A Short Note on the Convexity of Interval Matrix-Vector Products *

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Abstract

We investigate the question under which circumstances the pointwise interval matrix-vector product $\boldsymbol{A}\boldsymbol{x} := \{Ax \mid A \in \boldsymbol{A}, x \in \boldsymbol{x}\}$ of a real interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m,n}$ and a real interval vector $\boldsymbol{x} \in \mathbb{IR}^n$ is convex.

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1 Introduction

Let m, n, k be positive natural numbers. For $(m \times n)$-matrices $A, B \in \mathbb{R}^{m,n}$ comparison is defined componentwise, i.e., $A \leq B$ means $A_{i,j} \leq B_{i,j}$ for all $i = 1, ..., n$ and $j = 1, ..., m$. For $\underline{A}, \overline{A} \in \mathbb{R}^{m,n}$ with $\underline{A} \leq \overline{A}$ the set $\boldsymbol{A} = \{A \in \mathbb{R}^{m,n} \mid \underline{A} \leq A \leq \overline{A}\}$ is called an *interval matrix*. We denote the set of all such interval matrices by $\mathbb{IR}^{m,n}$. Elements of $\mathbb{IR}^m := \mathbb{IR}^{m,1}$ are called *interval vectors*. For $\boldsymbol{A} \in \mathbb{IR}^{m,n}$ and $\boldsymbol{B} \in \mathbb{IR}^{n,k}$ the *pointwise product*

$$\boldsymbol{A}\boldsymbol{B} := \{AB \mid A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$$

is in general neither an interval matrix nor convex, cf. [2] Section 3.1, [1].

Kelsey[3] asked for conditions under which $\boldsymbol{A}\boldsymbol{B}$ is convex. Here, the case $k = 1$ where $\boldsymbol{B} = \boldsymbol{x} \in \mathbb{IR}^n$ becomes an interval vector and $\boldsymbol{A}\boldsymbol{x}$ a pointwise interval matrix-vector product is of special interest; this is the subject of this note. In that case

$$\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n} \boldsymbol{x}(i)\boldsymbol{A}(:, i) \quad (1)$$

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is the Minkowski sum[1] of the sets $\boldsymbol{x}(i)\boldsymbol{A}(:,i) \subseteq \mathbb{R}^m$, $i = 1,...,n$, where $\boldsymbol{x}(i) \in \mathbb{IR}$ denotes the i-th component of $\boldsymbol{x}$ and $\boldsymbol{A}(:,i) \in \mathbb{IR}^m$ the i-th column of $\boldsymbol{A}$ in MATLAB notation. In (1) the inclusion from left to right is clear. For the opposite inclusion take $b := \sum_{i=1}^n x_i a_i \in \sum_{i=1}^n \boldsymbol{x}(i)\boldsymbol{A}(:,i)$ where $x_i \in \boldsymbol{x}(i)$ and $a_i \in \boldsymbol{A}(i,:)$. Then, $x := (x_1, \dots, x_n)^T \in \boldsymbol{x}$ and $A := [a_1, ..., a_n] \in \boldsymbol{A}$ so that $b = Ax \in \boldsymbol{Ax}$.

If all $\boldsymbol{x}(i)\boldsymbol{A}(:,i)$ in (1) are convex, then clearly $\boldsymbol{Ax}$ is convex as well since the Minkowski sum of convex sets is convex. For example, the former holds true if all component intervals $\boldsymbol{x}(i)$ do not contain zero in their interior, i.e., if the interval vector $\boldsymbol{x}$ is contained in one orthant of $\mathbb{R}^n$, cf. [1] Corollary 2.4. But this is only a sufficient and not a necessary condition ensuring that all $\boldsymbol{x}(i)\boldsymbol{A}(:,i)$ are convex. A complete characterization of the special situation where all addends in (1) are convex is given in the following lemma which will be proved in the next section:

Lemma 1.1 *Let $\boldsymbol{u} = [\underline{u}, \overline{u}] \in \mathbb{IR}$ be an interval and let $\boldsymbol{v} \in \mathbb{IR}^m$, $m \in \mathbb{N}$, be an interval vector, then $\boldsymbol{uv}$ is convex if, and only if, at least one of the following conditions holds true:*

(a) *$\boldsymbol{v}$ is a point-vector.*[2]

(b) *$\boldsymbol{v}$ is a line segment on an axis, i.e., $\boldsymbol{v}$ has at most one nonzero component.*

(c) *$0 \notin \mathring{\boldsymbol{u}}$, i.e., $\boldsymbol{u}$ does not contain zero in its interior.*

