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Bounds of high quality for first kind Volterra integral equations

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E–Methods for solving linear Volterra integral equations of the first kind with smooth kernels are considered. E–Methods are a new type of numerical algorithms computing numerical approximations together with mathematically guaranteed close error bounds. The basic concepts from verification theory are sketched and such self-validating numerics derived. Computational experiments show the efficiency of these procedures being an advance in numerical methods.

Тесные границы решений интегральных уравнений Вольтерра первого рода

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Рассматриваются E-методы решения линейных интегральных уравнений Вольтерра первого рода с гладким ядром. E-методы представляют собой новый тип численных алгоритмов, позволяющих получить одновременно с численными приближениями математически гарантированные тесные границы погрешностей. В работе излагаются основные понятия теории верификации и ее приложения в области самоверифицирующих вычислений. Численные эксперименты показывают эффективность этих новых вычислительных методов.

1. Introduction

We deal with the following linear Volterra integral equation of the first kind

$$\int_0^s k(s,t)y(t)\,dt = g(s), \quad 0 \le s \le a \tag{1}$$

which will be treated under the same assumptions as in the paper of De Hoog and Weiss [1]:

$$g \quad \text{is continuously differentiable on} \quad [0,a], \tag{2}$$

$$g(0) = 0, \tag{3}$$

$$k(s,t) \quad \text{is continuous for} \quad 0 \le t \le s \le a, \tag{4}$$

$$\frac{\partial k(s,t)}{\partial s} \quad \text{exists and is continuous for } 0 \le t \le s \le a, \tag{5}$$

$$\bigwedge_{s\in[0,a]} k(s,s) \neq 0. \tag{6}$$

Then (1) is equivalent to the linear Volterra integral equation of the second kind

$$x(s) = \frac{g'(s)}{k(s,s)} - \int_0^s \frac{1}{k(s,s)}\frac{\partial k(s,t)}{\partial s}x(t)\,dt, \quad 0 \le s \le a. \tag{7}$$

The solution of (7) is uniquely determined and continuous.

Remark. Equation (1) is ill posed in the sense of Hadamard (cf. Hadamard [5]). The equivalence of (1) with (7) shows us, that a small perturbation in g can effect a great change in x resp. y. Furthermore, the criterion (3) may be violated, so that (1) has no solution at all.

If we introduce the new quantities

$$\bigwedge_{s\in[0,a]} h(s) := \frac{g'(s)}{k(s,s)} \tag{8}$$

$$\bigwedge_{(s,t)\in[0,a]\times[0,a]} q(s,t) := -\frac{1}{k(s.s)}\frac{\partial k(s,t)}{\partial s} \tag{9}$$

then (7) has the form

$$x(s) = h(s) + \int_0^s q(s,t)x(t)dt. \quad 0 \le s \le a. \tag{10}$$

In operator notation, where I stands for identity, (1) and (10) can be shortly written as

$$k(y) = g$$

resp.

$$x = h + q(x) \quad \text{or} \quad (I - q)x = h.$$

The enclosure of (1) is determined via an enclosure of the equivalent equation (10). In Section 2 the basic ideas from inclusion theory are explained. Then in Section 3 enclosure methods for linear Volterra integral equations of the second kind are derived. Some modifications and generalizations are established in Section 4. Numerical computations are given in Section 5, these are compared with other results obtained by discretization schemes.

2. Enclosure theory

We will discuss existence and enclosure methods also called E−Methods for (1). These are methods providing the existence of a solution y of (1) whithin explicitly computable tight bounds.

To reach this goal the following tools are necessary:

2.1. Computer arithmetic

Precise formulation of the floating point operations $+$, $-$, $\cdot$, $/$, i.e. "no floating point number lies between the exact and the rounded result of such a single floating point operation." Furthermore, there is need of an exact scalar product $\bullet$ as fifth floating point operation (see Kulisch and Miranker [9]).

