Math. 350-01, Summer, 2017, Third exam answers, page 1

1. $a_n = \frac{2^n}{n}$. We use the ratio test: $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

$$R = \lim_{n\to\infty}\left|\frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}}\right| = \lim_{n\to\infty}\frac{2^n}{2^{n+1}}\,\frac{n+1}{n} = \cancel{2}\,\frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \boxed{\frac{1}{2}}.$$

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2. $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$, so $e^{2x} = \sum_{n=0}^{\infty}\frac{1}{n!}(2x)^n = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$,

so $e^{2x}-1 = \sum_{n=1}^{\infty}\frac{2^n}{n!}x^n$, so $\frac{e^{2x}-1}{x} = \sum_{n=1}^{\infty}\frac{2^n}{n!}x^{n-1}$

$$= \boxed{\sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^n}.$$

The degree 4 Taylor approximation is thus:

$$\frac{e^{2x}-1}{x} \approx P_4(x) = 2 + \frac{2^2}{2!}x + \frac{2^3}{3!}x^2 + \frac{2^4}{4!}x^3 + \frac{2^5}{5!}x^4$$

$$= \boxed{2 + 2x + \frac{4}{3}x^2 + \frac{2}{3}x^3 + \frac{4}{15}x^4}$$

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3. $y(x) = \sum_{n=0}^{\infty}a_n x^n$. $y' = \sum_{n=1}^{\infty}na_nx^{n-1} = \boxed{\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = y'}$

$y'' = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} = \boxed{\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = y''}$

$2xy = \sum_{n=0}^{\infty}2a_nx^{n+1} = \boxed{\sum_{n=1}^{\infty}2a_{n-1}x^n = 2xy}$

Thus, $y''+y'+2xy = 2(1)(a_2) + 1a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} + (n+2)(n+1)a_{n+2} + 2a_{n-1}\right]x^n = 0.$

$a_0 = y(0) = \boxed{1 = a_0}$ $y'(0) = \boxed{a_1 = 0}$.

We thus have $2a_2 + a_1 = 0 \Rightarrow a_2 = -\frac{1}{2}a_1 = \boxed{0 = a_2}$.

For $n \ge 1$, we have:

$$a_{n+2} = \frac{-1}{(n+1)(n+2)}\left[(n+1)a_{n+1} + 2a_{n-1}\right].$$

This gives, for $n=1$:

$$a_3 = \frac{-1}{(2)(3)}\left[2a_2 + 2a_0\right] = \frac{-1}{6}[2] = \boxed{-\frac{1}{3} = a_3}.$$

and, for $n=2$:

$$a_4 = \frac{-1}{(3)(4)}\left[3a_3 + 2a_1\right] = \frac{-1}{12}[-1] = \boxed{\frac{1}{12} = a_4}.$$

The degree 4 approximation to the solution is thus:

$$y(x) \approx P_4(x) = 1 - \frac{1}{3}x^3 + \frac{1}{12}x^4$$