TECHNICAL MEMORANDUM: MATHEMATICS

INTERVAL INTEGRALS

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Lockheed

MISSILES and SPACE DIVISION

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TECHNICAL MEMORANDUM: MATHEMATICS

INTERVAL INTEGRALS

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FOREWORD

This report presents results obtained in connection with research on automatic error analysis.

ABSTRACT

An arithmetic for intervals was studied by Moore and Yang (Ref. 1) as a basis for error analysis in digital computations. In that study, an integral $\int_A F$ was defined for certain functions F whose domain and range are contained in the space of bounded closed intervals of real numbers. The present paper establishes further properties of the integral defined there and relates this integral to integration of ordinary real-valued functions.

CONTENTS

Section 1
INTRODUCTION

R will be used to denote the real line. Whenever a and b are real numbers with $a \leqq b$, $[a, b]$ will denote the subset of R consisting of all real numbers x such that $a \leqq x \leqq b$. By $\mathcal{I}$ we will denote the set of all intervals $[a, b]$ contained in some interval. (Most of our results hold for $\tilde{\mathcal{I}} = \{[a, b] \mid a \in R \text{ and } b \in R\}$. (Arguments involving compactness and uniform continuity lead us to work in $\mathcal{I}$.) We use the usual metric topology in $\mathcal{I}$. $\mathcal{S} = \mathcal{S}_{[a, b]}$ will denote a subdivision of $[a, b]$;

$$\mathcal{S} = \{a = x_o, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

When the points of the subdivision are equally spaced, it will sometimes be denoted by $\mathcal{S}_n$. By $[a, b]\ r$, where r is a real number, we mean the set of all xr such that $x \in [a, b]$. We denote by $\Sigma(F, \mathcal{S})$ the summation

$$\Sigma(F, \mathcal{S}) = F[x_o, x_1](x_1 - x_o) + F[x_1, x_2](x_2 - x_1) + \cdots$$

$$+ F[x_{n-1}, x_n](x_n - x_{n-1})$$

Then Moore and Yang's integral is defined for each continuous function $F: \mathcal{I} \to \mathcal{I}$ for which $F(A) \subset F(B)$ whenever A is an interval contained in the interval B, by

$$\int_{[a, b]} F = \bigcap_{\mathcal{S}} \{\Sigma(F, \mathcal{S})\}$$

Section 2
CONTINUITY OF INTEGRATION

Theorem 1. Let $F: \mathcal{J} \to \mathcal{J}$ be a continuous function such that $A \subset B \Rightarrow F(A) \subset F(B)$. Define $G: \mathcal{J} \to \mathcal{J}$ by $G[a, b] = \int_{[a,b]} F$. Then G is continuous.

Proof: Consider

$$G[a_o, b_o] = \int_{[a_o, b_o]} F = \bigcap_{\mathcal{S}_{[a_o, b_o]}} \left\{ \Sigma\left(F, \mathcal{S}_{[a_o, b_o]}\right) \right\},$$

$$\epsilon > 0, \quad G[a_o, b_o] \in V_\epsilon$$

where V_ϵ = the open interval beginning at the left end-point of $G[a_o, b_o]$ minus ϵ and ending at the right end-point plus ϵ. The term $G[a_o, b_o]$ is an interval properly contained in V_ϵ and is the intersection of intervals I_α; therefore, one of these intervals I_1 has a right end-point less than the right end-point of $G[a_o, b_o] + \epsilon/2$. Similarly, there exists I_2 with left end-point greater than $G[a_o, b_o] - \epsilon/2$. The I_1 and I_2 came from subdivisions $\mathcal{S}^1_{[a_o, b_o]}$ and $\mathcal{S}^2_{[a_o, b_o]}$. Then $\mathcal{S}^3_{[a_o, b_o]}$ = the subdivision, consisting of all end-points of both original subdivisions so that

$$I_3 = \Sigma\left(F, \mathcal{S}^3_{[a_o, b_o]}\right) \subset \left[\Sigma\left(F, \mathcal{S}^1_{[a_o, b_o]}\right)\right] \cap \left[\Sigma\left(F, \mathcal{S}^2_{[a_o, b_o]}\right)\right] = I_1 \cap I_2$$