2.2. Interval arithmetic

The set of all real closed intervals A

$$A = [A] = [\underline{A}, \overline{A}] = \{x \in \mathbb{R} | \underline{A} \le x \le \overline{A}\}, \quad \underline{A}, \overline{A} \in \mathbb{R}$$

is abbreviated with $I(\mathbb{R})$. The standard operations $+, -, \cdot, /$ are defined as usual (with $0 \notin B$ in case of division)

$$A * B := \{a * b | a \in A, b \in B\}, \quad * \in \{+, -, \cdot, /\} \tag{11}$$

for the explicit representation of this operations cf. Moore [11]. The relations $=, \subseteq, \cup, \cap$ are defined in a set theoretic manner (for vectors and matrices componentwise). For $A \in I(\mathbb{R})$ midpoint, diameter and absolute value are defined as

$$\begin{aligned} \operatorname{mid}(A) &:= \frac{1}{2}(\underline{A} + \overline{A}), \\ \operatorname{diam}(A) &:= \overline{A} - \underline{A}, \\ |A| &:= \max\{|a| : a \in A\}. \end{aligned}$$

For intervals A, B, C, D the important rules hold

$$A \cdot (B + C) \subseteq A \cdot B + A \cdot C, \tag{12}$$

$$\operatorname{diam}(A \pm B) = \operatorname{diam}(A) + \operatorname{diam}(B), \tag{13}$$

$$\bigwedge_{* \in \{+,-,\cdot,/\}} (a \in A, b \in B \Rightarrow a * b \in A * B), \tag{14}$$

$$A \subseteq C, B \subseteq D \Rightarrow A * B \subseteq C * D. \tag{15}$$

2.3. Ultra arithmetic

The concept is founded on series expansion techniques as an arithmetic methodology. The pair (B, Ω) consists of a real Banach space B with norm $\|.\|$ and a set of (dyadic and monadic) operations Ω defined in B:

$$\Omega = \left\{+, -, \cdot, /, \int_0^s, \int, \exp, \sin, \cos, \ldots\right\}.$$

In this paper B is always the space $C[0, a]$ equipped with the maximum norm.

If $\{\varphi_i\}_{i \in \mathbb{N}}$, is a generating system of B, that is

$$\overline{\{\varphi_1, \varphi_2, \ldots\}} = B$$

then the set $I_n(B)$, $n \in \mathbb{N}$,

$$I_n(B) = \left\{ \sum_{j=1}^{n} A_j \varphi_j | A_j \in I(\mathbb{R}) \right\}. \tag{16}$$

is called interval screen. The pair $\mathcal{F}_n = \left(I_n(B), \diamondsuit\right)$ is called (interval) functoid, if the enclosure operations $\diamondsuit, * \in \Omega$, are explained such that

$$\bigwedge_{* \in \Omega} \bigvee_{\diamondsuit \in \diamondsuit} \bigwedge_{\substack{u \in U \in I_n(B) \\ v \in V \in I_n(B)}} u * v \in U \diamondsuit V \tag{17}$$

holds; here $\bigwedge$, $\bigvee$ are universal and existential quantifier respectively. It is important to note that the objects of $\mathcal{F}_n$ are sets of functions, whose graph is lying between two boundary functions; so $U \in \mathcal{F}_n$ characterizes the following subset in B

$$U = U(s) = \left\{ u \in B \mid \bigwedge_{s\in[0,a]} u(s) \in U(s) \right\}. \tag{18}$$

$U \in \mathcal{F}_n$ is called function enclosure or interval extension of $u \in B$ if

$$\bigwedge_{s\in[0,a]} u(s) \in U(s). \tag{19}$$

In the same way enclosures for kernels $k(s,t)$ or operators k are declared. Throughout this paper we agree that small letters $u, v, \ldots$ always denote (real valued) elements of B, whereas capital letters $U, V, \ldots$ indicate the corresponding (set valued) elements of $\mathcal{F}_n$.

For $U = \sum_{j=1}^{n} u_j\varphi_j$, $V = \sum_{j=1}^{n} v_j\varphi_j \in \mathcal{F}_n$ we define the coefficientwise enclosure $U \subseteq_n V$,

$$\bigwedge_{j=1}^{n} u_j \subseteq v_j. \tag{20}$$

Trivially we have

$$U \subseteq_n V \Rightarrow U \subseteq V. \tag{21}$$

For $U \in \mathcal{F}_n$ we put

$$\operatorname{diam}(U) := \sup_{0\le s\le a} \{\operatorname{diam} U(s)\}.$$

Further descriptions of the underlying ideas can be found in Kaucher and Baumhof [6], Kaucher and Miranker [7].