(d) *$0 \in \mathring{\boldsymbol{u}}$, $0_m \in \boldsymbol{v}$, and at least one of the following conditions holds true:*

  (i) *$\alpha\boldsymbol{v} \subseteq \beta\boldsymbol{v}$ where $\alpha, \beta \in \{\underline{u}, \overline{u}\}$ satisfy $\alpha \neq \beta$, $|\alpha| = \min|\boldsymbol{u}|$, $|\beta| = \max|\boldsymbol{u}|$.*

  (ii) *$\boldsymbol{u} = -\boldsymbol{u}$ and $\exists i \in \{1, \dots, m\} \forall j \in \{1, \dots, m\} \backslash \{i\} : \boldsymbol{v}_j = -\boldsymbol{v}_j$.*

Thus, if $\boldsymbol{A} \in \mathbb{IR}^{m,n}$ is an interval matrix and $\boldsymbol{x} \in \mathbb{IR}^m$ is an interval vector such that for all $i \in \{1,...,n\}$, $\boldsymbol{u} := \boldsymbol{x}(i)$ and $\boldsymbol{v} := A(:,i)$ fulfill one of the conditions (a)-(d)*, then the pointwise interval matrix-vector product $\boldsymbol{Ax}$ is convex.*

The question arises if $\boldsymbol{Ax}$ can be convex if some or even all $\boldsymbol{x}(i)\boldsymbol{A}(:,i)$ are not. The answer is yes, and in the next section we will give simple illustrative examples for that. According to these examples it seems not very promising to search for reasonable necessary conditions for $\boldsymbol{Ax}$ being convex.

Before proceeding we want to say clearly that all results and examples are completely elementary and of school level mathematics. Still we consider them as interesting and worth mentioning.

2 Examples and Proofs

Before we prove Lemma 1.1 we illustrate in Figures 1-6 in dimension $m = 2$ mainly all different shapes that the product $\boldsymbol{uv}$ of an interval $\boldsymbol{u} \in \mathbb{IR}$ and an interval vector $\boldsymbol{v} \in \mathbb{IR}^2$ may have. First note that if $\boldsymbol{v}$ is a point-vector, or if it is contained in an axis, or if $\boldsymbol{v}$ is an axis parallel line segment and $\boldsymbol{u}$ is a point-interval, then $\boldsymbol{uv}$ is a line segment which possibly might consist of a single point if $\boldsymbol{v}$ is a point-vector and $\boldsymbol{u}$ is a point-interval. These situations might be considered as trivial. In all other cases $\boldsymbol{uv}$ contains a nonzero area. Figures 1-6 characterize such non-trivial cases.

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[1] For subsets X, Y of a vector space V the pointwise sum $X + Y := \{x + y \mid x \in X, y \in Y\}$ is called *Minkowski sum* of X and Y. Recall that for real vector spaces $X + Y$ is convex if X and Y are convex.

[2] An interval vector $\boldsymbol{v} = [\underline{v}, \overline{v}] \in \mathbb{IR}^m$ is called a *point-vector* if $\underline{v} = \overline{v} =: v$. In this case $\boldsymbol{v}$ is identified with $v \in \mathbb{R}^m$. Analogously *point-matrices* are defined.