Therefore, $I_3 \subset V_\epsilon$. Denote the subdivision $\mathcal{S}^3_{[a_o, b_o]}$ as $\{a_o = x_o, x_1, x_2, \ldots, x_n = b_o\}$. Then,

$$I_3 = F[x_o, x_1](x_1 - x_o) + F[x_1, x_2](x_2 - x_1)$$
$$+ \cdots + F[x_{n-1}, x_n](x_n - x_{n-1}) \subset V_{\epsilon_1} \subset V_\epsilon$$

where V_{ϵ_1} is a symmetric neighborhood of I_3 with $\epsilon_1 \leqq \epsilon/2$. Let δ stand for any positive number less than min $(x_1 - x_o, x_n - x_{n-1})$. By continuity of F , continuity of subtraction, and continuity of multiplication, we can choose δ_o such that if $\delta < \delta_o$, the interval $F[x_o \pm \delta, x_1]\left(x_1 - (x_o \pm \delta)\right)$ differs from $F[x_o, x_1](x_1 - x_o)$ by an amount less than $\epsilon_1/2$. Similarly, $F[x_{n-1}, x_n \pm \delta](x_n \pm \delta - x_{n-1})$ differs from $F[x_{n-1}, x_n](x_n - x_{n-1})$ by an amount less than $\epsilon_1/2$ and, hence, the expression

$$F[x_o \pm \delta, x_1](x_1 - x_o) + F[x_1, x_2](x_2 - x_1) + \ldots$$
$$+ F[x_{n-2}, x_{n-1}](x_{n-2} - x_{n-1})$$
$$+ F[x_{n-1}, x_n \pm \delta](x_n \pm \delta - x_{n-1})$$

differs from the summation I_3, which is identical except for the first and last terms, by an amount less than $\epsilon_1/2 + \epsilon_1/2 = \epsilon_1$. Then if $[a, b] \in N_{\delta_0}[a_0, b_0]$, we have a subdivision

$$\bar{\mathcal{S}} = \bar{\mathcal{S}}_{[a\,,\,b]} = \{x_0 \pm \delta,\, x_1,\, x_2,\, \ldots,\, x_{n-1},\, x_n \pm \delta\}$$

such that

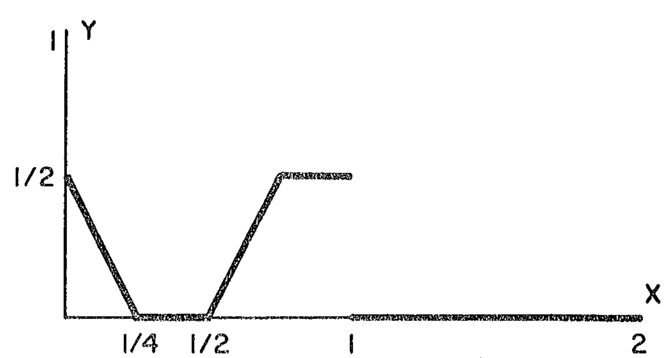
$$\Sigma\left(F, \bar{\mathcal{S}}_{[a\,,\,b]}\right) \subset V_{\epsilon_1} \subset V_\epsilon$$

Hence,

$$G[a\,,\,b] = \cap\left\{\left(F\,,\,\mathcal{S}_{[a\,,\,b]}\right)\right\} \subset V_\epsilon$$

and G is continuous at $[a_0\,,\,b_0]$. This completes the proof.

The following example shows that $A \subset B$ does not imply that $G(A) \subset G(B)$. Define $f\colon [0\,,\,2] \to B$ as indicated by the graph and $F\colon \mathcal{J} \to \mathcal{J}$ to be the united extension (Ref. 2, p. 552) of f.



For $\mathcal{S}^{1}_{[0,2]} = \{0, 1/4, 1/2, 1, 2\}$ and $B = [0, 2]$,

$$\Sigma\left(F, \mathcal{S}^{1}_{[0,2]}\right) = F[0,1/4](1/4) + F[1/4,1/2](1/4) + F[1/2, 1](1/2) + F[1,2](1)$$

$$= [0,1/2](1/4) + [0,0](1/4) + [0,1/2](1/2) + [1/2,1/2](1)$$

$$= [0,1/8] + [0,0] + [0,1/4] + [1/2,1/2]$$

$$= [1/2, 7/8]$$

Now for $A = [1/4, 1/2]$, $A \subset B$. But for $\mathcal{S}_{[1/4, 1/2]} = \{1/4, 1/2\}$,

$$\Sigma\left(F, \mathcal{S}_A\right) = F[1/4,1/2](1/2 - 1/4) = [0,0](1/4) = [0,0]$$

But every further subdivision of B will include the summation over $[1,2]$ and hence the summand $[1/2, 1/2]$, and hence have left end-point $\geqq 1/2$. Therefore, $G(B) = [1/2 + r, k]$ where $r \geqq 0$, and $G(A) = [0, 0]$ is not contained in $G(B)$.