2.4. Automatic differentiation

Let u, v be real valued functions, being sufficiently smooth in a neighbourhood of t_0. The Taylor coefficients $(u)_k$ of a function u are defined by

$$\bigwedge_{k\ge0} (u)_k := \frac{1}{k!}\frac{d^k u}{dt^k}(t_0) \quad \text{resp.} \quad \big(u(\tau)\big)_k := \frac{1}{k!}\frac{d^k u}{dt^k}(\tau). \tag{22}$$

For a constant c and the independent variable t we have

$$(c)_0 = c, \qquad \bigwedge_{k\ge1} (c)_k = 0, \tag{23}$$

$$(t)_0 = t_0, \quad (t)_1 = 1, \qquad \bigwedge_{k\ge2} (t)_k = 0. \tag{24}$$

Furthermore, the following rules hold

$$\bigwedge_{k\ge0} (u \pm v)_k = (u)_k \pm (v)_k, \tag{25}$$

$$\bigwedge_{k\ge0} (u \cdot v)_k = \sum_{j=0}^{k} (u)_j (v)_{k-j}, \tag{26}$$

$$\bigwedge_{k\ge0} (u/v)_k = \frac{1}{v}\left((u)_k - \sum_{j=1}^{k} (v)_j \cdot (u/v)_{k-j} \right). \tag{27}$$

Using the chain rule we obtain additionally

$$\bigwedge_{k\geq 0}(u')_k = (k+1)\cdot(u)_{k+1} \tag{28}$$

so that we are able to calculate Taylor coefficients for a wide class of functions (Rall [12]). If employing interval arithmetic in the recurrence formulae (23)–(28) there is available a constructive way to determine enclosures for given real quantities:

$$\bigwedge_{t\in[t_0,t_1]} u(t)\in U(t) = \sum_{k=0}^{n}(u)_k(t-t_0)^k + u\Big(([t_0,t_1])\Big)_{n+1}(t-t_0)^{n+1}. \tag{29}$$

To bound rounding errors the Taylor coefficients are computed additionally with interval analytical means.

2.5. Fixed point theorems for verified numerics

In this context we restrict ourselves to a small setting concerning selfvalidating algorithms; here it suffices to validate the existence of a solution with a theorem corresponding to Schauder's fixed point theorem:

Theorem 1. *Let $f : B \longrightarrow B$ be a nontrivial compact mapping, F an enclosure of f, $\emptyset \neq X$, $X \in \mathcal{F}_n$ such that the condition*

$$F(X) \subseteq_n X \tag{30}$$

prevails, then there exists a fixed point $\hat{x}$ of f, which is enclosed within $f(X)$

$$\bigvee_{\hat{x}\in X} \hat{x}\in f(X). \tag{31}$$

Proof. X describes an interval in the ordered Banach space $B = (C[0,a], \|.\|_\infty)$, hence X is closed and convex. Due to (18), (19), (21) and the theorem of Schauder the foregoing assertion follows. □

2.6. Programming languages

We need programming languages so that all the concepts of the Sections 2.1 and 2.2 are available. Such a language is PASCAL–XSC (cf. Klatte et al. [8]), an extension of PASCAL. The advantage of this new language is among others that programs written in PASCAL-XSC are easily readable, because all operations have been implemented as operators and can be used in mathematical notation.

3. Mathematically guaranteed bounds of high quality for linear Volterra integral equations of the first kind

As outlined in Section 1, under the hypothesis (2)–(6), the solution of (1) is also a solution of (10) and vice versa. So we consider the problem

$$x(s) = h(s) + \int_0^s q(s,t)x(t)\,dt, \quad 0\leq s\leq a. \tag{32}$$

The solution of (32) will be determined iteratively. To reach our main goal—guaranteed high precision bounds—the iteration is performed in the functoid $\mathcal{F}_n$ using function enclosures Q, H for the kernel and the inhomogeneity resp. (cf. Sections 2.3, 2.4). Let $\tilde{x}$ be an initial guess for the solution of (32) gained for example by a conventional numerical scheme and $\tilde{X} \in \mathcal{F}_n$ its interval extension. Consider the residuum

$$U = \Diamond(I - Q)(\tilde{X}) \diamondminus H \tag{33}$$

then we iterate with a fixed, appropriate $p \in \mathbb{N}$ according to

$$X^{(i+1)} = -\Big\langle \sum_{\nu=0}^{p} \Big\rangle \oint_0^s Q_\nu(s,t) \diamondsuit U(t)\,dt, \tag{34}$$

$$\diamondminus \oint_0^s Q_{p-1}(s,t) \diamondsuit X^{(i)}(t)\,dt =: F(X^{(i)}) \in \mathcal{F}_n, \quad i = 0, 1, \ldots$$

where the enclosures Q_ν of the iterated kernels q_ν satisfy the relation

$$\begin{aligned} Q_0 &:= I, \\ \bigwedge_{\nu \in \mathbb{N}} Q_\nu(s,t) &:= \oint_0^s Q(s,\tau) \diamondsuit Q_{\nu-1}(\tau,t)\,d\tau. \end{aligned} \tag{35}$$

Now we formulate the first enclosure theorem.

