Math 301-02, Summer, 2005, Final Exam Answers, page 1

1 (a) $$\int_{x=0}^{1} \frac{\arctan x}{x^2+1} = \int_{u=0}^{\pi/4} u\,du$$

$$\boxed{\begin{array}{l} u = \arctan(x) \\ du = \frac{1}{1+x^2}dx \end{array}}$$

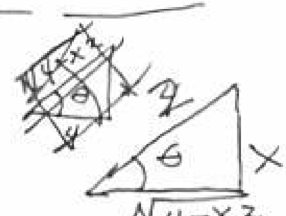
$$= \frac{u^2}{2}\Big|_{u=0}^{\pi/4} = \boxed{\frac{\pi^2}{\cancel{16}32}}$$

(b) $$\int_{x=0}^{\pi/2} x\,\overbrace{\sin(2x)dx}^{dv} = -\frac{1}{2}x\cos 2x\Big|_{x=0}^{\pi/2} + \frac{1}{2}\int_{x=0}^{\pi/2}\cos(2x)\,dx$$

$$= -\frac{1}{2}\frac{\pi}{2}\cos\pi + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)\Big|_{x=0}^{\pi/2}$$

$$= \frac{\pi}{4}$$

(c) $$\int_{x=-2}^{2} \frac{dx}{\sqrt{4-x^2}} = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{2\cos(\theta)d\theta}{2\cos(\theta)}$$




$$= \theta\Big|_{\theta=-\frac{\pi}{2}}^{\pi/2} = \boxed{\pi}$$

$x = 2\sin\theta$

$dx = 2\cos(\theta)\,d\theta$

$\sqrt{4-x^2} = 2\cos(\theta)$

(2) (a) $\int_{x=0}^{1} \frac{x+1}{\sqrt{x}}\,dx = \int_{x=0}^{1} \frac{dx}{\sqrt{x}} + \int_{x=0}^{1} \sqrt{x}\,dx$, since each of these converges.

$$= 2x^{1/2}\Big|_{x=0}^{1} + \frac{2}{3}x^{3/2}\Big|_{x=0}^{1} = 2+\frac{2}{3} = \boxed{\frac{8}{3}}$$ ✓

(b) Set $x = u^2$, $dx = 2u\,du$, $\sqrt{x} = u$:

$$\int_{x=0}^{1} \frac{\sqrt{x}}{x+1}\,dx = \int_{u=0}^{1} \frac{u(2u\,du)}{u^2+1} = 2\int_{u=0}^{1} \frac{u^2}{u^2+1}\,du = 2\left\{\int_{u=0}^{1} 1\,du - \int_{u=0}^{1} \frac{du}{u^2+1}\right\}$$

$$= 2\left\{1 - \arctan(u)\Big|_{u=0}^{1}\right\} = \boxed{2\left\{1-\frac{\pi}{4}\right\}}$$

(c) $\int_{x=\pi}^{\infty} \frac{\sin(\sin(x))}{x^2}\,dx$ since $\left|\frac{\sin(\sin(x))}{x^2}\right| \le \frac{1}{x^2}$ and $\int_{x=\pi}^{\infty} \frac{1}{x^2}\,dx$ converges. However, there is no ~~f~~ closed form expression for the integral.

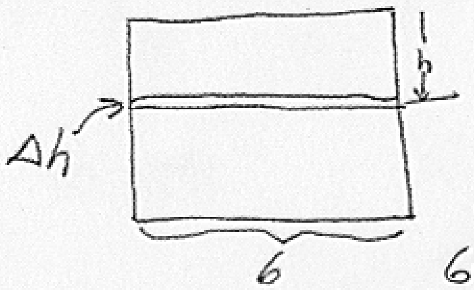
(d) $\int_{x=0}^{\infty} xe^{-x}\,dx = \lim_{M\to\infty} \int_{x=0}^{M} xe^{-x}\,dx = \lim_{M\to\infty}\left[-xe^{-x}\Big|_{x=0}^{M} + \int_{x=0}^{M} e^{-x}\,dx\right]$

$$= \lim_{M\to\infty}\left[-xe^{-x} - e^{-x}\right]\Big|_{x=0}^{M} = 1$$

(e) $\int_{x=-\infty}^{0} xe^{-x}\,dx$ does not converge, since the integrand does not tend to 0 as $x \to -\infty$.

(f) $\int_{x=-\infty}^{\infty} xe^{-x}\,dx$ does not converge since $\int_{-\infty}^{0} xe^{-x}\,dx$ does not converge.

(3)




The pressure at depth $h$ is $62.4h$ pounds per square foot, so the force of a strip at this depth and of height $\Delta h$ is approximately $6\cdot(62.4h)\Delta h$.