However, if for every $X \subset A$ we have $F_1(X) \subset F_2(X)$, then

$$\int_A F_1 \subset \int_A F_2$$

Section 3
DEPENDENCE ON DEGENERATE INTERVALS

Theorem 2. Let F_2 and F_1 be continuous functions from $\mathcal{J}$ to $\mathcal{J}$ such that $M \subset N$ implies $F_i(M) \subset F_i(N)$, $(i = 1, 2)$. Let $G_2 = \int F_2$ and $G_1 = \int F_1$ and $F[x, x] = F_1[x, x]$ for all $x \epsilon A$. Then $G_2(A) = G_1(A)$.

Proof: It is sufficient to show that every neighborhood of $G_2(A)$ contains $G_1(A)$ and conversely. As the arguments are identical, only one of these will be shown.

Lemma 1. If $\{I_\alpha\}$ is a descending collection of closed intervals (i.e., for any two of them there is a third one contained in their intersection) and $I_o = \cap \{I_\alpha\}$, then every neighborhood of I_o contains some $I_{\alpha_o} \epsilon \{I_\alpha\}$.

Proof of Lemma 1: Let I_o be contained in $N_\epsilon = N_\epsilon(I_o)$, $I_o = [a, b]$. Assume that every I_α has left end-point $a_\alpha \leqq a_o - \epsilon$. Then $\dot{\bigcap\limits_\alpha} \{I_\alpha\}$ has left end point $\leqq a_o - \epsilon$; therefore, $a_o \leqq a_o - \epsilon$. This is a contradiction and the assumption is false. Hence, there exists I_{α_1} with left end-point $a_{\alpha_1} > a_o - \epsilon$. Similarly, there exists I_{α_2} with right end-point $b_{\alpha_2} < b_o + \epsilon$. Hence, $\left[a_{\alpha_1}, b_{\alpha_2}\right] \subset N_\epsilon$. But in this descending collection, there exists α_3 such that

$$I_{\alpha_3} \subset I_{\alpha_1} \cap I_{\alpha_2} \subset \left[a_{\alpha_1}, b_{\alpha_2}\right] \subset N_\epsilon$$

Lemma 2. Let $\mathcal{S}_A$ be a subdivision of $A = [a, b]$, and $\bar{\mathcal{S}}_A$ a subdivision of A including all the points of $\mathcal{S}_A$ and more. Then $\Sigma\left(F_2, \bar{\mathcal{S}}_A\right) \subset \Sigma\left(F_2, \mathcal{S}_A\right)$.

Proof: Let

$$\mathcal{S}_A = \{ a = a_o , a_1, \cdots, a_n = b \}$$

and let

$$\mathcal{S}_{\bar{A}} = \{ a = a_o, a_o^1, \cdots, a_o^{m_o}, a_1, a_1^1, \cdots, a_1^{m_o}, a_2, \cdots a_3, \cdots, a_n = b \}$$

Then

$$\begin{aligned} \Sigma(F_2, \mathcal{S}_A) = & F_2 [a_o, a_o^1] (a_o^1 - a_o) + \cdots + F_2 [a_o^{m_o}, a_1] (a_1 - a_o^{m_o}) \\ & + F_2 [a_1, a_1^1] (a_1^1 - a_1) + \cdots + F_2 [a_1^{m_1}, a_2] (a_2 - a_1^{m_1}) \\ & + \cdots \\ & + F_2 [a_{n-1}, a_{n-1}^1] (a_{n-1}^1 - a_{n-1}) + \cdots + F_2 [a_{n-1}^{m_{n-1}}, a_n] (a_m - a_{m-1}^{m_{n-1}}) \end{aligned}$$

