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On the computational complexity of the solution of linear systems with moduli

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A problem of solvability for the system of equations of the form $Ax = D|x| + \delta$ is investigated. This problem is proved to be NP-complete even in the case when the number of equations is equal to the number of variables, the matrix A is nonsingular, $A \geq D \geq 0$, $\delta \geq 0$, and it is initially known that the system has a finite (possibly zero) number of solutions. For an arbitrary system of m equations of n variables, under additional conditions that the matrix D is nonnegative and its rank is one, a polynomial-time algorithm (of the order $O((\max\{m,n\})^3)$) has been found which allows to determine whether the system is solvable or not and to find one of such solutions in the case of solvability.

О вычислительной сложности решения линейных систем с модулями

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Изучается задача разрешимости для системы уравнений вида $Ax = D|x| + \delta$. Показано, что эта задача является NP-полной даже в случае, когда число уравнений равно числу переменных, матрица A невырождена, $A \geq D \geq 0$, $\delta \geq 0$ и заранее известно, что система имеет конечное (возможно, равное нулю) число решений. Для произвольной системы m уравнений от n переменных при дополнительном условии, что матрица D не отрицательна и ее ранг равен единице, найден полиномиальный алгоритм (порядка $O((\max\{m,n\})^3)$), позволяющий выяснить разрешимость этой системы и, в случае разрешимости, найти одно из решений.

1. Introduction

This work deals with the computational complexity of the solution of equations of the form

$$Ax = D|x| + \delta \tag{1}$$

where A, D are $m \times n$-matrices, δ, x are an m-vector and an n-vector respectively, $|x|$ is the n-vector made up of the moduli of the components of x. The elements of the matrices A, D and the components of the vector δ are integers.

The interest to investigation of systems of the form (1) is due to the fact that the following well-known problems of computational and interval mathematics reduce to them:

- the linear complementarity problem (see, for example, Berman and Plemmons [1]);
- the problem of computing vertices of the convex hull of the united solution set for a regular system of linear interval equations (see Rohn [7] and also Neumaier [6]);
- the compatibility problem for systems of linear equations in Kaucher interval arithmetic under some additional conditions (see Lakeyev [5]).

Concerning the latter problem, note that Shary [8] and Kupriyanova [4] have proved that algebraic interval solutions for systems of linear equations in Kaucher interval arithmetic allow us to obtain maximal interval inner estimates for various sets of solutions of interval linear algebraic systems (namely, for united, tolerable, controlled, and some other solution sets, see Shary [8]).

The question we solve in our work is as follows:

Is there a polynomial-time algorithm that finds out whether the system (1) is solvable for given matrices A, D and vector δ and, in case of solvability, computes a solution to the system?

The basic concepts and definitions of computational complexity theory (polynomial-time algorithm, classes P and NP, NP-completeness) can be found in [2].

2. Main results

In [7], a finite algorithm was proposed (sign-accord algorithm) that computes solutions of the system (1) if $m = n$, A is nonsingular and the system (1) is solvable for any vector δ. Nonetheless, as shown there, the algorithm may operate in an exponential number of steps. The fact that the latter result is not accidental may probably be seen from the following statement.

Theorem 1. *The following problem is NP-complete:*

Instance. We are given integer $n \times n$-matrices A, D and an integer n-vector δ, such that

- *A is nonsingular;*
- *$A \geq D \geq 0$, $\delta \geq 0$ (the inequality "$\geq$" is understood componentwise);*
- *the number of solutions of the system (1) is finite (possibly zero).*

Question. Is there a solution to the system (1)?

To prove the theorem we construct a polynomial reduction of the problem Partition, which is known to be NP-complete [8], to that problem.

The problem Partition implies:

Instance. Given μ integer positive numbers $d_1, \ldots, d_\mu$.

Question. Does there exist a sequence of signs $\varepsilon_1, \ldots, \varepsilon_\mu \in \{-1, 1\}$ such that $\sum_{i=1}^{\mu} \varepsilon_i d_i = 0$?

The desired reduction is based on the following two lemmas.

