Tolerable Solution Set for Interval Linear Systems with Constraints on Coefficients*†

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Abstract

We develop a technique for computing tolerable solution set for interval linear systems with convex polyhedral ties upon its coefficients. The tolerable solution set in the above problem is proved to be the intersection of finite number of hyperstripes, i.e. a solution set to a finite system of two-sided nonstrict linear inequalities. For some specific ties on the coefficients, we propose simplified variants of the general method.

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Interval linear algebraic system of equations (shortly ILAS) of the form $\boldsymbol{A}x = \boldsymbol{b}$ with an interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and an interval right-hand side $\boldsymbol{b} \in \mathbb{IR}^{m}$ is a family of point systems of linear algebraic equations of the same form $Ax = b$, in which the matrix A passes through the interval matrix $\boldsymbol{A}$ and the right-hand side vector b passes through the interval vector $\boldsymbol{b}$.

For the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, various solution sets can be defined [10], and the most popular of them are united solution set and tolerable solution set. We are going to consider *tolerable solution set*, TSS in brief. It is denoted as $\varXi_{tol}(\boldsymbol{A}, \boldsymbol{b})$ and consists of all such vectors x that, for any coefficient matrix A from the interval matrix $\boldsymbol{A}$, the product Ax falls into the interval $\boldsymbol{b}$:

$$\varXi_{tol}(\boldsymbol{A}, \boldsymbol{b}) := \left\{ x \in \mathbb{R}^n \;\middle|\; (\forall A \in \boldsymbol{A})\,(Ax \in \boldsymbol{b}) \right\}. \tag{1}$$

A *tie on the coefficients* of ILAS is meant to be a set of $m\times n$-matrices that imposes an additional constraint on the entries of A from $\boldsymbol{A}$. The tie will be denoted as $\mathcal{G}$. The

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set $\mathcal{G}$ may be described in various ways. In the situations where the misunderstanding cannot occur, we refer to any description of the set $\mathcal{G}$ as the *tie on the coefficients* too.

In our work, we restrict ourselves only to the ties that are convex polyhedrons, and we call them *convex polyhedral ties.*

Interval linear system of algebraic equations $\boldsymbol{A}x = \boldsymbol{b}$ with the tie $\mathcal{G}$ on the coefficients is a family of the equations $Ax = b$ in which the coefficient matrix passes through the set $\boldsymbol{A} \cap \mathcal{G}$ and the right-hand side passes through the interval $\boldsymbol{b}$.

Tolerable solution set for the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with the coefficients tie $\mathcal{G}$ is defined to be the set

$$\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) := \begin{cases} \varnothing, & \text{if } \boldsymbol{A} \cap \mathcal{G} = \varnothing, \\ \big\{\, x \in \mathbb{R}^n \;\big|\; (\forall S \in (\boldsymbol{A} \cap \mathcal{G}))(Sx \in \boldsymbol{b}) \big\}, & \text{otherwise.} \end{cases} \tag{2}$$

Interval linear systems of equations with dependent parameters have been studied in a good deal of papers. See, for example, [1, 2, 3, 5, 6, 7, 8, 9, 11, 12] and bibliography to them. But so far only united solution set was the object under study. Our work considers tolerable solution set.

The structure of the paper is as follows: Section 1 presents notation and necessary preliminaries, Section 2 is devoted to derivation of the technique for computing the tolerable solution set to interval linear systems with convex polyhedral ties on the coefficients, and, in Sections 3–4, simplified versions of the general method for particular forms of ties are developed.

1 Notation and necessary preliminaries

In this paper, starting from Introduction, we follow the notation proposed in [4]. In particular, $\mathbb{IR} := \{[\underline{z}, \overline{z}] \mid \underline{z}, \overline{z} \in \mathbb{R},\ \underline{z} \le \overline{z}\}$ is the set of intervals over the real axis, $^*\mathbb{IR} := \{[\underline{z}, \overline{z}] \mid \underline{z} \in \mathbb{R} \cup \{-\infty\},\ \overline{z} \in \mathbb{R} \cup \{\infty\},\ \underline{z} \le \overline{z}\}$ is the set of extended intervals.

Boldface letters are used to designate intervals, interval vectors and matrices; for example, $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ is an interval $m\times n$-matrix. Calligraphic letters are used to designate sets; for example, $\mathcal{S} \subset \mathbb{R}^{m\times n}$ is a subset in the set of all the rectangular point $m\times n$-matrices. The lower index denotes projection of a set onto a coordinate subspace; for example, if $\mathcal{S} \subset \mathbb{R}^{m\times n}$ then $\mathcal{S}_{i:} = \{(S_{i1}, \ldots, S_{in}) \mid S \in \mathcal{S}\}$ is an orthogonal projection of the set $\mathcal{S}$ onto the coordinate subspace of the i-th row (i.e. the set of possible values of the i-th row for the matrices from the set $\mathcal{S}$).