Denote the lines in the above summation as $S_o^1, S_1^2, \cdots, S_{m-1}^m$ respectively. Now,

$$\Sigma(F_2, \mathcal{S}_A) = F_2 [a_o, a_1] (a_1 - a_o) + \cdots + F_2 [a_{n-1}, a_n] (a_n - a_{n-1})$$

Since $I_n \subset J_n$ implies that $\Sigma I_n \subset \Sigma J_n$, it will be sufficient to show that

$$S_o^1 \subset F_2 [a_o, a_1] (a_1 - a_o),\ S_1^2 \subset F_2 [a_1, a_2] (a_2 - a_1),$$

$$\ldots\ S_{n-1}^n \subset F_2 [a_{n-1}, a_n] (a_n - a_{n-1})$$

Consider the first term, S_2^1. We have

$$S_o^1 = F_2 [a_o, a_o^1] (a_o^1 - a_o) + \cdots + F_2 [a_o^{m_o}, a_1] (a_1 - a_o^{m_o})$$

$$\subset F_2 [a_o, a_1] (a_o^1 - a_o) + \cdots + F_2 [a_o, a_1] (a_1 - a_o^{m_o})$$

$$= F_2 [a_o, a_1] (-a_o + a_o^1 - a_o^1 + a_o^2 - a_o^2 + \cdots + a_1)$$

$$= F_2 [a_o, a_1] (a_1 - a_o)$$

The same argument holds for the general term and the lemma is proved.

Lemma 3. The term $\{\Sigma(F, \mathcal{S}_A)\}$ is a descending collection of closed intervals.

Proof: Let $\Sigma\left(F, \mathcal{S}_A^1\right)$ and $\Sigma\left(F, \mathcal{S}_A^2\right)$ be elements of the collection. Denote by $\mathcal{S}_A^3$ the collection of points in both $\mathcal{S}_A^1$ and $\mathcal{S}_A^2$. By lemma 2, $\Sigma\left(F, \mathcal{S}_A^3\right) \subset \Sigma\left(F, \mathcal{S}_A^1\right)$ and $\Sigma\left(F, \mathcal{S}_A^3\right) \subset \Sigma\left(F, \mathcal{S}_A^2\right)$. Hence $\Sigma\left(F, \mathcal{S}_A^3\right)$ is contained in their intersection.

Proof of Theorem 2: Let $\overline{N}_{\epsilon_o} = N_{\epsilon_o}\left(G_2(A)\right)$ be an ϵ_o neighborhood of $G_2(A)$. Each $\Sigma\left(F_2, \mathcal{S}_A\right)$ is a closed interval and lemma 2 states that $\{\Sigma(F_2, \mathcal{S}_A)\}$ is a descending collection. Hence we may apply lemma 1 to insure that there exists $\mathcal{S}_A^o$ such that $\Sigma\left(F_2, \mathcal{S}_A^o\right) \subset \overline{N}_{\epsilon_o}$. This is an interval $\Sigma\left(F_2, \mathcal{S}_A^o\right)$ contained in an open set $\overline{N}_{\epsilon_o}$, therefore there exists and $\epsilon > 0$ such that $\overline{N}_{\epsilon}\, \Sigma\left(\left(F_2, \mathcal{S}_A^o\right)\right) \subset \overline{N}_{\epsilon_o}$. Denote the subdivision $\mathcal{S}_A^o = \{a = x_o, x_1, x_2, \cdots, x_n = b\}$. Then

$$\overline{N}_{\epsilon_o} = N_{\epsilon_o}\left(G(A)\right) \supset \overline{N}_{\epsilon}\left[F_2\left[x_o, x_1\right](x_1 - x_o) + \cdots \right.$$

$$\left. + F_2\left[x_{m-1}, x_m\right](x_m - x_{m-1})\right] = \overline{N}_{\epsilon}\left[\Sigma\left(F_2, \mathcal{S}_A^o\right)\right]$$

Let $N^o = N^o_{\epsilon/n}$ be an ϵ/n neighborhood of the interval $F_2\left[x_o, x_1\right](x_1 - x_o)$. Assume that there is no subdivision $\mathcal{S}_{[x_o, x_1]}$ of $[x_o, x_1]$ such that $\Sigma\left(F_1, \mathcal{S}_{[x_o, x_1]}\right) \subset N^o$.