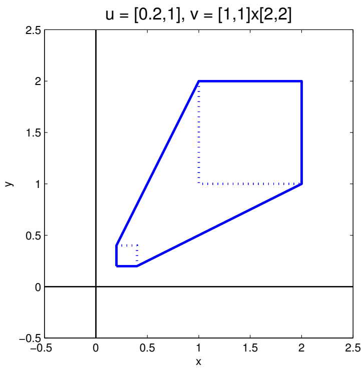

Figure 1: cone shape

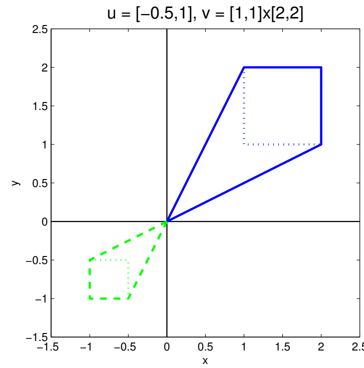

Figure 2: double cone shape

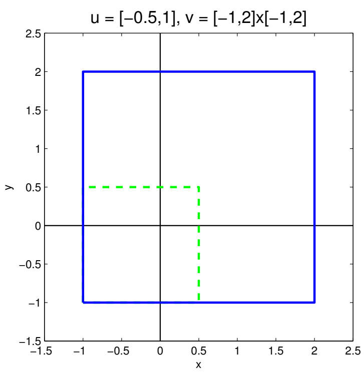

Figure 3: nested boxes

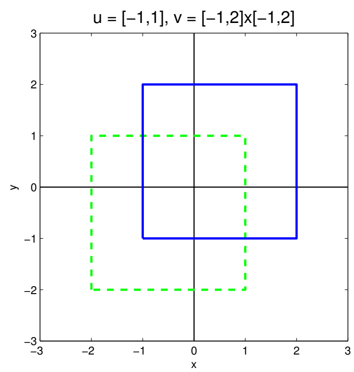

Figure 4: crossing boxes

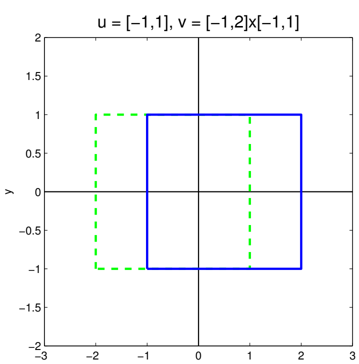

Figure 5: supplementing boxes

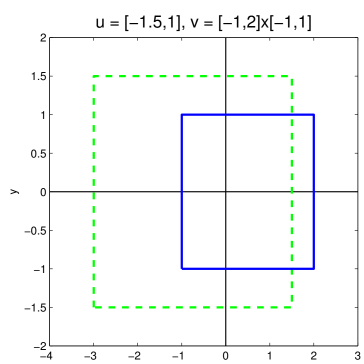

Figure 6: shrink-crossing boxes

A cone shape like in Figure 1 typically occurs if $\boldsymbol{v}$ does not contain the origin and zero is not an inner point of $\boldsymbol{u}$. If zero is an inner point of $\boldsymbol{u}$, then a double cone shape like in Figure 2 appears. If $\boldsymbol{v}$ contains the origin and if zero is not an inner point of $\boldsymbol{u} = [\underline{u}, \overline{u}]$, then

$$\boldsymbol{uv} = \begin{cases} \underline{u}\boldsymbol{v} & \text{if } |\underline{u}| > |\overline{u}|, \\ \overline{u}\boldsymbol{v} & \text{else,} \end{cases}$$

is again a box, namely the scaling of $\boldsymbol{v}$ by the interval bound of $\boldsymbol{u}$ which is of largest absolute value. If $\boldsymbol{v}$ contains the origin and if zero is an interior point of $\boldsymbol{u} = [\underline{u}, \overline{u}]$, then $\boldsymbol{uv} = \underline{u}\boldsymbol{v} \cup \overline{u}\boldsymbol{v}$ is the union of the two boxes $B_1 := \underline{u}\boldsymbol{v}$ and $B_2 := \overline{u}\boldsymbol{v}$. In Figures 3-6 B_1 is plotted with dashed and B_2 with solid boundary. The shapes shown in Figures 1,3, and 5 are convex, and those of Figures 2,4, and 6 are not. In principle these are the shapes that can occur for the product of an interval $\boldsymbol{u}$ and an interval vector $\boldsymbol{v}$. According to (1) pointwise interval matrix-vector products are Minkowski sums of these shapes. The following example E1 shows that non-convex shapes can add up to a convex one, see the first row of Figure 7. Example E2 shows that also a non-convex and a convex shape can add up to a convex one, see the second row of Figure 7.

E1) $\boldsymbol{A} := \begin{bmatrix} [-1,1] & [1,2] \\ [1,2] & [-1,1] \end{bmatrix}, \ \boldsymbol{x} := \begin{bmatrix} [-1,1] \\ [-1,1] \end{bmatrix} \ \boldsymbol{Ax} = \begin{bmatrix} [-3,3] \\ [-3,3] \end{bmatrix}$

E2) $\boldsymbol{A} := \begin{bmatrix} [-1,1] & [-1,1] \\ [1,2] & [-2,2] \end{bmatrix}, \ \boldsymbol{x} := \begin{bmatrix} [-1,1] \\ [-1,1] \end{bmatrix} \ \boldsymbol{Ax} = \begin{bmatrix} [-2,2] \\ [-4,4] \end{bmatrix}$