The symbol "$\odot$" will stand for memberwise multiplication of the sets; as an example, for $\mathcal{S} \subset \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, $\mathcal{S} \odot x = \{Sx \mid S \in \mathcal{S}\}$ means the set of all the products Sx in which the point matrix S is from $\mathcal{S}$. The symbol "$\bigotimes$" will denote usual direct (Cartesian) product of the sets; in particular, for the set $\mathcal{S} \subset \mathbb{IR}^n$, the equality $\mathcal{S} = \bigotimes_j \mathcal{S}_j$ means that the set $\mathcal{S}$ coincides with the direct product of its projections onto the coordinate axes.

The set from $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}$) is called *convex polyhedron* if it can be described by a finite system of nonstrict linear inequalities. Intersection of finite number of convex polyhedrons is a convex polyhedron. Bounded convex polyhedron is called *convex polytope.*

Vertex of a convex polyhedron is its point such that there does not exist a straight line segment within the set for which the point under consideration is an interior one. The vertices set of a convex polytope $\mathcal{V}$ will be denoted by $\operatorname{vert}\mathcal{V}$. This set is finite. Furthermore, any convex polytope is the convex hull of its vertices set: $\mathcal{V} = \operatorname{conv}(\operatorname{vert}\mathcal{V})$.

The inclusion

$$c^\top x \in \boldsymbol{d}, \qquad \text{where } c, x \in \mathbb{R}^n,\ \boldsymbol{d} \in \mathbb{IR}, \tag{3}$$

or, put it differently, two-sided nonstrict linear inequality of the form

$$\underline{d} \le c^\top x \le \overline{d}, \quad \text{where } c, x \in \mathbb{R}^n,\ \underline{d}, \overline{d} \in \mathbb{R},\ \underline{d} \le \overline{d},$$

will be referred to as *elementary linear inclusion*. *Hyperstripe* is defined as solution set to an elementary linear inclusion with respect to x.

The inclusion of the form

$$Cx \in \boldsymbol{d}, \qquad \text{where } \ x \in \mathbb{R}^n,\ C \in \mathbb{R}^{m\times n},\ \boldsymbol{d} \in \mathbb{IR}^m,$$

will be called *matrix linear inclusion*.

For vectors $x \in \mathbb{R}^n$, the relationship of the form

$$c^\top x \in \boldsymbol{d}, \qquad \text{where } c \in \mathbb{R}^n,\ \boldsymbol{d} \in {}^*\mathbb{IR},$$

will be referred to as *linear constraint*. It is an elementary linear inclusion in case of $\boldsymbol{d} \in \mathbb{IR}$, and it is a nonstrict linear inequality in case of $\boldsymbol{d} = [d, \infty]$ or $\boldsymbol{d} = [-\infty, d]$ with $d \in \mathbb{R}$.

2 Method for computing TSS for ILAS with convex polyhedral tie

2.1 Reduction to solution of inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$, where $\mathcal{V}$ is a convex polytope

We turn to determination of the tolerable solution set of an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with the tie $\mathcal{G}$ on its coefficients. First, we reformulate the definition of the solution set so as to get rid of the quantifier prefix in (2). We arrive at

$$\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \begin{cases} \varnothing, & \text{if } \boldsymbol{A} \cap \mathcal{G} = \varnothing, \\ \left\{ x \in \mathbb{R}^n \,\middle|\, (\boldsymbol{A} \cap \mathcal{G}) \odot x \subseteq \boldsymbol{b} \right\}, & \text{otherwise.} \end{cases} \tag{4}$$

Let us consider the intersection of $\boldsymbol{A}$ and $\mathcal{G}$. The tie $\mathcal{G}$ is taken only as a convex polyhedron. The interval matrix $\boldsymbol{A}$ is a convex polytope. Hence, the intersection $\boldsymbol{A} \cap \mathcal{G}$ is a convex polytope too. We denote it through $\mathcal{V}$.

Therefore, to compute TSS of the system $\boldsymbol{A}x = \boldsymbol{b}$ with a convex polyhedral tie $\mathcal{G}$ on the coefficients we have to develop a technique that solves, with respect to x, the inclusion

$$\mathcal{V} \odot x \subseteq \boldsymbol{b}, \quad \text{where } \mathcal{V} \text{ is a convex polytope in } \mathbb{R}^{m\times n}.$$

2.2 Properties of inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$

In this subsection, we consider properties of the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$, and they help us to construct the algorithm for its solution later. The first property is valid for an arbitrary set of matrices from $\mathbb{R}^{m\times n}$, not only for convex polytopes.

