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Computation of the stability radius of a Schur polynomial: an orthogonal projection approach

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The robust Schur stability of a polynomial with uncertain coefficients will be investigated. A formula for the stability radius of a Schur polynomial is established. The result is the counterpart of [1] for linear discrete-time systems

Вычисление радиуса стабильности многочлена Шура: метод ортогональных проекций

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Изучается робастная Шурова устойчивость многочлена с неопределенными коэффициентами. Дается формула для радиуса стабильности многочлена Шура. Результат дополняет работу [1] для случая линейных систем дискретного времени.

1. Introduction

Given a polynomial

$$\varphi(z) = (a_n + \delta_n)z^n + (a_{n-1} + \delta_{n-1})z^{n-1} + \cdots + (a_1 + \delta_1)z + (a_0 + \delta_0)$$

whose parameter vector $\mathbf{a} = [a_n\ a_{n-1}\ \ldots\ a_0]^T \in \mathcal{R}^{n+1}$, the uncertainties δ_i are real and within a hypersphere

$$\delta_n^2 + \delta_{n-1}^2 + \cdots + \delta_1^2 + \delta_0^2 < \delta^2$$

under the assumption that the nominal polynomial

$$\varphi_0(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

is Schur stable, i.e. the roots of $\varphi_0(z)$ are all within the open unit disc, we are interested in determining the largest δ so that $\varphi(z)$ remains stable. In the parameter space $\mathcal{R}^{n+1}$, the stability boundary is described by (i) the n-dimensional subspace $\varphi(-1) = 0$, (ii) the n-dimensional subspace $\varphi(1) = 0$, and (iii) the $(n-1)$-dimensional hypersurface $\varphi(e^{j\theta}) = 0$ and $\varphi(e^{-j\theta}) = 0$ for $\theta \in [0, \pi]$ [2]. Denote by r_{d_1}, r_{d_2}, and r_{d_3}, respectively, the distance from $\mathbf{a}$ to the stability boundaries (i), (ii), and (iii), and $r_d = \min\{r_{d_1}, r_{d_2}, r_{d_3}\}$. Since $\varphi_0(z)$ is stable,

$\varphi(z)$ will be stable if and only if $\delta < r_d$. Since the distance from a point $\mathbf{a}$ to a subspace can always be calculated using the orthogonal projection approach (see [3]), the main problem is then how to identify the distance between $\mathbf{a}$ and the hypersurface, which, for a fixed θ, is also a subspace $\mathcal{X}_S$ of $\mathcal{R}^{n+1}$ with the basis vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n-1$. Denote by $\mathcal{X}_N$ the orthogonal complement of $\mathcal{X}_S$ with a basis $\mathbf{x}_i$, $i = n,\ n+1$, then $\mathcal{R}^{n+1} = \mathcal{X}_S \oplus \mathcal{X}_N$, and every $\mathbf{a} \in \mathcal{R}^{n+1}$ can be uniquely decomposed as $\mathbf{a} = \mathbf{x}_N + \mathbf{x}_S$, where $\mathbf{x}_N \in \mathcal{X}_N$ and $\mathbf{x}_S \in \mathcal{X}_S$. The distance from $\mathbf{a}$ to $\mathcal{X}_S$ is then the euclidean norm of $\mathbf{x}_N$, denoted by $\|\mathbf{x}_N\|_2$. r_{d_3} is then the minimum of $\|\mathbf{x}_N\|_2$ which is a function of θ. In [3], $\mathbf{x}_N$ is represented in terms of the inverse of the gramian matrix of the vectors $\mathbf{x}_i$ $(i = 1, 2, \ldots, n-1)$, which is an $(n-1) \times (n-1)$ matrix, and $\|\mathbf{x}_N\|_2^2$ is determined in a quadratic form which involves this inverse matrix. The approach proposed by Soh et al. [4] is based on this mehtod. On the other hand, $\mathbf{x}_N$ can be also represented as a linear combination of the vectors $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$, and this combination can be fully determined by the inverse of the 2×2 gramian matrix of $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$. Hence, $\|\mathbf{x}_N\|_2$ can be determined in terms of some real rational function $\|\mathbf{x}_N\|_2^2 = \dfrac{q(x)}{p(x)}$. In a recent paper [5], the polynomials $p(x)$ and $q(x)$ are determined in terms of the Chebyshev polynomials $U_k(x)$.