Theorem 2. *If an iterate $X^{(i)} \in \mathcal{F}_n$ of the process (34) satisfies the inclusion condition*

$$X^{(i+1)} \subseteq_n X^{(i)} \tag{36}$$

then it is assured by computational means that a solution $\hat{x} \in B$ of (32) exists within $\hat{X} \in \mathcal{F}_n$:

$$\bigwedge_{s \in [0,a]} \hat{x}(s) \in \hat{X}(s) = \tilde{X}(s) \diamondminus X^{(i+1)}(s). \tag{37}$$

Proof. By (19) and (35) we see that the compact operator f

$$f(x) := -\sum_{\nu=0}^{p} \int_0^s q_\nu(s,t)u(t)\,dt + \int_0^s q_{p+1}(s,t)x(t)\,dt, \quad 0 \le s \le a$$

maps the nonvoid closed convex set $X^{(i+1)}$ into itself, so that the existence of a fixed point $\hat{y}$ of f with $\hat{y} \in X^{(i+1)}$ is guaranteed according to Theorem 1. With an approximative inverse r for the Fréchet derivative of $(I - q)$ we rewrite (32) in the form

$$x = x - r\Big((I - q)x - h\Big)$$

a mean value argumentation now gives

$$x = \tilde{x} - r\Big((I - q)\tilde{x} - h\Big) + \Big(I - r[(I - q) - h]'(\xi)\Big)(x - \tilde{x}); \quad \xi \text{ between } \tilde{x} \text{ and } x. \tag{38}$$

Subtracting $\tilde{x}$ on both sides and using the linearity of $(I - q) - h$ yields

$$x - \tilde{x} = -r\Big((I - q)\tilde{x} - h\Big) + \Big(I - r(I - q)\Big)(x - \tilde{x}). \tag{39}$$

In addition we can derive $\lim_{\nu\to\infty} \sqrt[\nu]{\|q^\nu\|} = 0$, hence the Neumann series $\sum_{\nu=0}^{\infty} q^\nu$ converges and coincides with $(I-q)^{-1}$. The operators q^ν can be represented as integral operators with kernels $q_\nu(s,t)$ according to (35). If r is choosen as

$$r = \sum_{\nu=0}^{p} q^\nu$$

then

$$I - r(I-q) = q^{p+1}$$

from which it can be seen that $\tilde{x}$ must be added to a fixed point of (39) to obtain a solution of (32).

From (34) the definition of the enclosure operations (17) and the function enclosures (19) we obtain (37), thus completing the proof. □

Remark. Because $\lim_{\nu\to\infty} \sqrt[\nu]{\|q^\nu\|} = 0$, therefore there exists an integer p with $\|q^{p+1}\| = \gamma < 1$. In practice (34) is performed with a value p, based on some rough estimates for $\|\int_0^s q_\nu(s,t)\,dt\|$. If the stopping criterion (36) can not be achieved after a prescribed number of iterations then the loop is repeated with an enlarged value of p.

The interplay between (1) and (10) is described in

Theorem 3. *Let the kernel and the forcing term of (1) satisfy (2)–(6). If $X \in \mathcal{F}_n$ such that for the solution $x \in B$ of (10) holds*

$$\bigwedge_{s\in[0,a]} x(s) \in X(s)$$

then the solution $y \in B$ of (1) exists and

$$\bigwedge_{s\in[0,a]} y(s) \in X(s). \tag{40}$$

Proof. Trivial because $y = x$ according to the assumptions (2)–(6). □

In practice there is often available only a disturbed right hand side g^δ:

$$\|g - g^\delta\| \le \delta \tag{41}$$

where δ is a known a priori error level. Unfortunately g^δ might not fulfill (3) or this perturbation causes great changes in the corresponding solution x. In the latter situation the diameter of the enclosing function X cannot be made arbitrarily small whereas the first case is trated in validation numerics by determing a set valued function $G \in \mathcal{F}_n$ (strategies for this purpose are developed in Dobner and Kaucher [3]), such that