Property 1 *Let $\mathbb{S} \subseteq \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{IR}^m$. The inclusion*

$$\mathbb{S} \odot x \subseteq \boldsymbol{b} \tag{5}$$

and the system of inclusions

$$\mathop{\&}_{i} \; (\mathbb{S}_{i:} \odot x \subseteq \boldsymbol{b}_i) \tag{6}$$

mean the systems of the same elementary linear inclusions. But the number of occurrences of each elementary linear inclusion in the system corresponding to (6) *is not greater than that in the system corresponding to* (5)*.*

▷ Proof. Using the definition of the operation "$\odot$", we rewrite the inclusion $\mathbb{S} \odot x \subseteq \boldsymbol{b}$ in the form $\mathop{\&}_{S\in\mathbb{S}} (Sx \in \boldsymbol{b})$. For the interval vector $\boldsymbol{b}$, any matrix inclusion $Sx \in \boldsymbol{b}$ is nothing but a system of elementary linear inclusions $\mathop{\&}_{i} (S_{i:}x \in \boldsymbol{b}_i)$. Therefore, $\mathbb{S} \odot x \subseteq \boldsymbol{b}$ is a concise form of

$$\mathop{\&}_{S\in\mathbb{S}} \mathop{\&}_{i} (S_{i:}x \in \boldsymbol{b}_i). \tag{7}$$

Let us rearrange the elementary linear inclusions in (7):

$$\mathop{\&}_{i} \mathop{\&}_{S\in\mathbb{S}} (S_{i:}x \in \boldsymbol{b}_i). \tag{8}$$

In (8), we change each block $\mathop{\&}_{S\in\mathbb{S}} (S_{i:}x \in \boldsymbol{b}_i)$ to $\mathop{\&}_{S_{i:}\in\mathbb{S}_{i:}} (S_{i:}x \in \boldsymbol{b}_i)$. This way, we eliminate the repetitions of the elementary linear inclusions that may take place due to the fact that different matrices S from the set $\mathbb{S}$ can have identical rows. We arrive at

$$\mathop{\&}_{i} \mathop{\&}_{S_{i:}\in\mathbb{S}_{i:}} (S_{i:}x \in \boldsymbol{b}_i). \tag{9}$$

Making use of the operation "$\odot$", we can represent (9) in a short form:

$$\mathop{\&}_{i} (\mathbb{S}_{i:} \odot x \subseteq \boldsymbol{b}_i).$$

◁

The sense of Property 1 is that the reduction from (5) to (6) eliminates obvious repetitions of elementary linear inclusions.

Property 2 *Let $\mathcal{V} \subseteq \mathbb{R}^{m\times n}$ be a convex polytope, $x \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{IR}^m$. Then*

$$\mathcal{V} \odot x \subseteq \boldsymbol{b} \iff (\operatorname{vert}\mathcal{V}) \odot x \subseteq \boldsymbol{b}. \tag{10}$$

▷ Proof.

$\Longrightarrow$) It is evident, since $\operatorname{vert}\mathcal{V} \subseteq \mathcal{V}$.

$\Longleftarrow$) The set $\boldsymbol{b}$ is convex, therefore, for every its subset, the result of taking the convex hull remains within the set $\boldsymbol{b}$:

$$(\operatorname{vert}\mathcal{V}) \odot x \subseteq \boldsymbol{b} \implies \operatorname{conv}\big((\operatorname{vert}\mathcal{V}) \odot x\big) \subseteq \boldsymbol{b}.$$

Multiplying by the vector x is a linear mapping. It transforms any convex combination of finite number of points to a combination of images of these points with the same coefficients. Therefore,

$$\big(\operatorname{conv}(\operatorname{vert}\mathcal{V})\big) \odot x \subseteq \operatorname{conv}\big((\operatorname{vert}\mathcal{V}) \odot x\big).$$

It remains to make use of the fact that the convex polytope $\mathcal{V}$ is the convex hull of its vertices set: $\operatorname{conv}(\operatorname{vert}\mathcal{V}) = \mathcal{V}$. ◁

Property 2 allows one to change the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$, which usually denotes an infinite system of elementary linear inclusions, by a **finite** system of elementary linear inclusions $(\operatorname{vert}\mathcal{V}) \odot x \subseteq \boldsymbol{b}$. We make this transition by deleting, from the system, those elementary linear inclusions that are implications of the rest of inclusions.

