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Rank of convex combinations of matrices

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Let A and B be complex rectangular matrices of the same rank r. We characterize the property that all convex combinations of A and B are of rank r. Moreover, for A and B of full rank, some conditions for the matrix set $r(A, B)$ ($c(A, B)$, resp.) whose rows (columns, resp.) are independent convex combinations of the rows (columns, resp.) of A and B are also proposed.

Ранг выпуклых комбинаций матриц

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Пусть A и B — комплексные прямоугольные матрицы одного и того же ранга r. Доказывается, что все выпуклые комбинации A и B также имеют ранг r. Кроме того, для матриц A и B полного ранга выводятся некоторые свойства множества матриц $r(A, B)$ (и, соответственно, $c(A, B)$), чьи строки (или столбцы) являются независимыми выпуклыми комбинациями строк (или столбцов) матриц A и B.

1. Introduction

Let A and B be m-by-n complex matrices, $m \geq n$, such that $\operatorname{rank}(A) = \operatorname{rank}(B) = r \leq n$.

Consider the matrix sets:

$$\begin{aligned} h(A,B) &= \{C\colon\ C = \alpha A + (1-\alpha)B,\ \alpha \in [0,1]\}, \\ r(A,B) &= \{C\colon\ C = TA + (I-T)B\}, \qquad \text{and} \\ c(A,B) &= \{C\colon\ C = AS + B(I-S)\} \end{aligned}$$

where T and S are diagonal m-by-m and n-by-n real matrices, respectively with diagonal entries from $[0, 1]$.

Our goal is to characterize the above sets with respect to the inheritance of rank r, which is meant that each matrix from these sets is of rank r.

It should be noted that, for square and nonsingular A and B, nonsingularity of our sets has been studied in [2].

We shall close this section with an extract of results from [2] which are basic for our considerations.

Theorem A (Johnson, Tsatsomeros [2]). *Let A and B be n-by-n complex nonsingular matrices.*

(a) *$h(A, B)$ is nonsingular iff the matrix BA^{-1} has no negative eigenvalues.*
(b) *If BA^{-1} is a P-matrix, i.e., if all the principal minors of BA^{-1} are positive, then $r(A, B)$ is nonsingular.*
(c) *If $B^{-1}A$ is a P-matrix then $c(A, B)$ is nonsingular.*

2. Full rank sets

Our first result characterizes $h(A, B)$ with respect to the inheritance of full rank of A and B.

Theorem 1. *Let A and B be m-by-n complex matrices, $m \geq n$, and let* $\operatorname{rank}(A) = \operatorname{rank}(B) = n$. *Then the set $h(A, B)$ is of full rank iff the matrix*

$$\begin{bmatrix} B^*B & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} A^*A & A^*B + B^*A - A^*A - B^*B \\ 0 & I \end{bmatrix}^{-1} \tag{1}$$

has no negative eigenvalues (here X^* denotes the hermitian conjugate of X).

Proof. "Necessity". Assume that $h(A, B)$ is of full rank. So, for each $\alpha \in [0, 1]$,

$$\operatorname{rank}\Big(\alpha A + (1 - \alpha)B\Big) = n.$$

Then, from a well known property of the rank, we get

$$n = \operatorname{rank}\Big(\big(\alpha A + (1 - \alpha)B\big)^*\big(\alpha A + (1 - \alpha)B\big)\Big). \tag{2}$$

So, $\big(\alpha A + (1 - \alpha)B\big)^*\big(\alpha A + (1 - \alpha)B\big)$ is nonsingular and after slight manipulations saving the nonsingularity it becomes

$$\alpha A^*A + (1 - \alpha)B^*B + \alpha(1 - \alpha)(A^*B + B^*A - A^*A - B^*B). \tag{3}$$

But the matrix (3) is the Schur complement [1] of I in the matrix

$$\begin{bmatrix} \alpha A^*A + (1 - \alpha)B^*B & \alpha(A^*B + B^*A - A^*A - B^*B) \\ -(1 - \alpha)I & I \end{bmatrix}. \tag{4}$$

So, (4) is nonsingular and therefore so is the matrix

$$\alpha \begin{bmatrix} A^*A & A^*B + B^*A - A^*A - B^*B \\ 0 & I \end{bmatrix} + (1 - \alpha) \begin{bmatrix} B^*B & 0 \\ -I & I \end{bmatrix}.$$

The assertion follows by applying to the last matrix the result of Johnson and Tsatsomeros.

"Sufficiency". If (1) has the mentioned spectral property then

$$\begin{bmatrix} B^*B & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} A^*A & A^*B + B^*A - A^*A - B^*B \\ 0 & I \end{bmatrix}^{-1} + \beta I$$

is nonsingular for each $\beta \geq 0$. From this we have that, for each $\alpha \in [0, 1]$,

$$\alpha \begin{bmatrix} A^*A & A^*B + B^*A - A^*A - B^*B \\ 0 & I \end{bmatrix} + (1 - \alpha) \begin{bmatrix} B^*B & 0 \\ -I & I \end{bmatrix}$$

is nonsingular. Applying some arguments used in the proof of "necessity" we get equality (2). So, the assertion follows. □

Remark 1. Comparing our result with (a) in Theorem A it is natural to ask if the condition that B^+A, where B^+ is the Moore-Penrose generalized inverse of B, has no negative eigenvalues is necessary and sufficient for $h(A,B)$ to be of full rank. To answer this question we consider an example.

