# Smoothness and Tractability of Multivariate Approximation

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## Smoothness and Tractability

**More Smoothness implies Easier to Approximate**

**Does More Smoothness imply Tractability ?**

# Formulation

$$f \in F_d \qquad \text{smooth } d\text{-variate functions}, \quad d = 1, 2, \dots$$

$$f \approx A_n(f) := \phi\big(L_1(f), L_2(f), \dots, L_n(f)\big), \quad L_j \in F_d^*$$

$$e(n,d) = \inf_{A_n} \sup_{f \in F_d,\ \|f\|_{F_d} \le 1} \|f - A_n(f)\|_{G_d}$$

$$n(\varepsilon, d) = \min\{\, n : \ e(n,d) \le \varepsilon \,\}$$

# Smoothness and Tractability

- **More Smoothness**

$e(n,d)$ goes faster to zero as $n \to \infty$

- **Tractability**

$n(\varepsilon,d)$ is *not* exponential in $d$ and $\varepsilon^{-1}$

# Infinite Smoothness

$$X \quad = \quad C^{\infty}([0,1]^d), \quad \beta = [\beta_1, \dots, \beta_d] \text{ with } \beta_j \in \{0, 1, \dots\}$$

$$F_d \quad = \quad \left\{ f \in X : \quad \|f\|_F = \left( \sum_{\beta} \frac{1}{\beta!} \|D^{\beta} f\|_{L_2}^2 \right)^{1/2} < \infty \right\}$$

$$G_d \quad = \quad \left\{ g : \quad \|g\|_G = \left( \sum_{\beta:\, |\beta_j| \le m} \frac{1}{\beta!} \|D^{\beta} f\|_{L_2}^2 \right)^{1/2} < \infty \right\}$$

**For $m = 0$ we approximate function values,**
**for $m > 0$ also partial derivatives up to order $m$.**

# Rate of Convergence

**For fixed $d$ and arbitrarily large $r > 0$**

$$e(n,d) = \mathcal{O}(n^{-r}) \quad \text{as} \quad n \to \infty$$

**All embedding inequalities are now ok**

**For fixed $d$ and arbitrarily small $p > 0$**

$$n(\varepsilon,d) = \mathcal{O}\left(\varepsilon^{-p}\right) \quad \text{as} \quad \varepsilon \to 0.$$

**But....**

# Curse of Dimensionality

**But...**

$$e(n,d) \quad = \quad 1 \qquad\qquad \text{for all} \quad n \in [0,(m+1)^d-1]$$

$$n(\varepsilon,d) \quad \geq \quad (m+1)^d \quad \text{for all} \quad \varepsilon \in (0,1)$$

**For $m \geq 1$ i.e., when we approximate partial derivatives,**

**Curse of Dimensionality !!!**

**Proof similar to Sloan +W. [1997]**

## $m = 0$

**Approximate** $f \in F_d$ **in** $G_d = L_2([0,1]^d)$

$$n(\varepsilon, d) = \Theta\left( \left( \ln \varepsilon^{-1} \right)^d \right)$$

$$\lim_{\varepsilon \to 0} \frac{n(\varepsilon, d)}{[\ln \varepsilon^{-1}]^d} = \frac{1}{\pi^2} \frac{(2\pi)^d}{d\,!}$$

**For small** $d$ **great,**

**for large** $d$ **the asymptotic constant exponentially small**

**But ....**

# Polynomial Tractability

**Polynomial Tractability: there are non-negative $C,\ p,\ q$ such that**

$$n(\varepsilon,d)\,\le\,C\,d^{\,q}\,\varepsilon^{-p}\quad\text{for all}\quad \varepsilon\in(0,1),\ d=1,2,\dots$$

**Do we have polynomial tractability?**

**NO !!!**

**For fixed $\varepsilon$ and $d\to\infty$**

$$n(\varepsilon,d)\,=\,\Theta\left(d^{\,\lceil \alpha\,\ln\,\varepsilon^{-1}\rceil-1}\right)\quad \alpha\,=\,\frac{1}{2\pi-\ln\,2}$$

**So the exponent of $d$ can be arbitrarily large**

**So ....**

# Generalized Tractability

$$T : [0,\infty)^2 \to [1,\infty) \qquad T \text{ increasing in both arguments}$$

$$T \text{ non-exponential} \qquad \lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln T(\varepsilon^{-1},d)}{\varepsilon^{-1}+d} = 0$$

as in Gnewuch + W. [2006]

**$T$-tractability:** **there are non-negative** $C$, $t$ such that

$$n(\varepsilon,d) \le C\,T(\varepsilon^{-1},d)^t \quad \text{for all} \quad \varepsilon \in (0,1),\ d = 1,2,\dots$$

**For**

$$T(\varepsilon^{-1},d) = \exp\big((1+\ln\varepsilon^{-1})\,(1+\ln d)\big)$$

**we have $T$-tractability !!!**

# Conclusions

- Smoothness and Tractability are **not** related

- Smoothness does not necessarily imply tractability

  as for the problem presented here

- Tractability does not necessarily imply large smoothness

  as for star discrepancy, weighted problems. etc...