$$g^\delta \in G, \tag{42}$$

$$\operatorname{diam}(G) = \delta, \tag{43}$$

$$\bigvee_{\overline{g}\in G} \overline{g}(0) = 0. \tag{44}$$

Now we can summarize

Theorem 4. *If $G \in \mathcal{F}_n$ suffices (42)–(44) and if for an iterate $X^{(i)} \in \mathcal{F}_n$ of (34) the enclosure*

$$X^{(i+1)} \subseteq_n X^{(i)}$$

is achieved, then it is guaranteed that there exists at least a pair $(\hat{x}, \hat{g}) \in X^{(i+1)} \times G$ such that

$$\int_0^s k(s,t)\hat{x}(t)\,dt = \hat{g}(s), \quad 0 \le s \le a$$

and at the same time

$$\hat{x}(s) = \frac{\hat{g}'(s)}{k(s,s)} - \int_0^s \frac{1}{k(s,s)} \frac{\partial k(s,t)}{\partial s} \hat{x}(t)\,dt, \quad 0 \le s \le a.$$

Proof. Follows from Theorem 2 and (42)–(44). □

4. Numerical experiments

The enclosure methods described in the foregoing chapters have been tested by solving several examples, where for sake of clarity the number of decimals in displaying the errors were reduced. There has been used the functoid $\mathcal{F}_n$, where the underlying interval screen has been chosen as

$$I_n(B) = \left\{ \sum_{j=0}^{n} A_j s^j | A_j \in I(\mathbb{R}) \right\}. \tag{45}$$

The enclosure operations (17) are defined with the help of the following rounding process

$$S^j = \Diamond s^j = \begin{cases} s^j, & j \le n \\ s^n[0, a^{j-n}], & \text{else.} \end{cases} \tag{46}$$

Example 1.

$$s\left(\frac{1}{2} + \frac{4}{5}s^4\right) = \int_0^s \left(\frac{1}{2} + s^4 - t^4\right) x(t)\,dt, \quad 0 \le s \le 1.5. \tag{47}$$

Solution: $x(s) = 1$.

We compare our enclosure result with the result of De Hoog and Weiss [1], computing the solution of (47) with a discretization method (cf. [1], method(34)) based on an implicit Runge-Kutta scheme.

	enclosure method		**discretization method**
	dimension $n = 10$	dimension $n = 40$	stepsize $h = 0.0375$
s	diam $X(s)$	diam $X(s)$	error of discretization method
0.3	$1.1 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$1.3 \cdot 10^{-9}$
0.6	$1.1 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$1.8 \cdot 10^{-8}$
0.9	$1.1 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$6.5 \cdot 10^{-8}$
1.2	$1.1 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$2.7 \cdot 10^{-7}$
1.5	$1.1 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$4.7 \cdot 10^{-6}$

Example 2.

$$1 + s - \sin(s) - \cos(s) = \int_0^s (1 + s - t)x(t)\,dt, \quad 0 \le s \le 1.2. \tag{48}$$

Solution: $x(s) = \sin(s)$.

This equation is also taken from De Hoog and Weiss [1]. The verified result and the solution obtained by De Hoog and Weiss with an implicit Runge-Kutta discretization (cf. [1], method (34)) are placed side by side for various dimensions resp. stepsizes at the endpoint of the considered domain.

enclosure method		**discretization method**	
dimension n	diam$X(1.2)$	stepsize h	error at $s = 1.2$
10	$1.4 \cdot 10^{-6}$	0.3	$-1.2 \cdot 10^{-5}$
20	$7.1 \cdot 10^{-12}$	0.15	$-8.0 \cdot 10^{-7}$
40	$7.1 \cdot 10^{-12}$	0.075	$-5.0 \cdot 10^{-8}$

Example 3.

$$-1 + s + e^{-s} = \int_0^s (1 + s - t)x(t)\,dt, \quad 0 \le s \le 9. \tag{49}$$

Solution: $x(s) = se^{-s}$.

Below we display the error of the enclosure method and of a discretization scheme, based on an implicit Runge-Kutta discretization (errors taken from De Hoog and Weiss [1], method (34)).

	enclosure method		**discretization method**
	dimension $n = 40$	dimension $n = 60$	stepsize $h = 0.3$
s	diam $X(s)$	diam $X(s)$	error
0.9	$7.9 \cdot 10^{-7}$	$5.0 \cdot 10^{-13}$	$-4.9 \cdot 10^{-5}$
4.5	$2.9 \cdot 10^{-5}$	$9.3 \cdot 10^{-10}$	$1.1 \cdot 10^{-6}$
7.2	$4.3 \cdot 10^{-4}$	$4.1 \cdot 10^{-8}$	$1.1 \cdot 10^{-7}$
9.0	$4.3 \cdot 10^{-3}$	$4.0 \cdot 10^{-7}$	$1.5 \cdot 10^{-8}$