Denote

by $\mathcal{S}_1$ the subdivision $\{x_o, x_1\}$ of $[x_o, x_1]$,

by $\mathcal{S}_2$ the subdivision $\{x_o = y_2^o, y_2^1, y_2^2 = x_1\}$ into 2 equal parts,

by $\mathcal{S}_3$ the subdivision $\{x_o = y_3^o, y_3^1, y_3^2, y_3^3 = x_1\}$ into 3 equal parts,

... by $\mathcal{S}_p$ the subdivision $\{x_o = y_p^o, y_p^1, \cdots, y_p^p = x_1\}$ into p equal parts, etc.

Then for each p, $\Sigma(F_1, \mathcal{S}_p) \not\subset N^o$. Denote the right end-points of $F_2\left[x_o, x_1\right](x_1 - x_o) + \epsilon/n$ and of $\Sigma(F_1, \mathcal{S}_p)$ by r_o and r_p respectively, for $p > 0$. For a set A we will also use the notation $r(A)$ and $\ell(A)$ for the right and left end-points of A. Either there are infinitely many p for which $r_p \geqq r_o$, or else a similar statement holds for left end-points. The former case will be treated here. In this case there is a subsequence $\{\mathcal{S}_m\} \subset \{\mathcal{S}_p\}$ such that, for all m, $r_m \geqq r_o$. In the summation $\Sigma(F_1, \mathcal{S}_m)$ there are m intervals $\left[y_m^i, y_m^{i+1}\right]$ all of the same length. Denote by $[a_m, b_m]$ one of these for which the right end-point of $F_2\left[a_m, b_m\right]$ is maximum.

Then

$$r_o \leqq r\left[(F_1, \mathcal{S}_m)\right] = r\left(F_1\left[x_o, y_m^1\right]\left(y_m^1 - x_o\right) + F_1\left[y_m^1, y_m^2\right]\left(y_m^2 - y_m^1\right)\right.$$

$$\left. + \cdots + F_1\left[y_m^{m-1}, y_m^m = x_1\right]\left(y_m^m - y_m^{m-1}\right)\right)$$

$$\leqq r\left(F_1\left[a_m, b_m\right]\left(y_m^1 - x_o\right) + F_1\left[a_m, b_m\right]\left(y_m^2 - y_m^1\right)\right.$$

$$\left. + \cdots + F_1\left[a_m, b_m\right]\left(y_m^m - y_m^{m-1}\right)\right)$$

$$= r\left(F_1\left[a_m, b_m\right]\left(y_m^1 - x_o + y_m^2 - y_m^1 + \cdots + x_1 - y_m^{m-1}\right)\right)$$

$$= r\left(F_1\left[a_m, b_m\right](x_1 - x_o)\right)$$

Now some subsequence of $\left\{[a_m, b_m]\right\}$ converges to a point $[\bar{x}, \bar{x}]$ with $x_o \leq \bar{x} \leq x_1$. We will retain the notation so as to write $\left\{[a_m, b_m]\right\} \to [\bar{x}, \bar{x}]$ as $m \to \infty$. Since F_1 is continuous, $\left\{F_1 [a_m, b_m]\right\}$ converges to $F_1 [\bar{x}, \bar{x}]$. It follows that $\left\{r F_1 [a_m, b_m] (x_1 - x_o)\right\}$, since this is still a continuous function of $[a_m, b_m]$, converges to

$$r F_1 [\bar{x}, \bar{x}] (x_1 - x_o) = r F_2 [\bar{x}, \bar{x}] (x_1 - x_o)$$

$$\leqq r F_2 [x_o, x_1] (x_1 - x_o) = r_o - \epsilon/n$$

But every $r F_1 [a_m , b_m] (x_1 - x_o) \geqq r_o$; therefore,

$$\lim_{n \to \infty} F_1 [a_m , b_m] (x_1 - x_o) = r_1 \geqq r_o$$

We now have

$$r_o \leqq r_1 = \lim_{m \to \infty} r F_1 [a_m , b_m] (x_1 - x_o) \leqq r_o - \epsilon/m$$

This is a contradiction. Hence our assumption was false and there exists a subdivision $\mathcal{S}^1$ of $[x_o , x_1]$ such that

$$\Sigma\left(F_1 , \mathcal{S}^1\right) \subset N_{\epsilon/n} F_1 [x_o , x_1] (x_1 , x_o)$$