Lemma 1. *Let* $A_0 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $D_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ *be 2×2-matrices and* $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$. *Consider the system*

$$A_0 x = D_0 |x| + a, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \tag{2}$$

Then:

i) if $a_2 > a_1$ then the system (2) has no solution;

ii) if $a_2 = a_1$ then the system (2) has the unique solution $x^0 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$;

iii) if $a_2 < a_1$ then the system (2) has exactly two solutions

$$x^1 = \begin{pmatrix} a_2 \\ a_1 - a_2 \end{pmatrix}, \quad x^2 = \frac{1}{3}\begin{pmatrix} 4a_1 - a_2 \\ a_2 - a_1 \end{pmatrix}. \tag{3}$$

For proving this lemma it is sufficient to consider the system (2) separately in the spaces $x_2 \geq 0$ and $x_2 \leq 0$, in which it transforms into an ordinary system of linear equations.

Let now some positive integer numbers $d_1, \ldots, d_\mu$ be given, and let $d_0 = 2\sum_{i=1}^{\mu} d_i$, $\delta_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \in \mathbb{R}^2$, $n = 2(\mu + 2)$.

Consider the following system of the form (1) composed of n equations of n variables:

$$\begin{cases} A_0\begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = D_0\begin{pmatrix} |x_{2i-1}| \\ |x_{2i}| \end{pmatrix} + \delta_0, \quad i = \overline{1,\mu}, & (4) \\ \sum_{j=1}^{\mu}\begin{pmatrix} d_j & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} + A_0\begin{pmatrix} x_{2\mu+1} \\ x_{2\mu+2} \end{pmatrix} = D_0\begin{pmatrix} |x_{2\mu+1}| \\ |x_{2\mu+2}| \end{pmatrix} + \begin{pmatrix} d_0 \\ 0 \end{pmatrix}, & (5) \\ \sum_{j=1}^{\mu}\begin{pmatrix} 0 & 0 \\ d_j & 0 \end{pmatrix}\begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} + A_0\begin{pmatrix} x_{2\mu+3} \\ x_{2\mu+4} \end{pmatrix} = D_0\begin{pmatrix} |x_{2\mu+3}| \\ |x_{2\mu+4}| \end{pmatrix} + \begin{pmatrix} 0 \\ d_0 \end{pmatrix}. & (6) \end{cases}$$

Lemma 2.

1) If the vector $x = (x_1, \ldots, x_n)^T$ is a solution of the system (4)–(6) then $x_{2\mu+1} = x_{2\mu+2} = x_{2\mu+3} = x_{2\mu+4} = 0$, the numbers $\varepsilon_i = \frac{1}{2}x_{2i-1} - 1$ belong to $\{-1, 1\}$ for all $i = \overline{1,\mu}$, $\sum_{i=1}^{\mu}\varepsilon_i d_i = 0$ and x has the form

$$x = (2\varepsilon_1 + 2, 1 - 2\varepsilon_1, \ldots, 2\varepsilon_\mu + 2, 1 - 2\varepsilon_\mu, 0, 0, 0, 0)^T. \tag{7}$$

2) Conversely, if $\varepsilon_i \in \{-1, 1\}$, $i = \overline{1,\mu}$ and $\sum_{i=1}^{\mu}\varepsilon_i d_i = 0$ then the vector x of the form (7) is a solution of the system (4)–(6).

Proof.

1) Let $x = (x_1, \ldots, x_n)^T$ be a solution of the system (4)–(6). Since (4) for any $i = \overline{1,\mu}$ is a particular case of (2) (for $a = \delta_0$) then from Lemma 1 we have that $\begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} \in \left\{\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\}$. Therefore, $x_{2i-1} \in \{0, 4\}$ and consequently, $\varepsilon_i = \frac{1}{2}x_{2i-1} - 1 \in \{-1, 1\}$.