In this paper, we shall show that $p(x) = \sum_{k=0}^{n-1}(n-k)U_k^2(x)$, $q(x) = \|H\mathbf{a}\|_2^2$, where H is the skew-symmetric Toeplitz matrix $(H)_{lm} = U_{m-l-1}(x)$. This will allow us to determine exactly the degrees of $p(x)$ and $q(x)$. Further, we shall find an orthogonal basis for $\mathcal{X}_n$ and the Pythagoras form for $\|\mathbf{x}_N\|_2^2$. This establishes the counterpart of the result in [1] for discrete-time systems.

Throughout this paper, j denotes the imaginary unit, i.e. $j^2 = -1$. For a square matrix A, adj$A$ denotes its adjoint matrix, and $\det(A)$ its determinant. Given the vectors $\mathbf{a}_i \in \mathcal{R}^n$, $i = 1, 2, \ldots, m$ with $m \le n$, $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is the linear span of $\mathbf{a}_i$ over $\mathcal{R}$, i.e.

$$\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \left\{\mathbf{a} \in \mathcal{R} \;:\; \mathbf{a} = \sum_{i=1}^{m} \alpha_i \mathbf{a}_i,\ \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathcal{R}\right\}$$

where $\mathcal{R}$ in the field of real numbers.

2. Background result

As stated in previous section, for a polynomial $\varphi(z)$ of degree n, the stability region in the parameter space $\mathcal{R}^{n+1}$ is bounded by

$$\begin{aligned}
\mathcal{P}_1 &: \varphi(-1) = 0, \\
\mathcal{P}_2 &: \varphi(1) = 0, \\
\mathcal{X}_S &: \varphi(e^{j\theta}) = 0 \ \text{ and } \ \varphi(e^{-j\theta}) = 0 \ \text{ for some } \ \theta \in [0, \pi].
\end{aligned}$$

Denote by $\mathbf{a}$ the parameter vector of a Schur polynomial $\varphi_0(z)$, by r_{d_1}, r_{d_2}, and r_{d_3} the distance from $\mathbf{a}$ to $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{X}_S$ respectively. Then the stability radius δ is

$$\delta = \min\{r_{d_1}, r_{d_2}, r_{d_3}\}. \tag{1}$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are hyperplanes, r_{d_1} and r_{d_2} can be readily determined:

$$r_{d_1} = \frac{|\varphi_0(-1)|}{\sqrt{n+1}}, \qquad r_{d_2} = \frac{|\varphi_0(1)|}{\sqrt{n+1}}. \tag{2}$$

The method to compute r_{d_3} will be summarised in the following.

Define the $(n-1)\times(n+1)$ matrix Φ_d:

$$\Phi_d := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -2x & 1 & 0 & \dots & 0 \\ 1 & -2x & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & 1 & -2x \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{3}$$

where $x = \cos\theta$ with $\theta \in [0, \pi]$. Denote by $\mathbf{x}_i$, $i = 1, 2, \dots, n-1$, the column vectors of Φ_d, and by $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ a basis of $\mathcal{X}_N$,—the zero space of Φ_d^T. Then,

$$\begin{aligned} \mathcal{X}_S &= \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-1}\}, \\ \mathcal{X}_N &= \text{span}\{\mathbf{x}_n, \mathbf{x}_{n+1}\} \end{aligned}$$

and $\mathcal{R}^{n+1} = \mathcal{X}_S \oplus \mathcal{X}_N$. Let $\mathbf{a} \in \mathcal{R}^{n+1}$, then $\mathbf{a}$ can be uniquely decomposed into