Similarly, there is a subdivision $\mathcal{S}^i$ of $[x_i , x_{i+1}]$ such that

$$\Sigma\left(F_1 , \mathcal{S}^i\right) \subset N_{\epsilon/n} F_1 [x_{i-1} , x_i] (x_i - x_{i-1})$$

for each $i = 1, 2, \cdots, n$. The set $\bar{\mathcal{S}}_A$ consisting of all the points in the union of the divisions

$$\mathcal{S}^1 \text{ of } [x_o , x_1] , \mathcal{S}^2 \text{ of } [x_1 , x_2] , \ldots , \mathcal{S}^m \text{ of } [x_{n-1}, x_n]$$

is a subdivision of $[x_o , x_n] = [a , b] = A$, and

$$
\begin{aligned}
G_1(A) &= G_1[a , b] \subset \Sigma(F_1 , \bar{\mathcal{S}}_A) \\
&= \sum_{i=1}^{n} \left(\Sigma\left(F_1 , \mathcal{S}^i\right)\right) \subset \sum_{i=1}^{n} N_{\epsilon/N}\left[\Sigma(F , \mathcal{S}^i)\right] \subset N_\epsilon\left[\sum_{i=1}^{n}\left(\Sigma(F , \mathcal{S}^i)\right)\right] \\
&= N_\epsilon\left[\Sigma(F , \bar{\mathcal{S}}_A)\right] \subset N_\epsilon\left[\Sigma\left(F , \mathcal{S}_A^o\right)\right] = \bar{N}_\epsilon = \bar{N}_{\epsilon_o}
\end{aligned}
$$

It follows immediately from theorem 2 that if F_1 and F_2 are continuous functions from $\mathcal{J}$ to $\mathcal{J}$ such that $M \subset N$ implies $F_i(M) \subset F_i(N)$, $(i = 1, 2)$ and $F_1[x , x] = F_2[x , x]$ for all $x \in A$, then for every interval $A' \subset A$ we have $G_1(A') = G_2(A')$ with $G_1(A') = \int_{A'} F_1$ and $G_2(A') = \int_{A'} F_2$. Accordingly, we have the following converse theorem.

<u>Theorem 3.</u> Suppose F_1 and F_2 are continuous functions from $\mathcal{J}$ to $\mathcal{J}$ such that $M \subset N$ implies $F_i(M) \subset F_i(N)$ $(i = 1, 2)$. Let $G_1 = \int F_1$ and $G_2 = \int F_2$. If for every interval $A' \subset A$ we have $G_1(A') = G_2(A')$ then $F_1[x , x] = F_2[x , x]$ for every $x \in A$.

<u>Proof</u>: Suppose the conclusion is false; i.e., that for some $x' \in A$ we have $F_1[x' , x'] \neq F_2[x' , x']$. It suffices to show that for some $A' \subset A$ such that $x' \in A'$ we have $G_1(A') \neq G_2(A')$. Denote by $r\,F_i(X)$ right end-point of $F_i(X)$ $(i = 1, 2)$. Without loss of generality we may assume that $r\,F_1[x' , x'] < r\,F_2[x' , x']$. The functions F_1 , F_2 , are continuous; hence, for some $A' \subset A$ such that $x' \in A'$, we have $A'' \subset A' \Rightarrow r\,F_1(A'') < r\,F_2(A'')$.

We assert that

$$r\, G_1(A') = r \int_{A'} F_1 < r\, G_2(A') = r \int_{A'} F_2$$

Otherwise (if $r\, G_1(A') \geq r\, G_2(A')$), for any $\epsilon > 0$ there is a subdivision $\mathcal{S}_2$ of A' such that $r\, G_2(A') > r \sum (F_2, \mathcal{S}_2) - \epsilon$ and a subdivision $\mathcal{S}_1$ of A' such that $r\, G_1(A') \leq r \sum (F_1, \mathcal{S}_1)$. Then let $\mathcal{S}_3$ be the subdivision of A' consisting of the points $x_o < x_1 < \cdots\ x_n$ in either $\mathcal{S}_1$ or $\mathcal{S}_2$ and we have $r \sum (F_1, \mathcal{S}_3) > r \sum (F_2, \mathcal{S}_3) - \epsilon$. But this inequality means that

$$r \{ F_1 [x_o, x_1] (x_1 - x_o) + \ldots + F_1 [x_{n-1}, x_n] (x_n - x_{n-1}) \}$$
$$> r \{ F_2 [x_o, x_1] (x_1 - x_o)$$
$$+ \ldots + F_2 [x_{n-1}, x_n] (x_n - x_{n-1}) \} - \epsilon$$

