Instructions: This portion of the exam is open-book, open notes; it is recommended that you have the Kincaid / Cheney and Gill / Murray / Wright texts, as well as your notes with you. Make sure to organize and check your work.

1. Compute the condition number of \( f(x) = \sin(x) \) near \( x = 10^6 \pi + \pi/4 \).

2. Show that the dual of

\[
\min x_1 - 2x_2 + x_3
\]

subject to

\[
x_1 + x_2 + x_3 = 3,
0 \leq x_1 \leq 1,
x_2 = 1,
x_3 \text{ arbitrary}
\]

is

\[
\max 3y_1 - y_2 + y_3
\]

subject to

\[
\begin{align*}
y_1 - y_2 + r_1 &= 1 \\
y_1 + y_3 &= -2 \\
y_1 &= 1
\end{align*}
\]

(Hint: You must show that the Lagrange multipliers of any solution of the primal problem form a solution to the dual problem. This involves showing that these Lagrange multipliers are both feasible for the dual and make the dual objective function optimal. Let \( y_1, y_2, y_3, \) and \( r_1 \) be the Lagrange multipliers corresponding to the constraints \( x_1 + x_2 + x_3 = 3, -x_1 \geq -1, x_2 = 1, \) and \( x_1 \geq 0, \) respectively. You can then follow the general argument on pp. 75–76 of Gill / Murray / Wright.)
3. If $A$ is an $m$ by $n$ matrix, with $m \gg n$, then we may wish to find a least squares solution to the (usually) overdetermined system of equations $Ax = b$. A common technique is the following algorithm.

(i) Form a QR factorization, i.e., write $A = QR$, where $Q^TQ = I$ and $R$ is right triangular.
(ii) form $w = Q^Tb$,
(iii) For $i = n$ to 1 by $-1$:
$$x_i \leftarrow \frac{w_i - \sum_{j=i+1}^{n} r_{i,j} x_j}{r_{i,i}}.$$ 

Note that $Q$ in the above algorithm is $m$ by $m$; in certain instances, $m$ may be so large that it is undesirable to store $Q$. An alternative is to factor $A$ in the form $A = LQ$, where $L$ is left or lower triangular, and $Q$ is a (different) $n$ by $n$ orthogonal matrix (i.e. $Q^TQ = I$). In such instances, only $Q$ and $L^T L$ are stored. Write an algorithm as above, assuming you are able to obtain $Q$ and $L^T L$. (Hint: In this case, you may need to work with a more ill-conditioned system.)

4. Discretize
$$u_{xx} = -u^2 \quad u(0) = u(1) = 0$$

with a finite element method with basis functions

$$\phi_1(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq .25 \\ 1 - 4(x - .25) & \text{if } .25 \leq x \leq .5 \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_2(x) = \begin{cases} 4(x - .25) & \text{if } .25 \leq x \leq .5 \\ 1 - 4(x - .5) & \text{if } .5 \leq x \leq .75 \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_3(x) = \begin{cases} 4(x - .5) & \text{if } .5 \leq x \leq .75 \\ 1 - 4(x - .75) & \text{if } .75 \leq x \leq 1 \\ 0 & \text{otherwise}. \end{cases}$$

Contrast the system that you get with that you get when you discretize a linear PDE. How might you approximate the solution(s) to the discretized system?