Here the accuracy of the E−Solution decreases as s increases, this arising from the fact that the diameter of the function enclosure for the inhomogeneity becomes very large at the end of the interval. If enlarging the functoid dimension, then the error of the enclosure solution decreases. The connection between a fixed functoid dimension and an increasing diameter of the domain is illustrated in the table below.

dimension	domain	diam $X(s)$
20	$[0, 1]$	$7 \cdot 10^{-13}$
20	$[0, 2]$	$1.1 \cdot 10^{-11}$
20	$[0, 4]$	$5.9 \cdot 10^{-6}$

Example 4.

$$s + \delta(s) = \lambda \int_0^s (1 + s - t)x(t)\,dt, \quad 0 \le s \le 2, \quad \lambda \in \mathbb{R}. \tag{50}$$

Type of solution: polynomial.

To study the stability behaviour of the enclosure solution, (50) is computed for the following perturbations δ:

$$\begin{aligned} \delta &= \delta_0 = 0, \\ \delta &= \delta_1 = [1, 1000000000001]s + [2 \cdot 10^{-12}, 3 \cdot 10^{-12}]s^2. \end{aligned}$$

Furthermore, x is enclosed for different values of the parameter λ.

	$n = 30,\ \lambda = 1$ $\delta = \delta_0$	$n = 30,\ \lambda = 1$ $\delta = \delta_1$	$n = 30,\ \lambda = 0.1$ $\delta = \delta_1$
diam $X(0.5)$	$1.0 \cdot 10^{-13}$	$2.5 \cdot 10^{-12}$	$1.6 \cdot 10^{-12}$
diam $X(1.0)$	$3.0 \cdot 10^{-12}$	$5.0 \cdot 10^{-12}$	$2.2 \cdot 10^{-12}$
diam $X(2.0)$	$2.2 \cdot 10^{-12}$	$1.7 \cdot 10^{-11}$	$3.5 \cdot 10^{-12}$

In the next table we display the results for a disturbed inhomogeneity.

	$n = 30,\ \lambda = 10$ $\delta = \delta_1$	$n = 60,\ \lambda = 10$ $\delta = \delta_1$
diam $X(0.5)$	$5.5 \cdot 10^{-10}$	$1.0 \cdot 10^{-12}$
diam $X(1.0)$	$2.4 \cdot 10^{-3}$	$1.6 \cdot 10^{-7}$
diam $X(2.0)$	$2.6 \cdot 10^{+6}$	$6.3 \cdot 10^{-3}$

Example 5.

$$1 - \cos(s) = \int_0^s \cos(s - t)x(t)\,dt, \quad 0 \le s \le 1.95. \tag{51}$$

Solution: $x(t) = t$.

This equation can be found in Linz [10], where x is approximated with a midpoint method and extrapolation. The results of the discretization schemes were adapted from Linz [10].

	enclosure method	**discretization method**	
	$n = 60$	$h = 0.1$	$h = 0.1$
s	diam$X(s)$	error in midpoint method	error in extrapolation
0.15	$2.0 \cdot 10^{-13}$	$6.0 \cdot 10^{-5}$	$1.0 \cdot 10^{-5}$
0.75	$2.0 \cdot 10^{-13}$	$3.1 \cdot 10^{-4}$	$1.0 \cdot 10^{-5}$
1.35	$2.0 \cdot 10^{-11}$	$5.6 \cdot 10^{-4}$	$1.0 \cdot 10^{-5}$
1.95	$5.0 \cdot 10^{-7}$	$8.1 \cdot 10^{-4}$	$1.0 \cdot 10^{-5}$

Again the enclosed solution is of high accuracy.

5. Some final remarks

In contrast to most conventional numerical methods the result of the enclosure algorithms are verified, i.e., it is mathematically guaranteed that the computed solution exists within a set of small diameter. Additionally E–Solutions are given in a functional form so that the value at each point is available with the same effort.

As shown before, the accuracy achieved with enclosure numerics is often better than the accuracy obtained, e.g., with discretization schemes.

The blowing up of X with increasing argument which can be observed for some examples, is affected by the choice of the basis and rounding according to (45), (46); it could be avoided if using more appropriate basis systems, e.g., Tschebyscheff systems and Tschebyscheff rounding (cf. Kaucher and Baumhof [6]).

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