Proposition 1 *Let $\mathcal{V} \subset \mathbb{R}^{m\times n}$ be a convex polytope, $\boldsymbol{b} \in \mathbb{IR}^m$, $x \in \mathbb{R}^n$. Then*

$$\mathcal{V} \odot x \subseteq \boldsymbol{b} \iff \mathop{\&}_{i} \Big(\big(\operatorname{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big). \tag{11}$$

▷ <u>Proof</u>. Applying Property 1, we rewrite the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$ in the form

$$\mathop{\&}_{i} \; (\mathcal{V}_{i:} \odot x \subseteq \boldsymbol{b}_i).$$

The projections $\mathcal{V}_{i:}$, $i = 1, \ldots, m$, of the convex polytope $\mathcal{V}$ are convex polytopes by themselves. Therefore, Property 2 holds true for them:

$$\mathcal{V}_{i:} \odot x \subseteq \boldsymbol{b}_i \iff \big(\operatorname{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i.$$

◁

Both Proposition 1 and Property 2 result in a finite system of elementary linear inclusions that is equivalent to the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$. Let us compare these systems, writing them out in similar forms, when, in each system, the rows with the right-hand side $\boldsymbol{b}_i$ are separated to the i-th block:

	from	original form	block form
System 1	(10)	$(\operatorname{vert}\mathcal{V}) \odot x \subseteq \boldsymbol{b}$	$\mathop{\&}_{i} \Big(\mathop{\&}_{V \in \operatorname{vert}\mathcal{V}} V_{i:}x \subseteq \boldsymbol{b}_i \Big)$
System 2	(11)	$\mathop{\&}_{i} \Big(\big(\operatorname{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big)$	$\mathop{\&}_{i} \Big(\mathop{\&}_{v \in \operatorname{vert}(\mathcal{V}_{i:})} vx \subseteq \boldsymbol{b}_i \Big)$

In such a block form, when all the elementary linear inclusions are explicitly written out, it is fairly simple to realize (see Fig. 1) that System 2 is a subsystem of System 1.

Indeed, each row of the coefficients of the i-th block in System 1 corresponds to the i:-th projection $V_{i:}$ of a vertex V of the polytope $\mathcal{V}$, while the rows of the coefficients of the i-th block in System 2 correspond only to the vertices v of the projection $\mathcal{V}_{i:}$ of the polytope $\mathcal{V}$.

Notice that Proposition 1 is very convenient for computation of TSS to interval linear systems without ties. Eliminating the quantifier prefixes in the definition (1), we can conclude that $\varXi_{tol}(\boldsymbol{A}, \boldsymbol{b})$ coincides with the solution set of the inclusion $\boldsymbol{A} \odot x \subseteq \boldsymbol{b}$. The interval matrix $\boldsymbol{A}$ is a convex polytope, therefore, by Proposition 1, $\varXi_{tol}(\boldsymbol{A}, \boldsymbol{b})$ is described by a finite system of elementary linear inclusions

$$\mathop{\&}_{i} \Big(\big(\operatorname{vert}(\boldsymbol{A}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big).$$

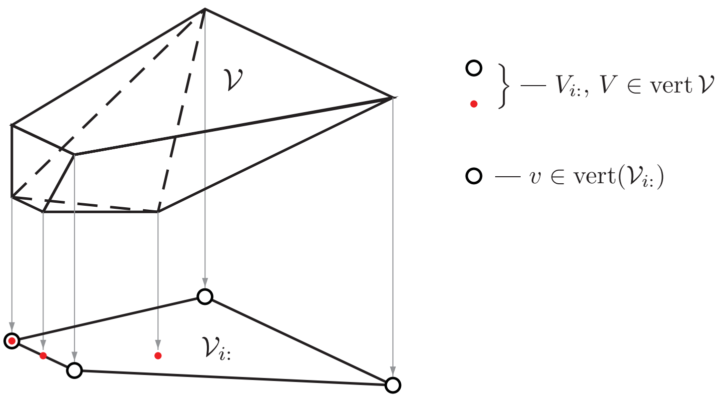

Figure 1: How projection of vertices set relates to vertices set of projection

2.3 A technique for computing TSS

Let us return from the properties of the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$ to computation of the tolerable solution set $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$ of the system $\boldsymbol{A}x = \boldsymbol{b}$ with a convex polyhedral tie $\mathcal{G}$ imposed on the coefficients of the system. Proposition 1 proved in Section 2.2 completes the following chain of arguments:

1. First, in Section 2.1, we transformed the definition (2) to the equality (4). The formula (4) means that if $\boldsymbol{A} \cap \mathcal{G} = \varnothing$ then $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$ is also empty, and if $\boldsymbol{A} \cap \mathcal{G} \neq \varnothing$ then $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$ coincides with the solution set of the inclusion $(\boldsymbol{A} \cap \mathcal{G}) \odot x \subseteq \boldsymbol{b}$ with respect to the unknown x.

2. Further, in Section 2.1, we pointed out that, for convex polyhedron $\mathcal{G}$, the intersection $\boldsymbol{A} \cap \mathcal{G}$ is a convex polytope.