$$\mathbf{a} = \mathbf{x}_N + \mathbf{x}_S \tag{4}$$

where $\mathbf{x}_N \in \mathcal{X}_N$, and $\mathbf{x}_S \in \mathcal{X}_S$. $\mathbf{x}_S$ is the orthogonal projection of $\mathbf{a}$ on $\mathcal{X}_S$. It is readily verified that the *length* $\|\cdot\|_2$ of $\mathbf{x}_N$, defined as

$$\|\mathbf{x}_N\|_2^2 := \langle \mathbf{x}_N, \mathbf{x}_N \rangle := \mathbf{x}_N^T \mathbf{x}_N \tag{5}$$

is the distance from $\mathbf{a}$ to $\mathcal{X}_S$ for a fixed $x \in [-1, 1]$. Also, denote $r_{d_3}(x) = \|\mathbf{x}_N\|_2$. Then the distance from $\mathbf{a}$ to $\mathcal{X}_S$ is given by

$$r_{d_3} = \min_{x \in [-1,1]} \{r_{d_3}(x)\}. \tag{6}$$

Define the matrix

$$X_N = [\mathbf{x}_n \ \mathbf{x}_{n+1}]. \tag{7}$$

Then, from the orthogonal projection approach described in [3], we obtain

$$\mathbf{x}_N = X_N G^{-1}(\mathbf{x}_n, \mathbf{x}_{n+1}) X_N^T \mathbf{a} \tag{8}$$

and

$$r_{d_3}^2(x) = \mathbf{x}_N^T \mathbf{x}_N = \mathbf{a}^T X_N G^{-1}(\mathbf{x}_n, \mathbf{x}_{n+1}) X_N^T \mathbf{a} \tag{9}$$

where

$$G(\mathbf{x}_n, \mathbf{x}_{n+1}) := X_N^T X_N = \begin{bmatrix} \mathbf{x}_n^T \mathbf{x}_n & \mathbf{x}_n^T \mathbf{x}_{n+1} \\ \mathbf{x}_{n+1}^T \mathbf{x}_n & \mathbf{x}_{n+1}^T \mathbf{x}_{n+1} \end{bmatrix}$$

is the Gramian of $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ [3]. Since $G(\mathbf{x}_n\ \mathbf{x}_{n+1})$ is a 2×2 matrix, (9) can be represented as a rational function

$$r_{d_3}^2(x) = \frac{q(x)}{p(x)} \tag{10}$$

with

$$\begin{aligned} p(x) &= \det G(\mathbf{x}_n, \mathbf{x}_{n+1}) = \mathbf{x}_n^T\mathbf{x}_n\mathbf{x}_{n+1}^T\mathbf{x}_{n+1} - \mathbf{x}_n^T\mathbf{x}_{n+1}\mathbf{x}_n^T\mathbf{x}_{n+1}, \\ q(x) &= \mathbf{a}^T X_N \begin{bmatrix} \mathbf{x}_{n+1}^T\mathbf{x}_{n+1} & -\mathbf{x}_n^T\mathbf{x}_{n+1} \\ -\mathbf{x}_{n+1}^T\mathbf{x}_n & \mathbf{x}_n^T\mathbf{x}_n \end{bmatrix} X_N^T\mathbf{a}. \end{aligned} \tag{11}$$

The following result is due to Wu and Mansour [5].

Proposition 1. *A basis for $\mathcal{X}_N$ is*

$$\begin{aligned} \mathbf{x}_n &= [U_{n-1}(x)\ U_{n-2}(x)\ \dots\ U_1(x)\ 1\ 0\ ]^T, \\ \mathbf{x}_{n+1} &= [-U_{n-2}(x)\ -U_{n-3}(x)\ \dots\ -U_0(x)\ 0\ 1]^T \end{aligned}$$

where $U_k(x)$ is the Chebyshev polynomial of the second kind.