Denote $d = \sum_{j=1}^{\mu} d_j x_{2j-1}$. Then (5), (6) may be represented in the form

$$A_0\begin{pmatrix} x_{2\mu+1} \\ x_{2\mu+2} \end{pmatrix} = D_0\begin{pmatrix} |x_{2\mu+1}| \\ |x_{2\mu+2}| \end{pmatrix} + \begin{pmatrix} d_0 - d \\ 0 \end{pmatrix}, \tag{8}$$

$$A_0\begin{pmatrix} x_{2\mu+3} \\ x_{2\mu+4} \end{pmatrix} = D_0\begin{pmatrix} |x_{2\mu+3}| \\ |x_{2\mu+4}| \end{pmatrix} + \begin{pmatrix} 0 \\ d_0 - d \end{pmatrix}. \tag{9}$$

Since the system (8) is a particular case of the system (2) $\left(\text{for } a = \begin{pmatrix} d_0 - d \\ 0 \end{pmatrix}\right)$ and $\begin{pmatrix} x_{2\mu+1} \\ x_{2\mu+2} \end{pmatrix}$ is its solution, from Lemma 1 we have $0 \le d_0 - d$. Similarly, when considering the system (9), we obtain $d_0 - d \le 0$, and consequently, $d_0 - d = 0$. Hence, again due to Lemma 1, we have that $x_{2\mu+1} = x_{2\mu+2} = x_{2\mu+3} = x_{2\mu+4} = 0$, and furthermore,

$$\sum_{j=1}^{\mu} d_j\varepsilon_j = \sum_{j=1}^{\mu} d_j \left(\frac{1}{2}x_{2j-1} - 1\right) = \frac{1}{2}\left(\sum_{j=1}^{\mu} d_j x_{2j-1} - 2\sum_{j=1}^{\mu} d_j\right) = \frac{1}{2}(d - d_0) = 0.$$

Note also that for any solution x of the system (4) the equality $x_{2i-1} + x_{2i} = 3$ holds, and $x_{2i-1} = 2\varepsilon_i + 2$. Thus $x_{2i} = 1 - 2\varepsilon_i$, $i = \overline{1,\mu}$ and x has the form (7).

2) Let us prove the opposite. Let $\varepsilon_i \in \{-1, 1\}$, $i = \overline{1,\mu}$, $\sum_{i=1}^{\mu} \varepsilon_i d_i = 0$ and the vector x be defined by (7). Then $d = \sum_{j=1}^{\mu} d_j x_{2j-1} = \sum_{j=1}^{\mu} d_j(2\varepsilon_j + 2) = 2\sum_{j=1}^{\mu} d_j\varepsilon_j + 2\sum_{j=1}^{\mu} d_j = d_0$. So, (8), (9) hold, and consequently, (5), (6). Since furthermore, $\begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 2\varepsilon_i + 2 \\ 1 - 2\varepsilon_i \end{pmatrix} \in \left\{\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\}$ for any $\varepsilon_i \in \{-1, 1\}$, $i = \overline{1,\mu}$, then (4) also holds for all $i = \overline{1,\mu}$. □

Proof of Theorem 1. Let us prove that the problem formulated in the condition of Theorem 1 is NP-hard. For this purpose, as noted above, it is sufficient to construct the polynomial reduction of the problem Partition to it.

Let us take positive integers $d_1, \dots, d_\mu$ and form the system (4)–(6) having the form (1). The matrix A obtained is nonsingular, since it is a block lower triangular matrix which has $\mu + 2$ blocks along the diagonal, each of which is equal to the nonsingular 2×2-matrix A_0. Obviously, the inequalities $A \ge D \ge \mathbf{0}$ and $\delta \ge \mathbf{0}$ hold.

By Lemma 2, any solution of the system (4)–(6) can be represented in the form (7), and consequently, the number of solutions is finite (not greater than 2^μ). From Lemma 2 it also follows that for given $d_1, \dots, d_\mu$ the problem Partition has a solution if and only if there is a solution of the system (4)–(6). Furthermore, it is obvious that the system (4)–(6) is constructed from $d_1, \dots, d_\mu$ by polynomial-time algorithms. Consequently, the problem under scrutiny is NP-hard. Its belonging to the NP class is obvious. □

Therefore, if $P \ne NP$ then there is no polynomial-time algorithm revealing the solvability of the system (1) even under the additional conditions of Theorem 1. On the other hand, there are other practically interesting classes of systems of the form (1), for which polynomial-time algorithms have been found. Let $\text{rank}(D)$ designate the rank of the matrix D, $\text{corank}(A) = \max\{m, n\} - \text{rank}(A)$.

Theorem 2. *There exists a polynomial-time algorithm (of the order $O((\max\{m,n\})^3)$) that*

(i) finds out whether the system (1) is solvable for given rational $m \times n$-matrices and a rational m-vector b satisfying

$$D \ge 0, \qquad \text{rank}(D) = 1$$

(ii) computes a solution to (1) in case of solvability.