3. Finally, relying on Proposition 1, we can change the inclusion $(\boldsymbol{A} \cap \mathcal{G}) \odot x \subseteq \boldsymbol{b}$, where $\boldsymbol{A} \cap \mathcal{G}$ is a convex polytope, by a finite system of elementary linear inclusions

$$\mathop{\&}_{i} \Big(\big(\operatorname{vert}((\boldsymbol{A} \cap \mathcal{G})_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big).$$

The above chain of arguments

- proves that the tolerable solution set for the interval system $\boldsymbol{A}x = \boldsymbol{b}$ with a convex polyhedral coefficients tie $\mathcal{G}$ is the intersection of finite number of hyper-stripes,

- and substantiates the following technique for finding this set.

A method for computing tolerable solution set
to interval linear system of equations $\boldsymbol{A}x = \boldsymbol{b}$
with convex polyhedral tie $\mathcal{G}$ on the coefficients:

Step I.	Find the sets $\operatorname{vert}\big((\boldsymbol{A} \cap \mathcal{G})_{i:}\big)$, $i = 1, \dots, m$. If at least one of them is empty, then $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \varnothing$. If all the sets are nonempty, go to Step II.	(12)
Step II.	Produce a description of the solution set to a finite system of elementary linear inclusions $\underset{i}{\&} \Big(\big(\operatorname{vert}((\boldsymbol{A} \cap \mathcal{G})_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big)$ in a necessary form.	

A few words about Step II. The complexity of execution of Step II depends on what is meant as 'nesessary form'. Possible variants:

- exact description of the solution set
  - optimal inequality system
  - set of vertices and directions of polyhedron
- estimate of the solution set
  - shape of the estimate (point, ellipse, interval)
  - disposition with respect to the estimated set (outer, inner, nearest in a certain metric)

So, we have a lot of options. But the problems of transition from the finite system of elementray linear inclusions to each variant of the nesessary form of the solution set are well known. They are considered, for instance, in identification theory of linear systems. Additionally, the problems of computing various interval estimates are treated in interval analysis when evaluating the united and tolerable solution sets of ILAS (in case of point coefficient matrix). This is why we do not consider Step II in detail and concentrate all our efforts on Step I only.

Let us agree that "we have found the set $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$" if Step I of the above method is fulfilled.

To carry out Step I of the method (12), the following plan of actions looks reasonable:

1. Find a description of the convex polyhedron $\mathcal{G}$ in the form of a finite system of linear inequalities.
2. Add to this system the inequalities that represent membership in the interval matrix $\boldsymbol{A}$.
3. Find the vertices set of the polytope $\boldsymbol{A} \cap \mathcal{G}$ from the inequalities system obtained.
4. If the resulting set is empty, then $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \varnothing$.
5. If the vertices set of the polytope $\boldsymbol{A} \cap \mathcal{G}$ is not empty, then project this set onto the coordinate subspaces of separate rows and delete excess points from the projections.

Although quite complicated, this way is sufficiently universal. Specific complexity of execution of Step I heavily depends on both the geometric properties and description of the set $\mathcal{G}$.

Taking into account particular features of the tie, we can develop simpler ways of executing Step I of the method (12) for specific problems. In Sections 3–4, we consider two special cases of the tie $\mathcal{G}$ and elaborate quite elementary algorithms for Step I of (12) that have polynomial complexity.

3 Case 1: row-independent linear constraints

Characteristics of the convex polyhedral tie $\mathcal{G}$:

1. The set $\mathcal{G}$ is defined as a finite system of linear constraints:

$$G \in \mathcal{G} \quad \Longleftrightarrow \quad \mathop{\&}_{k=1,\dots,q} \left(\sum_{i,j} C_{kij} G_{ij} \in \boldsymbol{d}_k \right), \tag{13}$$

   where $q \in \mathbb{N}$, $C_{kij} \in \mathbb{R}$, $\boldsymbol{d}_k \in {}^*\mathbb{IR}$.

2. Each entry of the matrix G can occur with a nonzero coefficient in only one of the linear constraints of the system (13), i.e.

$$C_{lij} \neq 0 \quad \Longrightarrow \quad \big(\forall k \in \{1,\dots,q\} \setminus \{l\}\big)\ \big(C_{kij} = 0\big).$$

3. No two entries from one row of the matrix G can get into a linear constraint of the system (13) with nonzero coefficients, i.e.

$$C_{kir} \neq 0 \quad \Longrightarrow \quad \big(\forall j \in \{1,\dots,n\} \setminus \{r\}\big)\ \big(C_{kij} = 0\big).$$

In particular, the sets of symmetric and skew-symmetric matrices can be represented in the form corresponding to Case 1. For example, for square $n \times n$-matrices the description of the skew-symmetric matrices that matches Case 1 looks as follows:

$$G_{ij} + G_{ji} = 0, \quad i = 1,\dots,n, \quad j = i+1,\dots,n.$$

How the tie affects Step I of the method (12).