We shall use this basis to define the matrix X_N throughout the rest of this paper. From Proposition 1 and (11), we see that

$$\begin{aligned} \deg[p(x)] &\le 2(n-1) + 2(n-2) = 4n-6, \\ \deg[q(x)] &\le 2(n-1) + 2(n-2) = 4n-6. \end{aligned} \tag{12}$$

Examples shown that there are cancellations in the coefficients of $p(x)$ resp. $q(x)$, and $2(n-1)$ should be the degree for both $p(x)$ and $q(x)$. We shall show in the following section that this is true.

Further, it should be noted that the basis of $\mathcal{X}_N$ given in Proposition 1 is not an orthogonal one. However, since for any nonsingular 2×2 matrix $V(x)$, the vectors

$$[\mathbf{y}_n \quad \mathbf{y}_{n+1}\ ] = X_N V(x) \tag{13}$$

also form a basis for $\mathcal{X}_N$, we can choose the matrix $V(x)$ such that the resulting $\mathbf{y}_n$ and $\mathbf{y}_{n+1}$ are orthogonal, i.e. $\mathbf{y}_n^T\mathbf{y}_{n+1} = 0$. It is clear that $r_{d_3}^2(x)$ is independent of the choice of the basis of $\mathcal{X}_N$. Hence, $r_{d_3}^2(x)$ can be represented in the Pythagoras form:

$$r_{d_3}^2(x) = \left(\frac{\mathbf{y}_n^T\mathbf{a}}{\|\mathbf{y}_n\|_2}\right)^2 + \left(\frac{\mathbf{y}_{n+1}^T\mathbf{a}}{\|\mathbf{y}_{n+1}\|_2}\right)^2 \tag{14}$$

by choosing a suitable $V(x)$ to orthogonalize the basis. In the following section we shall show how to choose the matrix $V(x)$.

3. The main results

Let us first define $U_{-k}(x)$ for $k = 1, 2, \dots$

$$U_{-k-1}(x) = 2xU_{-k}(x) - U_{-k+1}(x). \tag{15}$$

It is readily verified that $U_{-k}(x)$ satisfies the recursive form for $U_k(x)$:

$$U_{-k+1}(x) = 2xU_{-k}(x) - U_{-k-1}(x).$$

Hence, $U_{-k}(x)$ extends the definition of the Chebyshev polynomials for negative indices. We claim that

$$U_{-k}(x) = -U_{k-2}(x). \tag{16}$$

Ideed, for $k = 1, 2, 3$ we have

$$\begin{aligned} U_{-1}(x) &= 2xU_0(x) - U_1(x) = 0, \\ U_{-2}(x) &= 2xU_{-1}(x) - U_0(x) = -U_0(x), \\ U_{-3}(x) &= 2xU_{-2}(x) - U_{-1}(x) = -U_1(x). \end{aligned}$$

If we assume

$$U_{-k}(x) = -U_{k-2}(x) \quad \text{and} \quad U_{-k-1}(x) = -U_{k+1-2}(x)$$

then

$$\begin{aligned} U_{-k-2}(x) &= 2xU_{-k-1}(x) - U_{-k}(x) \\ &= -2xU_{k-1}(x) + U_{k-2}(x) = -U_k(x) = U_{(k+2)-2}(x). \end{aligned}$$

With this extension, X_N^T can be represented as

$$X_N^T = \begin{bmatrix} U_{n-1}(x) & U_{n-2}(x) & \dots & U_{-1}(x) \\ -U_{n-2}(x) & -U_{n-3}(x) & \dots & -U_{-2}(x) \end{bmatrix} =: \begin{pmatrix} U_{n-1-(i-1)}(x) \\ -U_{n-2-(i-1)}(x) \end{pmatrix}_{i=1,2,\dots,n+1}.$$