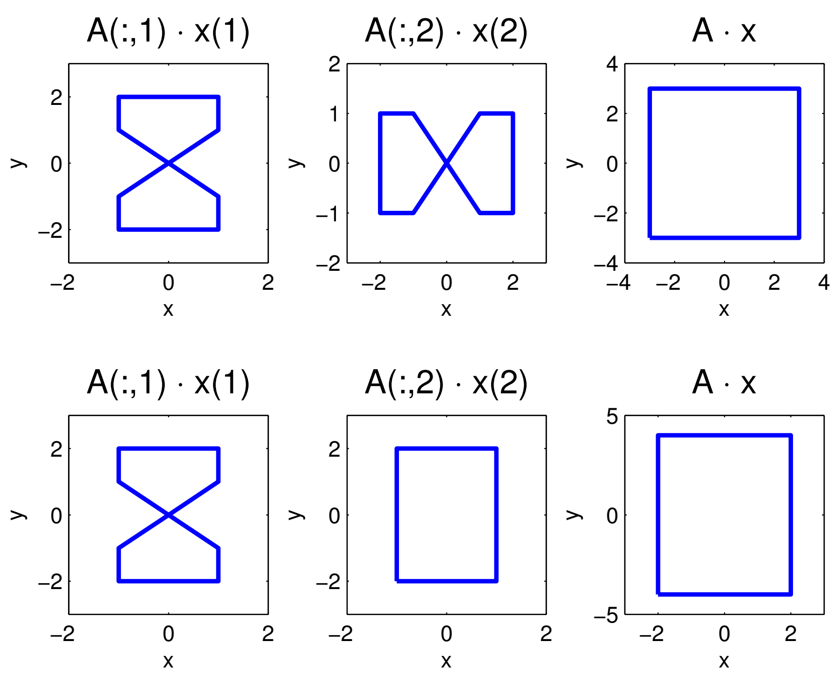

Figure 7: Examples E1 and E2

Remark 2.1 *Let $\boldsymbol{u} = [\underline{u}, \overline{u}] \in \mathbb{IR}$ be an interval containing zero in its interior and let $\boldsymbol{v} \in \mathbb{IR}^m$ be an interval vector which is not a point vector. Then, $\boldsymbol{w} := \boldsymbol{u}\boldsymbol{v}$ is convex if, and only if, $\boldsymbol{w}$ is an interval vector. In this case $\boldsymbol{v}$ is either a line segment on an axis or contains the origin.*

Proof: Suppose that $\boldsymbol{w}$ is convex. We have $\boldsymbol{w} = Q_1 \cup Q_2$ for convex $Q_1 := [\underline{u}, 0]\boldsymbol{v}$ and $Q_2 := [0, \overline{u}]\boldsymbol{v}$. Clearly, Q_1 is an interval vector if, and only if $Q_2 = (\overline{u}/\underline{u}) \cdot Q_1$ is an interval vector. Thus, if Q_1 and Q_2 are interval vectors, i.e., axis parallel rectangular solids, then $\boldsymbol{w}$ must also be an axis parallel rectangular solid in order to be convex. If Q_1 and Q_2 are not interval vectors, then $\boldsymbol{v}$ does not contain the origin and is also not a line segment on an axis. Therefore, Q_1 and Q_2 build a double cone C in a vicinity of the origin. The double cone C is not degenerated to a line because $\boldsymbol{v}$ is neither a point vector nor a line segment on an axis. Hence, C is not convex, a contradiction. This argument also proves the statement in the final sentence of the assertion. □

Remark 2.2 *Let $\boldsymbol{u} = [\underline{u}, \overline{u}] \in \mathbb{IR}$ be an interval containing zero in its interior and let $\boldsymbol{v} \in \mathbb{IR}^m$ be an interval vector containing the origin. Suppose that $\boldsymbol{v}$ is not contained in an axis, i.e., has at least two nonzero components. Then, $\boldsymbol{w} := \boldsymbol{u}\boldsymbol{v}$ is an interval vector if, and only if, at least one of the following conditions holds true:*

(i) *$\alpha\boldsymbol{v} \subseteq \beta\boldsymbol{v}$ where $\alpha, \beta \in \{\underline{u}, \overline{u}\}$ satisfy $\alpha \neq \beta$, $|\alpha| = \min|\boldsymbol{u}|$, and $|\beta| = \max|\boldsymbol{u}|$.*

(ii) *$\boldsymbol{u} = -\boldsymbol{u}$ and $\exists i \in \{1, \ldots, m\} \forall j \in \{1, \ldots, m\} \backslash \{i\} : \boldsymbol{v}_j = -\boldsymbol{v}_j$.*