We denote the set of all index pairs (ij) for the entries of the matrix G by Idx:

$$\mathrm{Idx} = \{(ij) \mid i = 1,\dots,m,\ j = 1,\dots,n\}.$$

Additionally, Idx_k will stand for the pairs of indices (ij) for which G_{ij} occurs with a nonzero coefficient in the k-th constraint:

$$\mathrm{Idx}_k = \{(ij) \in \mathrm{Idx} \mid C_{ijk} \neq 0\}.$$

The intersection of the sets $\boldsymbol{A}$ and $\mathcal{G}$ will be denoted as $\mathcal{S}$.

By virtue of the first condition on the set $\mathcal{G}$, the matrices S from the set $\mathcal{S}$ are described by the system of linear constraints

$$\begin{cases} \mathop{\&}\limits_{k=1,\dots,q} \left(\sum\limits_{(ij)\in\mathrm{Idx}_k} C_{kij} S_{ij} \in \boldsymbol{d}_k \right), \\ \mathop{\&}\limits_{(ij)\in\mathrm{Idx}} \ (S_{ij} \in \boldsymbol{A}_{ij}). \end{cases}$$

Using the second condition on the set $\mathcal{G}$, we split up the system to the blocks so as different blocks do not have common variables:

$$\begin{cases} \underset{k=1,\dots,q}{\&} \begin{cases} \sum\limits_{(ij)\in \mathrm{Idx}_k} C_{kij} S_{ij} \in \boldsymbol{d}_k, \\ S_{ij} \in \boldsymbol{A}_{ij}, \ \text{where } (ij) \in \mathrm{Idx}_k; \end{cases} \\ \underset{(ij)\in \mathrm{Idx}\setminus(\bigcup_k \mathrm{Idx}_k)}{\&} (S_{ij} \in \boldsymbol{A}_{ij}). \end{cases} \tag{14}$$

In the above formula, a separate block corresponds to each index k. Additionally, a separate block that consists of one constraint $S_{ij} \in \boldsymbol{A}_{ij}$ corresponds to each pair of indices (ij) for which $(\forall k)(C_{kij} = 0)$. So, the values of the variables S_{ij} entering into one block do not depend on the values of the variables from the other blocks.

Using the third condition on the set $\mathcal{G}$, we conclude that the entries of one row of the matrix S cannot occur in the same block. Therefore, for every i, the variables S_{ij}, $j = 1, \dots, n$, are not interdependent, wich implies

$$(\forall i) \ \Big(\mathcal{S}_{i:} = \bigotimes_j \mathcal{S}_{ij} \Big).$$

Let us define the set $\mathcal{S}_{ij}$ from the unique block of the system (14) that contains the variable S_{ij}.

If $(ij) \in \mathrm{Idx}_k$, then the only block that contains the variable S_{ij} has the form

$$\begin{aligned} &\sum_{(lr)\in \mathrm{Idx}_k} C_{klr} S_{lr} \in \boldsymbol{d}_k, \\ &S_{lr} \in \boldsymbol{A}_{lr}, \ \text{where } (lr) \in \mathrm{Idx}_k \,. \end{aligned} \tag{15}$$

We rewrite (15) taking out the variable S_{ij}:

$$S_{ij} \in \Bigg(\boldsymbol{d}_k - \sum_{(lr)\,\in\, \mathrm{Idx}_k \setminus \{(ij)\}} C_{klr} S_{lr} \Bigg) \Bigg/ C_{kij}, \tag{16a}$$

$$S_{lr} \in \boldsymbol{A}_{lr}, \ \text{where } (lr) \in \mathrm{Idx}_k \setminus \{(ij)\}, \tag{16b}$$

$$S_{ij} \in \boldsymbol{A}_{ij}. \tag{16c}$$

Bearing in mind the special form of the constraint (16c), we obtain

$$\mathcal{S}_{ij} = \boldsymbol{A}_{ij} \cap \widetilde{\mathcal{S}}_{ij}, \tag{17}$$

where $\widetilde{\mathcal{S}}_{ij}$ is the ij-th projection of the solution set to the system (16a)&(16b).

The constraints (16a) and (16b) allow us to think of the set $\widetilde{\mathcal{S}}_{ij}$ as the range of the multivalued function

$$\Bigg(\boldsymbol{d}_k - \sum_{(lr)\,\in\, \mathrm{Idx}_k \setminus \{(ij)\}} C_{klr} S_{lr} \Bigg) \Bigg/ C_{kij}$$

over the interval $\bigotimes\limits_{(lr)\in \mathrm{Idx}_k \setminus \{(ij)\}} \boldsymbol{A}_{lr}$. Such a viewpoint on $\widetilde{\mathcal{S}}_{ij}$ makes it clear that

$$\widetilde{\mathcal{S}}_{ij} = \bigcup_{\substack{S_{lr} \in \boldsymbol{A}_{lr}, \\ (lr)\in \mathrm{Idx}_k \setminus \{(ij)\}}} \Bigg(\boldsymbol{d}_k - \sum_{(lr)\,\in\, \mathrm{Idx}_k \setminus \{(ij)\}} C_{klr} S_{lr} \Bigg) \Bigg/ C_{kij}.$$