The total force is thus ~~approximately~~:

$$6\cdot\int_{h=0}^{6} 62.4h\,dh = (6)\,62.4\frac{h^2}{2}\Big|_{h=0}^{6} = (6)(62.4)(18) = \boxed{\cancel{1123.2\text{ pounds}}}\ \boxed{6739.2\text{ pounds}}$$

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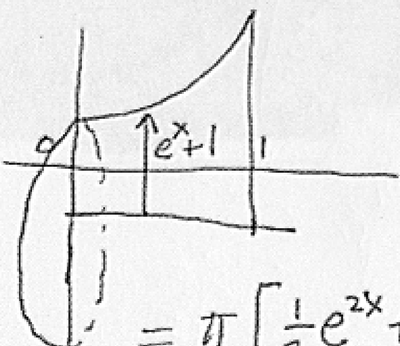
(4)




The volume is

$$\int_{x=0}^{1}\pi(e^x+1)^2\,dx = \pi\int_{x=0}^{1} e^{2x}+2e^x+1\,dx$$

$$= \pi\left[\frac{1}{2}e^{2x}+2e^x+x\right]\Big|_{x=0}^{1} = \pi\left\{\left[\frac{1}{2}e^2+2e+1\right]-\left[\frac{1}{2}+2\right]\right\}$$

$$= \boxed{\pi\left(\frac{1}{2}e^2+2e-\frac{3}{2}\right)}.$$

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5 (a)

We need values of $\frac{\arctan(x)}{x^2+1}$ at the depicted points.


| $x$ | $\arctan(x)/(x^2+1)$ |
|---|---|
| 0 | 0 |
| .25 | .2306 |
| .5 | .3709 |
| .75 | .4118 |
| 1 | .3927 |

| # subintervals | approx. value |
|---|---|
| 1 | $(0+.3927)/2 \approx .1964$ |
| 2 | $\approx .2836$ |
| 4 | $\approx .3024$ |

Computations with 2 subintervals:

$$\frac{1}{2}\left(\frac{0+.3709}{2} + \frac{.3709+.3927}{2}\right) \approx .2836$$

Computations with 4 subintervals:

$$\frac{1}{4}\left(\frac{0+.2306}{2} + \frac{.2306+.3709}{2} + \frac{.3709+.4118}{2} + \frac{.4118+.3927}{2}\right) \approx .3024$$

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(b) In Problem 1(a), we obtained $\pi^2/32 \approx .3084$. Thus, the trapezoidal rule with 4 subintervals is accurate to slightly less than 2 digits. Analyzing the error:

| # subintervals $n$ | approx − exact $= E_n$ | $E_{n+1}/E_n^2$ |
|---|---|---|
| 1 | $\cancel{-.1060}$ $-.1120$ | $-1.977$ |
| 2 | $-.0248$ | $-9.7555$ |
| 4 | $-.0060$ | |

There isn't an apparent pattern, but this may be because only four digits were taken.

(6) $e^{u} = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!}e^{c}$, so

(a) $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120}e^{+k}$ where $k \in (-x^2, 0)$

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(b) $\left| e^{-x^2} - p_8(x) \right| = \left| \frac{-x^{10}}{120} \right| \left| e^{k} \right| \leq \frac{1}{120}$.

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(c) $\int_{-1}^{1} e^{-x^2}dx \approx \int_{-1}^{1}\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right)dx = 2\int_{0}^{1}\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}\right)dx$

$= 2\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216}\right)\Big|_{x=0}^{1} = 2\left(1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}\right)$

$\approx 1.495$

$\cancel{= 2\left(1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{192}\right) \approx 1.4961}$.

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(d) $\left| [\text{True}] - [\text{Approx}] \right| = \left| \int_{-1}^{1} e^{-x^2} - p_8(x)\, dx \right|$

$\leq \int_{-1}^{1} \left| e^{-x^2} - p_8(x) \right| dx \leq \int_{-1}^{1} \frac{1}{120}\, dx = \frac{1}{60} \approx .0167.$

The actual error is much smaller than this, as can be seen by more sophisticated estimation methods. A better bound can be obtained by observing the Taylor series is an alternating series with terms whose absolute values decrease, for $-1 \leq x \leq 1$. Therefore,

$\left| e^{-x^2} - p_8(x) \right| \leq \frac{x^{10}}{120}$, so

$\left| [\text{True}] - [\text{Approx}] \right| \leq \int_{-1}^{1} \frac{x^{10}}{120}\, dx = 2\left(\frac{1}{11(120)}\right) \approx .0015.$