Proof of Theorem 2. Let us give an informal description of the desired algorithm. Since, according to the condition, $\text{rank}(D) = 1$ and $D \ge 0$, there are vectors $a \ge \mathbf{0}$, $b \ge \mathbf{0}$, $a \in \mathbb{R}^m$,

$b \in \mathbb{R}^n$ with rational components such that $D = ab^T$. It is furthermore obvious that the vectors a, b may be found from the matrix D by an algorithm of linear complexity. Then the system (1) may be written in the form

$$Ax = (b, |x|)a + \delta \tag{10}$$

where $(\cdot, \cdot)$ is the scalar product, which is obviously equivalent to the system

$$\begin{cases} Ax = \lambda a + \delta, & (11) \\ (b, |x|) = \lambda & (12) \end{cases}$$

where $\lambda \in \mathbb{R}^1$ is a new variable.

Introduce the notation $X = \{(\lambda, x) \mid Ax = \lambda a + \delta\}$. Let us find out (say, with the aid of the Gaussian algorithm) whether $X = \emptyset$. If $X = \emptyset$ then the system (11), and consequently the system (10), have no solutions; otherwise we find one of the solutions $(\lambda_0, x_0) \in X$. Next, consider two cases.

1^0. $\lambda_0 \leq (b, |x_0|)$ or there exists a solution of the system (11) for $\lambda = 0$.

In this case, if there exists a solution $\tilde{x}$ of the system (11) for $\lambda = 0$ then (since $(b, |\tilde{x}|) \geq 0$ due to nonnegativity of b) it is possible to take $\lambda_0 = 0$, $x_0 = \tilde{x}$, i.e. in this case, the inequality $\lambda_0 \leq (b, |x_0|)$ holds.

Consider the following system of linear equations and inequalities of the variables $\lambda \in \mathbb{R}^1$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$:

$$\begin{cases} A(u - v) = \lambda a + \delta, \\ (b, u + v) \leq \lambda, \\ u \geq 0, \quad v \geq 0. \end{cases} \tag{13}$$

Let us find out (using Khatchiyan's polynomial algorithm [3]) whether the system (13) is solvable. If it is not solvable, then the system (11), (12) is also unsolvable, since if (λ_1, x_1) is the solution of (11), (12) then obviously $u_1 = \max\{x_1, 0\}$, $v_1 = u_1 - x_1$, $\lambda = \lambda_1$ is the solution of the system (13) ($\max\{.,.\}$ is understood coordinatewise).

If the system (13) is solvable, then let us find one of its solutions (λ_1, u_1, v_1) (also by Khatchiyan's algorithm).

Let us prove that in the present case the system (11), (12) is also solvable.

Denote $x_1 = u_1 - v_1$. Then $Ax_1 = A(u_1 - v_1) = \lambda_1 a + \delta$ and $(b, |x_1|) = (b, |u_1 - v_1|) \leq (b, u_1 + v_1) \leq \lambda_1$.

Consider a segment in $\mathbb{R}^{n+1}$ connecting the points (λ_0, x_0) and (λ_1, x_1), i.e. take $x_\tau = x_0 + \tau(x_1 - x_0)$, $\lambda_\tau = \lambda_0 + \tau(\lambda_1 - \lambda_0)$ for $\tau \in [0, 1]$. It is furthermore obvious that $Ax_\tau = \lambda_\tau a + \delta$, and the function $\varphi(\tau) = (b, |x_\tau|) - \lambda_\tau = (b, |x_0 + \tau(x_1 - x_0)|) - \tau(\lambda_1 - \lambda_0) - \lambda_0$ satisfies the inequalities $\varphi(0) \geq 0$, $\varphi(1) \leq 0$.

