.
(a) For the trapezoidal rule, show that $\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)] \right| \leq \frac{M(b-a)^3}{12}$.
(b) Use the idea of Romberg integration to derive Simpson's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right]$$

from the trapezoidal rule.

Q5: Suppose $A, \Delta A \in \mathbb{R}^{n\times n}$, and $b, \Delta b \in \mathbb{R}^n$ are such that $Ax = b$, and $(A + \Delta A)y = b + \Delta b$, where A is nonsingular. Let $\kappa(A)$ be the condition number of A. Assume that $\|\Delta A\| \leq \delta\|A\|$, $\|\Delta b\| \leq \delta\|b\|$, where $\delta\kappa(A) = r < 1$. Show that $A + \Delta A$ is nonsingular, and $\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}$.

Q6: Consider the solution of the linear system $Ax = b$, where

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is a standard 1-D finite difference discretized Laplacian.

(a) Show that A is symmetric positive definite, and give the Cholesky decomposition of A.

(b) Let $A = L + D + U$ be the standard splitting of A. Write down the Jacobi and Gauss-Seidel methods for $Ax = b$ in *matrix form*, and use the Gerschgorin circle theorem to show that both methods converge to the true solution.

(c) Find the spectral radius of the iteration matrix of the SOR method

$$x_{k+1} = (\omega L + D)^{-1}[(1-\omega)D - \omega U]x_k + (L + \frac{1}{\omega}D)^{-1}b, \text{ with optimal } \omega = \frac{4}{2+\sqrt{2}}.$$

Q7: Let $A = \begin{bmatrix} 1 & 5 & 7 \\ 3 & 0 & 6 \\ 4 & 3 & 1 \end{bmatrix}$. Consider using the QR algorithm to find the spectrum of A.

(a) Use a Householder reflector U_0 to reduce A to an upper Hessenberg matrix $H_0 = U_0^T A U_0$.

(b) Perform QR steps $H_k - \mu_k I = Q_k R_k$, $H_{k+1} = R_k Q_k + \mu_k I$ with scalars μ_k $(k = 0, 1, 2, \dots)$. Show that H_m $(m \geq 1)$ is unitarily similar to H_0 and is also an upper Hessenberg.

(c) Let $\tilde{Q}_m = Q_0 Q_1 \cdots Q_m$ and $\tilde{R}_m = R_m R_{m-1} \cdots R_0$. It can be shown that

$$\tilde{Q}_m \tilde{R}_m = (H_0 - \mu_m I)(H_0 - \mu_{m-1} I) \cdots (H_0 - \mu_0 I).$$

Assume that H_0 has a unique real dominant eigenvalue λ_1, and e_1 has a nonzero component in the direction of the eigenvector of H_0 corresponding to λ_1. Show that the first column of $\tilde{Q}_m$ generated by the QR algorithm with $\mu_k = 0$ converges to this eigenvector in direction, and the $(1,1)$ entry of H_m converges to λ_1 as $m \to \infty$.

Q8: Consider the solution of $F(z) \equiv F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + y - 3 \\ 3x - \frac{y^2}{2} - 1 \end{bmatrix} = 0$. Define $D_0 = [0.5, 1.5] \times [1.5, 2.5]$, and let γ be the Lipschitz constant such that $\|F'(z_\alpha) - F'(z_\beta)\| \leq \gamma \|z_\alpha - z_\beta\|$ for all $z_\alpha, z_\beta \in D_0$. Let z_0 be the initial Newton iterate, $\beta = \|F'(z_0)^{-1}\|$ and $\eta = \|F'(z_0)^{-1}F(z_0)\|$. The Newton-Kantorovich theorem states that if

$$2\gamma\beta\eta \leq 1, \tag{1}$$

then there is a unique solution $z^* \in D_0$ of $F(z) = 0$ near z_0, and Newton's method starting with z_0 converges to z^*.

(a) Does (1) hold in *Frobenius norm* for $z_0 = (1.5, 1.5)^T$ and $z_0 = (1.2, 1.8)^T$, respectively?

(b) Perform three steps of Newton's method with the z_0 that does *not* satisfy (1). What do you observe? Is (1) a necessary condition for the local convergence of Newton's method?

Q9: Consider the multistep method for the IVP $y' = f(t, y)$ with $y(0) = y_0$:

$$y_{n+2} - y_{n+1} = h\left(\frac{5}{12}f_{n+2} + \frac{2}{3}f_{n+1} - \frac{1}{12}f_n\right), \quad \text{where } f_k = f(t_k, y_k).$$

(a) Show that this method is consistent. What is the order of accuracy of this method?

(b) Show that this method is (zero) stable. Describe its region of absolute stability, and find the interval of absolute stability on the negative real axis. Is this method A-stable? A(0)-stable?