Proposition 2. *For Chebyshev polynomials of the second kind, there holds*

$$U_n(x)U_{n-i}(x) - U_{n+1}(x)U_{n-1-i}(x) = U_i(x).$$

Proof. From $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$, we get

$$\begin{aligned} U_{n+1}(x)U_{n-1-i}(x) &= \big(2xU_n(x) - U_{n-1}(x)\big)U_{n-1-i}(x) \\ &= 2xU_{n-1-i}(x)U_n(x) - U_{n-1}(x)U_{n-1-i}(x) \\ &= U_n(x)\big(U_{n-i}(x) + U_{n-2-i}(x)\big) - U_{n-1}(x)U_{n-1-i}(x) \end{aligned}$$

and

$$\begin{aligned} U_n(x)U_{n-i}(x) - U_{n+1}(x)U_{n-1-i}(x) &= U_{n-1}(x)U_{n-1-i}(x) - U_n(x)U_{n-2-i}(x) \\ &= U_{n-1}(x)U_{n-1-i}(x) - U_{n+1-1}(x)U_{n-1-1-i}(x). \end{aligned}$$

Repeating this process, we obtain

$$U_n(x)U_{n-i}(x) - U_{n+1}(x)U_{n-1-i}(x) = U_{n-k}(x)U_{n-k-i}(x) - U_{n+1-k}(x)U_{n-1-k-i}(x)$$

where $k = 0, \pm 1, \pm 2, \dots$ Setting $k = n - 1 - i$, we get finally

$$\begin{aligned} U_n(x)U_{n-i}(x) - U_{n+1}(x)U_{n-1-i}(x) &= U_{i+1}(x)U_1(x) - U_{i+2}(x)U_0(x) \\ &= 2xU_{i+1}(x) - U_{i+2}(x) = U_i(x). \end{aligned}$$

□

Now, let us define the matrices

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ H_p &= X_N P_1 X_N^T. \end{aligned} \tag{17}$$

Based on Proposition 2, the matrix H_p can be determined.

Corollary 1. *The matrix H_p defined in (17) is given by*

$$H_p = \begin{bmatrix} 0 & U_0(x) & U_1(x) & \dots & U_{n-1}(x) \\ -U_0(x) & 0 & U_0(x) & \dots & U_{n-2}(x) \\ -U_1(x) & -U_0(x) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & U_0(x) \\ -U_{n-1}(x) & -U_{n-2}(x) & \dots & -U_0(x) & 0 \end{bmatrix}.$$

Proof. The (l, m)-th element of H_p is

$$\begin{aligned} (H_p)_{lm} &= U_{n-2-(l-1)}(x)U_{n-1-(m-1)}(x) - U_{n-1-(l-1)}(x)U_{n-2-(m-1)}(x) \\ &= U_{n-l-1}(x)U_{n-l-1-(m-l-1)}(x) - U_{n-l}(x)U_{n-l-2-(m-l-1)}(x) \\ &= U_{m-l-1}(x) \qquad l, m = 1, 2, \dots, n+1. \end{aligned}$$

The last equation follows from Proposition 2. The equality $(H_p)_{ml} = -(H_p)_{lm}$ follows from (16). □

Note that H_p is a skew-symmetric Toeplitz matrix, and $PH_p = -H_pP$, where P is the rotation matrix such that $\mathbf{a}^TP = [a_0\ a_1\ \dots\ a_n]$ for all $\mathbf{a} = [a_n\ a_{n-1}\ \dots\ a_0]^T$, i.e. the elements on the secondary diagonal of P are all equal to one, and the other elements are all zero.

We are now in a position to determine $\deg[p(x)]$ and $\deg[q(x)]$.

Theorem 1. $p(x) = \sum_{k=0}^{n-1}(n-k)U_k^2(x)$, *and* $q(x) = \|H_p\mathbf{a}\|_2^2$. *Hence*