Consequently, there may be found $\tau_0 \in [0, 1]$ such that $\varphi(\tau_0) = 0$. But then $(\lambda_{\tau_0}, x_{\tau_0})$ is a solution of the system (11), (12), and consequently, x_{τ_0} is a solution of the system (10). Note that in order to find this solution it is sufficient to solve the scalar equation (x_0, x_1, λ_0, λ_1 are known)

$$\sum_{i=1}^{n} b_i |x_{0i} + \tau(x_{1i} - x_{0i})| = \tau(\lambda_1 - \lambda_0) + \lambda_0. \tag{14}$$

The solution of this equation can easily be found if one considers the decomposition of the real line $\mathbb{R}^1$ into intervals by roots of the equations $x_{0i} + \tau(x_{1i} - x_{0i}) = 0$, $i = \overline{1, n}$ (for i such that $x_{1i} \neq x_{0i}$), since the number of these intervals is not greater than $n + 1$, and on each of these intervals, (14) transforms into a linear equation.

2^o. $(b, |x_o|) < \lambda_0$ and there is no solution of the system (11) for $\lambda = 0$.

Note, in this case, if $(\lambda, x) \in X$ then $\lambda = \lambda_0$. Indeed, if there exists $(\lambda_1, x_1) \in X$ such that $\lambda_1 \neq \lambda_0$ then for $\tilde{x} = \frac{1}{\lambda_1 - \lambda_0}(\lambda_1 x_0 - \lambda_0 x_1)$ we obtain $A\tilde{x} = \delta$. The latter is in contradiction with the fact of unsolvability of the system (11) for $\lambda = 0$.

Consider now the set $X_0 = \{x \in \mathbb{R}^n \mid Ax = 0\}$. Let $n - \text{rank}(A) = k$. Then (for example, again with the use of the Gaussian algorithm) it is possible to find the $n \times k$-matrix C (with rational elements) such that $X_0 = \{Cy \mid y \in \mathbb{R}^k\}$. Note, in this case $X = \{(\lambda_0, x_0 + Cy) \mid y \in \mathbb{R}^k\}$. Let $c_1, \ldots, c_n \in \mathbb{R}^k$ be the vectors formed by rows of the matrix C, and $c_i = (c_{i1}, \ldots, c_{ik})$. Let us show that in this case there is a solution of the system (10) if and only if $i \in \overline{1,n}$, $j \in \overline{1,k}$ may be found such that $b_i c_{ij} \neq 0$.

Indeed, if for all $i = \overline{1,n}$, $j = \overline{1,k}$, it is true that $b_i c_{ij} = 0$, then obviously for any $y \in \mathbb{R}^k$ $(b, |x_0 + Cy|) = \sum_{i=1}^{n} b_i |x_{0i} + (c_i, y)| = \sum_{i=1}^{n} b_i |x_{0i}| = (b, |x_0|) < \lambda_0$ holds and consequently, there are no solutions of the system (11), (12).

Let i_0, j_0 be such that $b_{i_0} c_{i_0 j_0} \neq 0$. Let $x_1 = (c_{1j_0}, \ldots, c_{nj_0})^T$ be the j_0-th column of the matrix C. Then it is easy to prove that the function $\varphi(\tau) = (b, |x_0 + \tau x_1|) - \lambda_0$ is not upper bounded for $\tau \in [0, \infty)$. And since $\varphi(0) = (b, |x_0|) - \lambda_0 < 0$, it is possible to find $\tau_0 \in [0, \infty)$ such that $\varphi(\tau_0) = 0$. Consequently, $(b, |x_0 + \tau_0 x_1|) = \lambda_0$, and $x_0 + \tau_0 x_1$ is a solution of the system (10).

For finding τ_0 we need to solve the equation $\varphi(\tau) = 0$, which is similar to (14).

Note also that in order to construct this algorithm we used only the Gaussian algorithm for solving systems of linear equations $Ax = \lambda a + \delta$, $Ax = \delta$, $Ax = 0$ and Khatchiyan's algorithm for solving the system (13). So, in general the complete algorithm has the order indicated in Theorem 2. □

Let us also formulate (without proving) the following (in some sense more general) theorem.

Theorem 3. *Let k be a fixed natural number. There is a polynomial algorithm (of the order $O((\max\{m, n\})^{k+5})$) that*

(i) finds out whether the system (1) is solvable for given rational $m \times n$-matrices A, D and a rational m-vector b satisfying

$$\text{rank}(D) + \text{corank}(A) \leq k$$

(ii) computes a solution to (1) in case of solvability.

Note, for comparison, that we do not know if the sign-accord algorithm mentioned above will complete its operation in a polynomial number of steps even under the conditions of Theorem 2.

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