Taking the union makes no difficulty, since the function under study is defined by a rational expression with only one occurrence of every variable. We get

$$\widetilde{\mathbb{S}}_{ij} = \left(\boldsymbol{d}_k - \sum_{(lr)\,\in\,\mathrm{Idx}_k\,\setminus\{(ij)\}} C_{klr}\boldsymbol{A}_{lr}\right) \Big/ \, C_{kij}. \tag{18}$$

(The same result can be obtained by a more complicated way, having carried out all the transformations of Fourier method that eliminates the unknowns S_{lr}, $(lr) \neq (ij)$, from the system (16a)&(16b).)

Hence, for $(ij) \in \mathrm{Idx}_k$, the set $\mathbb{S}_{ij}$ is determined from (17) and (18).

If $(ij) \in \mathrm{Idx} \setminus \left(\bigcup_k \mathrm{Idx}_k\right)$, then the only block that contains the variable S_{ij} has the form $S_{ij} \in \boldsymbol{A}_{ij}$, and so $\mathbb{S}_{ij} = \boldsymbol{A}_{ij}$. Overall, we have

$$\mathbb{S}_{ij} = \begin{cases} \boldsymbol{A}_{ij} \bigcap \left(\left(\boldsymbol{d}_k - \sum\limits_{(lr)\,\in\,\mathrm{Idx}_k\,\setminus\{(ij)\}} C_{klr}\boldsymbol{A}_{lr}\right) \Big/ \, C_{kij}\right) & \text{for } (ij) \in \mathrm{Idx}_k, \\ \boldsymbol{A}_{ij} & \text{for } (ij) \in \mathrm{Idx} \setminus \left(\bigcup\limits_k \mathrm{Idx}_k\right). \end{cases} \tag{19}$$

If the set $\mathbb{S}_{ij}$ is not empty, then it is an interval.

We have demonstrated that, for the convex polyhedral tie $\mathcal{G}$ corresponding to Case 1, the set $(\boldsymbol{A} \cap \mathcal{G})_{i:}$ is a direct product of the sets $\mathbb{S}_{ij}$ computed by the formula (19). Therefore, Step I of the method (12) for Case 1 looks as follows:

1. Find the sets $\mathbb{S}_{ij}$, $(ij) \in \mathrm{Idx}$, according to the formula (19).
2. If any one of the sets obtained is empty, then $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \varnothing$.

   If all the sets $\mathbb{S}_{ij}$, $(ij) \in \mathrm{Idx}$, are nonempty, then, for every index i, find $\mathrm{vert}\,((\boldsymbol{A} \cap \mathcal{G})_{i:})$ as the vertices set of the interval vector $\bigotimes_j \mathbb{S}_{ij}$ and go to Step II of the method (12).

4 Case 2: row-independent proportional parametric tie

Characteristics of the convex polyhedral tie $\mathcal{G}$:

1. The set $\mathcal{G}$ is determined in parametric form

$$\mathcal{G} = \bigcup_{p \in \mathbb{R}^q} G(p), \quad \text{where } p \text{ is a parameter vector,}$$

   and, additionally, each element $G_{ij}(p)$ of the matrix $G(p)$ is proportional to one of the parameters:

$$G_{ij}(p) = c_{ij} p_{k(i,j)}, \quad c_{ij} \in \mathbb{R} \setminus \{0\}, \quad p_{k(i,j)} \in \{p_1, \dots, p_q\}.$$

2. No two entries of the matrix $G(p)$ that are proportional to one of the parameters belong to one row, i.e. for every $k = 1, \dots, q$ the implication

$$(il) \in \mathrm{Idx}_k \quad \Longrightarrow \quad \big(\forall j \in \{1, \dots, n\} \setminus \{l\}\big)\ \big((ij) \notin \mathrm{Idx}_k\big)$$

   holds, where $\mathrm{Idx}_k := \{(ij) \mid G_{ij}(p) = c_{ij} p_k\}$ is the set of index pairs for such entries of the matrix $G(p)$ that are proportional to the parameter p_k.

The set $\mathcal{G}$ is evidently a linear subspace in $\mathbb{R}^{m\times n}$.