$$\deg[p(x)] = \deg[q(x)] = 2 \cdot \deg[U_{n-1}(x)] = 2(n-1).$$

Proof. It is clear that

$$\begin{aligned} p(x) &= \mathbf{x}_n^T\mathbf{x}_n\mathbf{x}_{n+1}^T\mathbf{x}_{n+1} - \mathbf{x}_n^T\mathbf{x}_{n+1}\mathbf{x}_n^T\mathbf{x}_{n+1} \\ &= \mathbf{x}_n^T(\mathbf{x}_n\mathbf{x}_{n+1}^T - \mathbf{x}_{n+1}\mathbf{x}_n^T)\mathbf{x}_{n+1} \\ &= \mathbf{x}_n^TX_NP_1X_N^T\mathbf{x}_{n+1} = \mathbf{x}_n^TH_p\mathbf{x}_{n+1}, \\ q(x) &= \mathbf{a}^TX_N \cdot \operatorname{adj} G(\mathbf{x}_n, \mathbf{x}_{n+1}) \cdot X_N^T\mathbf{a} \\ &= \mathbf{a}^TX_N \begin{bmatrix} \mathbf{x}_{n+1}^T\mathbf{x}_{n+1} & -\mathbf{x}_{n+1}^T\mathbf{x}_n \\ -\mathbf{x}_n^T\mathbf{x}_{n+1} & \mathbf{x}_n^T\mathbf{x}_n \end{bmatrix} X_N^T\mathbf{a} \\ &= \mathbf{a}^T(X_NP_1X_N^T)^T(X_NP_1X_N^T)\mathbf{a} \\ &= \|H_p\mathbf{a}\|_2^2. \end{aligned} \tag{18}$$

Consider (18). From Corollary 1 follows

$$\mathbf{x}_n^TH_p\mathbf{x}_{n+1} = \sum_{k=0}^{n-1}\left(U_k(x) \cdot \sum_{i=1}^{n-k}(x_iy_{i+k+1} - x_{i+k+1}y_i)\right)$$

where x_i and y_l are, respectively, the i-th and the l-th entry of $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$. Since $x_i = U_{n-i}(x)$, $y_l = -U_{n-1-l}(x)$,

$$\begin{aligned} x_iy_{i+k+1} - x_{i+k+1}y_i &= -U_{n-i}(x)U_{n-1-i-k-1}(x) + U_{n-i-k-1}(x)U_{n-1-i}(x) \\ &= U_m(x)U_{m-k}(x) - U_{m+1}(x)U_{m-1-k}(x) = U_k(x). \end{aligned}$$

Hence,

$$\begin{aligned} p(x) &= \sum_{k=0}^{n-1}\left(U_k(x)\sum_{i=1}^{n-k}U_k(x)\right) \\ &= \sum_{k=0}^{n-1}(n-k)U_k^2(x). \end{aligned}$$

□

Remark. The above equation gives a recursive form for $p(x)$. Indeed, if we rewrite the distance function $r_{d_3}(x)$ of a polynomial $\varphi_0(z)$ of degree n as

$$p_{d_3}^2(x) = \frac{q^{(n)}(x)}{p^{(n)}(x)}$$

then

$$p^{(n)}(x) = p^{(n-1)}(x) + \|\mathbf{x}_n\|_2^2.$$

In the rest of this section, we shall find an orthogonal basis for $\mathcal{X}_N$ by choosing a suitable $V(x)$. Let us first introduce the 2×2 matrix $U_{i,i+k}$ composed of the i- and $(i+k)$-th columns of X_N^T:

$$U_{i,i+k} = \begin{bmatrix} U_{n-1-(i-1)}(x) & U_{n-1-(i+k-1)}(x) \\ -U_{n-2-(i-1)}(x) & -U_{n-2-(i+k-1)}(x) \end{bmatrix} \tag{19}$$

where $k = 1, 2, \ldots, n+1-i$. The following result can be also obtained using Proposition 2.

Corollary 2. *For $k = 1, 2, \ldots, n+1-i$, $\det(U_{i,i+k}) = U_{k-1}(x)$, and*

$$(U_{i,i+k}P_1)\cdot(P_1U_{i,i+k})^T = U_{k-1}(x)\cdot I_2.$$

Proof. The first part follows directly from Proposition 2, since

$$\begin{aligned} \det(U_{i,i+k}) &= U_{n-1-(i+k-1)}(x)U_{n-2-(i-1)}(x) - U_{n-1-(i-1)}(x)U_{n-2-(i+k-1)}(x) \\ &= U_{n-i-1-(k-1)}(x)U_{n-i-1}(x) - U_{n-i}(x)U_{n-i-2-(k-1)}(x). \end{aligned}$$