Proof: Since $\boldsymbol{v}$ contains the origin we have

$$\boldsymbol{w} \quad = \quad [\underline{u}, 0]\boldsymbol{v} \cup [0, \overline{u}]\boldsymbol{v} = \underline{u}\boldsymbol{v} \cup \overline{u}\boldsymbol{v} = \alpha\boldsymbol{v} \cup \beta\boldsymbol{v}. \tag{2}$$

"$\Leftarrow$". If (i) holds true, then by (2) $\boldsymbol{w} = \beta\boldsymbol{v}$ is an interval vector. If (ii) holds true, then without loss of generality we may assume that $i = 1$. Set $u := \overline{u} = -\underline{u}$ so that (2), (ii), and $0 \in (\boldsymbol{v}_1 \cap -\boldsymbol{v}_1)$ imply

$$\begin{aligned} \boldsymbol{w} \quad &= \quad -u\boldsymbol{v} \cup u\boldsymbol{v} = u(-\boldsymbol{v} \cup \boldsymbol{v}) = u((-\boldsymbol{v}_1 \cup \boldsymbol{v}_1) \times \boldsymbol{v}_2 \times \cdots \times \boldsymbol{v}_m) \\ &= \quad u([-\max|\boldsymbol{v}_1|, \max|\boldsymbol{v}_1|] \times \boldsymbol{v}_2 \times \cdots \times \boldsymbol{v}_m) \in \mathbb{IR}^m. \end{aligned}$$

"$\Rightarrow$". Suppose that $\boldsymbol{w}$ is an interval vector. Then,

$$\boldsymbol{w} \quad = \quad \boldsymbol{u}\boldsymbol{v}_1 \times \cdots \times \boldsymbol{u}\boldsymbol{v}_m = (\alpha\boldsymbol{v}_1 \cup \beta\boldsymbol{v}_1) \times \cdots \times (\alpha\boldsymbol{v}_m \cup \beta\boldsymbol{v}_m). \tag{3}$$

Now, suppose that (i) does not hold true. Then, there is an $i \in \{1, \ldots, m\}$ such that

$$\alpha\boldsymbol{v}_i \nsubseteq \beta\boldsymbol{v}_i. \tag{4}$$

Take $j \in \{1, ..., m\} \backslash \{i\}$ such that $\boldsymbol{v}_j \neq [0, 0]$. Note that by assumption at least one such j exists and since $0 \in \boldsymbol{v}_j$ necessarily $\operatorname{rad}(\boldsymbol{v}_j) > 0$ [3]. From (2), (3), and (4) it follows that

$$\emptyset \neq (\alpha\boldsymbol{v}_i \backslash \beta\boldsymbol{v}_i) \times \beta\boldsymbol{v}_j \subseteq \alpha(\boldsymbol{v}_i \times \boldsymbol{v}_j)$$

so that $\beta\boldsymbol{v}_j \subseteq \alpha\boldsymbol{v}_j$. Since $|\beta| \geq |\alpha|$ and $\operatorname{rad}(\boldsymbol{v}_j) > 0$, this yields $|\alpha| = |\beta|$, $\alpha = -\beta$, and hence also $\boldsymbol{v}_j = -\boldsymbol{v}_j$. Thus, (ii) holds true. □

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[3]For an interval $\boldsymbol{v} = [\underline{v}, \overline{v}]$, $\operatorname{rad}(v) := (\overline{v} - \underline{v})/2$ denotes the *radius* of $\boldsymbol{v}$.

Now, Lemma 1.1 follows easily from Remark 2.1 and Remark 2.2: First, suppose that $\boldsymbol{uv}$ is convex and that conditions (a)-(c) of Lemma 1.1 do not hold true. Then, $0 \in \mathring{u}$ and according to Remark 2.1 $\boldsymbol{uv}$ is an interval vector containing the origin. Thus, by Remark 2.2 condition (d) is fulfilled.

Next suppose that one of the conditions (a)-(d) hold true. If (a) or (b) holds true, then $\boldsymbol{uv}$ is a point or a line segment and therefore convex. If (c) holds true, then $\boldsymbol{uv}$ has convex cone shape. If (d) holds true and if (a) and (b) do not hold true, then Remark 2.2 says that $\boldsymbol{uv}$ is convex. This finishes the proof of Lemma 1.1.

References

[1] Bünger, F.: A note on the boundary shape of matrix polytope products, *Reliable Computing*, 20:73–88, 2014.

[2] Neumaier, A.: *Interval Methods for Systems of Equations*, Cambridge University Press, 1990.

[3] Kelsey, R.: private communication, September 5th, 2015.