Example. Let

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ -\varepsilon \end{bmatrix}.$$

Then it is easy to see that $\text{rank}(A) = \text{rank}(B) = \text{rank}\big((A,B)\big) = 1$. But an inspection yields

$$B^{+}A = \left[\frac{-\varepsilon}{1+\varepsilon^2}\right].$$

So, for $\varepsilon > 0, B^{+}A$ does not possess the mentioned spectral property.

The example suggests our next three results.

Theorem 2. *Let A and B be m-by-n complex matrices, $m \geq n$, and let* $\text{rank}(A) = \text{rank}(B) = n$. *If $B^{+}A$ has no negative eigenvalues then $h(A,B)$ is of full rank.*

Proof. By (a) of Theorem A we obtain that the set $h(B^{+}A, I)$ is nonsingular. So, for each $\alpha \in [0,1]$,

$$\text{rank}\big(\alpha B^{+}A + (1-\alpha)I\big) = n. \tag{5}$$

It is well known that, for B of full column rank, $B^{+} = (B^{*}B)^{-1}B^{*}$. Using this formula and some rank properties, (5) becomes

$$\text{rank}\Big(B^{*}\big(\alpha A + (1-\alpha)B\big)\Big) = n.$$

Observing that $\text{rank}(B^{*}) = n$ the assertion follows by the property of the rank of the product of matrices. □

Theorem 3. *Let A and B be m-by-n complex matrices, $m \geq n$, and let* $\text{rank}(A) = \text{rank}(B) = n$. *If $A^{+}B$ is a P-matrix then the set $c(A,B)$ is of full rank.*

Proof. Since $(A^{+}B)^{-1}$ is also a P-matrix, by (c) of Theorem A we get that

$$c(I, A^{+}B) \quad \text{is nonsingular.}$$

The assertion follows by reasoning used in the proof of Theorem 2. □

We shall close this section with a partial characterization of full rank property of the set $r(A,B)$.

Theorem 4. *Let A and B be m-by-n complex matrics, $m \leq n$, and let* $\text{rank}(A) = \text{rank}(B) = m$. *If AB^{+} is a P-matrix then $r(A,B)$ is of full rank.*

Proof. Using again the fact that the inverse of a P-matrix is a P-matrix, by (b) of Theorem A, we get

$$r(AB^{+}, I) \quad \text{is nonsingular.}$$

Since, by the assumptions, B is of full row rank therefore $B^{+} = B^{*}(BB^{*})^{-1}$. The assertion follows by reasoning used in the proof of Theorem 2. □

3. Constant rank sets

Theorem 5. *Let A and B be m-by-n complex matrices, $m \geq n$, and let*

$$0 < \operatorname{rank}(A) = \operatorname{rank}(B) = r < n.$$

Moreover, let for an r-by-n complex matrix W both

$$Y = \begin{bmatrix} A \\ I - W^*(W^*)^- \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} B \\ I - W^*(W^*)^- \end{bmatrix}$$

be of full column rank and let the matrix

$$\begin{bmatrix} Z^*Z & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} Y^*Y & Y^*Z + Z^*Y - Y^*Y - Z^*Z \\ 0 & I \end{bmatrix}^{-1}$$

have no negative eigenvalues (here W^- denotes any weak inverse of W, i.e., a matrix satisfying $W = WW^-W$). The set $h(A,B)$ is of rank r iff the matrix

$$\begin{bmatrix} WB^*BW^* & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} WA^*AW^* & W(A^*B + B^*A - A^*A - B^*B)W^* \\ 0 & I \end{bmatrix}^{-1} \tag{6}$$

has no negative eigenvalues.

Proof. "Necessity". Assume that for each $\alpha \in [0,1]$,

$$\operatorname{rank}\big(\alpha A + (1-\alpha)B\big) = r. \tag{7}$$

Following the assumptions, by Theorem 1, the matrix

$$\alpha Y + (1-\alpha)Z = \begin{bmatrix} \alpha A + (1-\alpha)B \\ I - W^*(W^*)^- \end{bmatrix}$$

has, for each $\alpha \in [0,1]$, full column rank. So, by Corollary 6.1 from [3] and by (7), we get

$$\operatorname{rank}\Big(\big(\alpha A + (1-\alpha)B\big)W^*\Big) = \operatorname{rank}\big(\alpha A + (1-\alpha)B\big) = r.$$

Observing that the matrix $\Big(\big(\alpha A + (1-\alpha)B\big)W^*\Big)^*\Big(\big(\alpha A + (1-\alpha)B\big)W^*\Big)$ is nonsingular the assertion follows by the argument used in the proof of "necessity" of Theorem 1.

"Sufficiency". Assume that (6) has no negative eigenvalues. Then, by reasoning used in the proof of "sufficiency" of Theorem 1, we arrive at the equality

$$r = \operatorname{rank}\big(\alpha AW^* + (1-\alpha)BW^*\big) = \operatorname{rank}\Big(\big(\alpha A + (1-\alpha)B\big)W^*\Big).$$

The assertion follows by Corollary 6.1 from [3]. □

Remark 2. It is well known [3] that an m-by-n complex matrix of a positive rank r admits a full rank decomposition into two matrices of full rank r. Using Theorem 5, W can be chosen as the matrix Q in a full rank decomposition PQ of A (or S in a full rank decomposition RS of B).

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