

① The equation is of the form  $my'' + \gamma y' + ky = \sin(t)$ , where  $\gamma$  is the resistance-to-motion coefficient and  $k$  is the spring constant. We will use kilograms, meters, and seconds. The spring constant  $k$  is:

$k = \text{Newtons/meter}$ , so  $k = 0.1 \text{ Newtons}/0.1 \text{ meter} = 1 \text{ Newton/meter}$ .  
 Similarly,  $\gamma = \text{Newtons}/(\text{meter/second})$ , so  $\gamma = 1/1 = 1 \text{ Newton}/(\text{meter/second})$ .

Thus, the initial value problem is:

$$y'' + y' + y = \sin(t), \quad y(0) = 0, \quad y'(0) = 0$$

② The characteristic equation is:  $r^2 + r + 1 = 0$ , giving  $r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$ .

This gives a general solution to the homogeneous equation of:  ~~$y_h(t) = C_1 e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) + C_2 e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t)$~~

$$y_h(t) = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \left(\sin\left(\frac{\sqrt{3}}{2}t\right)\right) e^{-\frac{1}{2}t}$$

We assume  $y(t) = A \cos(t) + B \sin(t)$  for a particular solution to the non-homogeneous equation:

|       |     |            |     |            |
|-------|-----|------------|-----|------------|
| $y$   | $A$ | $\cos(t)$  | $B$ | $\sin(t)$  |
| $y'$  |     | $-\sin(t)$ |     | $\cos(t)$  |
| $y''$ |     | $-\cos(t)$ |     | $-\sin(t)$ |

$$y'' + y' + y = -A \sin(t) + B \cos(t) = \sin(t)$$

$$\Rightarrow A = -1, \quad B = 0 \Rightarrow y_p(t) = -\cos(t)$$

Thus, the general solution to the non-homogeneous equation is:

$$y(t) = e^{-\frac{1}{2}t} \left[ C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right] - \cos(t)$$

$$y(0) = C_1 - 1 = 0 \Rightarrow \boxed{C_1 = 1}$$

$$y'(t) = -\frac{1}{2} e^{-\frac{1}{2}t} \left[ C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right] + e^{-\frac{1}{2}t} \left[ -\frac{\sqrt{3}}{2} C_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2} C_2 \cos\left(\frac{\sqrt{3}}{2}t\right) \right] + \sin(t)$$

$$\text{so } y'(0) = -\frac{1}{2} C_1 + \frac{\sqrt{3}}{2} C_2 = 0 \Rightarrow \frac{\sqrt{3}}{2} C_2 = \frac{1}{2} \Rightarrow \boxed{C_2 = \frac{1}{\sqrt{3}}}$$

Thus,

$$y(t) = e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) - \cos(t)$$