In particular, the following types of matrices can be represented in the form that corresponds to Case 2: symmetric matrices, skew-symmetric matrices, circulant matrices, Hankel matrices, Hurwitz matrices, Toeplitz matrices. For instance, one of possible parametrizations of skew-symmetric matrices that meets Case 2 has the form

$$G(p) = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ -p_2 & p_{n+1} & p_{n+2} & \cdots & p_{2n-1} \\ -p_3 & -p_{n+2} & p_{2n} & \cdots & p_{3n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_n & -p_{2n-1} & -p_{3n-3} & \cdots & p_{n(n+1)/2} \end{pmatrix}, \quad p_1, \dots, p_{n(n+1)/2} \in \mathbb{R}.$$

<u>How the tie affects Step I of the method (12)</u>.

If the tie $\mathcal{G}$ meets the first condition of Case 2, then the set $\mathcal{S} := \boldsymbol{A} \cap \mathcal{G}$ consists of all such matrices S that

$$\begin{cases} \mathop{\&}\limits_{k=1,\dots,q} \big(S_{ij} = c_{ij} p_k, \text{ for } (ij) \in \mathrm{Idx}_k\big), \\ S_{ij} \in \boldsymbol{A}_{ij}, \text{ for } i = 1, \dots, m, \ j = 1, \dots, n. \end{cases} \tag{20}$$

We rewrite (20) as a system of q blocks:

$$\mathop{\&}\limits_{k=1,\dots,q} \begin{cases} S_{ij} = c_{ij} p_k, & \text{for } (ij) \in \mathrm{Idx}_k, \\ S_{ij} \in \boldsymbol{A}_{ij}, & \text{for } (ij) \in \mathrm{Idx}_k\,. \end{cases} \tag{21}$$

In (21), no two blocks have common variables, so the values of the variables S_{ij}, $(ij) \in \mathrm{Idx}_k$, from the k-th block do not depend on the values of the variables from the other blocks.

Due to the second requirement on the set $\mathcal{G}$, the entries of one row of the matrix S cannot occur in one block. Hence, for every i, the variables S_{ij}, $j = 1, \dots, n$, are independent from each other, i.e. $(\forall i)\ \big(\mathcal{S}_{i:} = \bigotimes_j \mathcal{S}_{ij}\big)$.

Let us find the set $\mathcal{S}_{ij}$ from the only block of the system (21) that contains the variable S_{ij}. For $(ij) \in \mathrm{Idx}_k$, this is the block

$$\begin{cases} S_{lr} = c_{lr} p_k, & \text{for } (lr) \in \mathrm{Idx}_k, \\ S_{lr} \in \boldsymbol{A}_{lr}, & \text{for } (lr) \in \mathrm{Idx}_k\,. \end{cases}$$

After obvious transformations, it can be reduced to the form

$$\begin{cases} S_{lr} = c_{lr} p_k, & \text{for } (lr) \in \mathrm{Idx}_k, \\ c_{lr} p_k \in \boldsymbol{A}_{lr}, & \text{for } (lr) \in \mathrm{Idx}_k\,. \end{cases}$$

For $p_k \in \mathbb{R}$, the equalities $S_{lr} = c_{lr} p_k$, $(lr) \in \mathrm{Idx}_k$, describe, in the space of the variables S_{lr}, $(lr) \in \mathrm{Idx}_k$, a straight line that goes through the origin of coordinates. As far as all the constants c_{lr}, $(lr) \in \mathrm{Idx}_k$, differ from zero, this straight line is not orthogonal to any of the coordinate axes. Every inclusion $c_{lr} p_k \in \boldsymbol{A}_{lr}$ with $c_{lr} \neq 0$ bounds the range of values of the parameter p_k by the interval $\boldsymbol{A}_{lr}/c_{lr}$. Taking into account the effect of all such constraints, we conclude that the range of values of the parameter p_k is equal to $\bigcap\limits_{(lr) \in \mathrm{Idx}_k} \boldsymbol{A}_{lr}/c_{lr}$. It is either empty or an interval in $\mathbb{IR}$.

Therefore,

$$\begin{cases} S_{lr} = c_{lr} p_k, & \text{for } (lr) \in \mathrm{Idx}_k, \\ p_k \in \bigcap\limits_{(lr)\in \mathrm{Idx}_k} \boldsymbol{A}_{lr}/c_{lr}. \end{cases}$$

As the result, the range of values of the variable S_{ij} that satisfy the above system can be found by the formula

$$\mathcal{S}_{ij} = c_{ij} \odot \left(\bigcap_{(lr)\in \mathrm{Idx}_k} \boldsymbol{A}_{lr}/c_{lr} \right). \tag{22}$$

We have demonstrated that Step I of the method (12) can be executed as follows:

1. Find the sets $\mathcal{S}_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$, according to the formula (22).
2. If any one of the sets obtained is empty, then $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \varnothing$.
   If all the sets obtained are nonempty, then, for every index i, determine vert $((\boldsymbol{A} \cap \mathcal{G})_{i:})$ as the vertices set of the interval vector $\bigotimes_j \mathcal{S}_{ij}$ and go to Step II of the method (12).

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