It follows from the above inequality that

$$0 \geq (x_1 - x_o) \left(r\, F_2 [x_o, x_1] - r\, F_1 [x_o, x_1] \right)$$
$$+ \ldots + (x_n - x_{n-1}) \left(r\, F_2 [x_{n-1}, x_n] - r\, F_1 [x_{n-1}, x_n] \right)$$

since ϵ is arbitrary. However, $[x_{i-1}, x_i] \subset A'$ for $i = 1, \cdots, n$, so $r\, F_2 [x_{i-1}, x_i] - r\, F_1 [x_{i-1}, x_i] > 0$ which is a contradiction. Therefore, $r\, G_1(A') < r\, G_2(A')$ and the proof is complete.

Theorem 2 and theorem 3 may be combined into the following statement:

Theorem (2-3): Let F_1, F_2 be continuous functions from $\mathcal{I}$ to $\mathcal{I}$ such that $M \subset N$ implies $F_i(M) \subset F_i(N)$ $(i = 1,2)$. Let $G_1 = \int F_1$ and $G_2 = \int F_2$. Then the following are equivalent:

(1) $F_1 [x, x] = F_2 [x, x]$ for all $x \in A$

(2) $G_1(A_1) = G_2(A_1)$ for all $A_1 \subset A$

Remark: Theorem 2 indicates that the integral is heavily dependent on the behavior of F on the special intervals $[x, x]$, and little dependent on the character of F on long intervals. It suggests that a related integral defined only in terms of intervals $[x, x]$ might be sought.

Section 4
REFERENCES

1. Lockheed Missiles and Space Division, Interval Analysis I, by R. E. Moore and C. T. Yang, LMSD-285875, Sunnyvale, Calif., Sep 1959

2. Wayman Strother, "Fixed Points, Fixed Sets, and M-Retracts," Duke Math. J., Vol. 22, No. 4, 1955, pp. 551–556

Appendix A
RELATION TO ORDINARY INTEGRATION

This appendix deals with the question raised in the remark following theorem 2.

Inasmuch as $\Sigma(F, \mathcal{S})$ is a descending collection of intervals, the left end-points form a nondecreasing sequence and, hence, the left end-point of the intersection $\bigcap_{\mathcal{S}} \{\Sigma(F, \mathcal{S})\} = \int_A F$ is the limit of the left end-points of the $\{\Sigma(F, \mathcal{S})\}$. In fact, if we start with $\mathcal{S}_1 = \{a , b\}$ and continue subdividing so that $\mathcal{S}_n$ consists of n equal parts, then the left end-point of $\int_A F$ is $\lim_{n\to\infty} \pi_1 \; \Sigma(F, \mathcal{S}_n)$, where π_1 is defined by $\pi_1(K)$ = the left end-point of K. Define for $F: \mathcal{J} \to \mathcal{J}$ a function $f_\ell: R_0 \to R$ by $f_\ell(x) = \pi_1 \; F[x, x]$, where R_0 is the largest interval in $\mathcal{J}$; i.e., $\mathcal{J}$ is the set of all subintervals of R_0. Define similarly π_2 and f_r for right end-points. Then we have

<u>Theorem 4:</u>

$$\int_{[a\,,\,b]} F = \left[\int_a^b f_\ell(x)\,dx \,, \int_a^b f_r(x)\,dx \right]$$

Proof:

$$\int_{[a\ ,\ b]} F = \bigcap_n \left\{\Sigma(F\ ,\ \mathcal{S}_n)\right\}$$

$$= \left[\lim_{n\to\infty} \Sigma\ \pi_1\ F\left[x_i\ ,\ x_{i+1}\right](x_{i+1} - x_i)\ ,\right.$$

$$\left.\lim_{n\to\infty} \Sigma\ \pi_2\ F\left[x_i\ ,\ x_{i+1}\right](x_{i+1} - x_i)\right]$$

$$= \left[\lim_{n\to\infty} \sum_{i=1}^{n} \pi_1\ F\left[x_i\ ,\ x_{i+1}\right]\left(\frac{1}{n}\right),\ \sum_{i=1}^{n} \pi_2\ F\left[x_i\ ,\ x_{i+1}\right]\left(\frac{1}{n}\right)\right]$$

$$= \left[\lim_{n\to\infty} \left(\frac{1}{n}\right)\sum_{i=1}^{n} \pi_1\ F\left[x_i\ ,\ x_{i+1}\right],\ \lim_{n\to\infty} \left(\frac{1}{n}\right)\sum_{i=1}^{n} \pi_2\ F\left[x_i\ ,\ x_{i+1}\right]\right]$$

Then

$$\int_a^b f_\ell(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(f_\ell(x_i)\right)(x_{i+1} - x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f_\ell(x_i)$$

and

$$\int_a^b f_r(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f_r(x_i)$$

Hence it is sufficient to prove that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \pi_1 F[x_i, x_{i+1}]$ equals $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f_\ell(x_i)$ and the similar result for $\pi_2 F$ and f_r. By definition of f_ℓ

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f_\ell(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \pi_1 F[x_i, x_i]$$

and, hence, what we need to prove is that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \pi_1 F[x_i, x_i] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \pi_1 F[x_i, x_{i+1}]$$

Since

$$F[x_i, x_i] \subset F[x_i, x_{i+1}]$$

then

$$\pi_1 F[x_i, x_i] \geqq \pi_1 F[x_i, x_{i+1}]$$

Hence, it is sufficient to prove that for every positive number ϵ there exists n_o such that $n > n_o$ implies

$$\left(\sum_{i=1}^{n} \frac{1}{n} \pi_1 F[x_i, x_i] - \sum_{i=1}^{n} \frac{1}{n} \pi_1 F[x_i, x_{i+1}]\right) < \epsilon$$

Then

$$\sum_{i=1}^{n} \frac{1}{n} \pi_1 F [x_i , x_i] - \sum_{i=1}^{n} \frac{1}{n} \pi_1 F [x_i , x_{i+1}]$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} \pi_1 F [x_i , x_i] - \sum_{i=1}^{n} \pi_1 F [x_i , x_{i+1}]\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\pi_1 F [x_i , x_i] - \pi_i F [x_i , x_{i+1}]\right)$$

But π_1 F uniformly continuous implies that there exists δ_ϵ such that $|x_i - x_{i+1}| < \delta_\epsilon$ implies that

$$\left(\pi_1 F [x_i , x_i] - \pi_1 F [x_i , x_{i+1}]\right) < \epsilon$$

There exists n_o such that $n > n_o$ implies $1/n < \delta_\epsilon$. Then $n > n_o$ implies

$$|x_i - x_{i+1}| < \delta_\epsilon \text{ and } \pi_1 F [x_i , x_i] - \pi_1 F [x_i , x_{i+1}] < \epsilon$$

Then $n > n_o$ implies

$$1/n = |x_i - x_{i+1}| < \delta_\epsilon \text{ and } \pi_1 F [x_i , x_i] - \pi_1 F [x_i , x_{i+1}] < \epsilon$$

Hence,

$$n > n_o \Rightarrow \frac{1}{n}\left[\sum_{i=1}^{n}\left(\pi_1 F[x_i , x_i] - \pi_1 F[x_i , x_{i+1}]\right)\right] < \frac{1}{n}\left[\sum_{i=1}^{n} \epsilon\right] = \frac{1}{n}(n\,\epsilon) = \epsilon$$

A similar statement holds for right end-points and the theorem is proved.

It follows trivially that if $F[x , x]$ is real for all x, the left and right end-points of $F[x , x]$ are identical and, hence,

$$\int_{[a , b]} F = \left[\int_a^b f_\ell(x)\, dx , \int_a^b f_\ell(x)\, dx\right]$$

i.e., $\int_{[a , b]} F$ reduces to ordinary integration.

Theorem 4 suggests a more general definition for $\int F$ may be feasible – namely the right side of the equality in theorem 4. This conclusion could lead to a deletion of the condition $A \subset B \Rightarrow F(A) \subset F(B)$. A definition of a derivative of F as the anti-integral involving f_ℓ and f_r may also prove interesting.