To prove the second part, we consider the matrix product $AP_1A^TP_1^T$ for any 2×2 matrix A. Obviously, $P_1A^TP_1^T$ is nothing else the adjoint matrix of A. Hence, $AP_1A^TP_1^T = \det(A)\cdot I_2$.□

Denote by $\mathbf{h}_i$ the i-th column vector of the matrix H_p, and define

$$\begin{aligned} \mathbf{h} &= \begin{cases} \frac{1}{2}\left(\mathbf{h}_{\frac{n+1}{2}} - \mathbf{h}_{\frac{n+1}{2}+1}\right) & n = \text{odd} \\ \frac{1}{2}\left(\mathbf{h}_{\frac{n}{2}+2} - \mathbf{h}_{\frac{n}{2}}\right) & n = \text{even}, \end{cases} \\ \mathbf{g} &= \begin{cases} \frac{1}{2}\left(\mathbf{h}_{\frac{n+1}{2}} + \mathbf{h}_{\frac{n+1}{2}+1}\right) & n = \text{odd} \\ \frac{1}{2x}\left(\mathbf{h}_{\frac{n}{2}+2} + \mathbf{h}_{\frac{n}{2}}\right) & n = \text{even}. \end{cases} \end{aligned} \tag{20}$$

Proposition 3. *For n = even, we have*

$$\begin{aligned} \mathbf{h} &= \left[T_{\frac{n}{2}}(x)\ T_{\frac{n}{2}-1}(x)\ \ldots\ T_0(x)\ \ldots\ T_{\frac{n}{2}-1}(x)\ T_{\frac{n}{2}}(x)\right]^T, \\ \mathbf{g} &= \left[U_{\frac{n}{2}-1}(x)\ U_{\frac{n}{2}-2}(x)\ \ldots\ U_{-1}(x)\ \ldots\ -U_{\frac{n}{2}-2}(x)\ -U_{\frac{n}{2}-1}(x)\right]^T \end{aligned}$$

where $T_k(x)$ is the Chebyshev polynomial of the first kind: $T_k(\cos\theta) = \cos k\theta$.

Proof. It is clear that the i-th element of $\mathbf{g}$, denoted by $(\mathbf{g})_i$, is given by

$$\begin{aligned}(\mathbf{g})_i &= \frac{U_{\frac{n}{2}-2-(i-1)}(x) + U_{\frac{n}{2}-(i-1)}(x)}{2x} \\ &= U_{\frac{n}{2}-1-(i-1)}(x).\end{aligned}$$

Further, we claim that

$$xU_{k+1}(x) - U_k(x) = T_{k+2}(x). \tag{21}$$

Hence, the i-th element of $\mathbf{h}$ is

$$\begin{aligned}(\mathbf{h})_i &= \frac{U_{\frac{n}{2}-(i-1)}(x) - U_{\frac{n}{2}-2-(i-1)}(x)}{2} \\ &= \frac{U_{\frac{n}{2}-(i-1)}(x) + U_{\frac{n}{2}-2-(i-1)}(x) - 2U_{\frac{n}{2}-2-(i-1)}(x)}{2} \\ &= \frac{2\left(xU_{\frac{n}{2}-1-(i-1)}(x) - U_{\frac{n}{2}-2-(i-1)}(x)\right)}{2} = T_{\frac{n}{2}-(i-1)}(x).\end{aligned} \tag{22}$$

To verify the claim, it suffices to show

$$U_k(x) - T_k(x) = xU_{k-1}(x).$$

Indeed, from

$$\begin{aligned}&U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \\ &T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1\end{aligned}$$

we see that

$$U_1(x) - T_1(x) = x = xU_0(x) \quad \text{and} \quad U_2(x) - T_2(x) = 2x^2 = xU_1(x).$$

Let us assume that (21) holds for $k-1$ and k, i.e.

$$U_{k-1}(x) - T_{k-1}(x) = xU_{k-2}(x) \quad \text{and} \quad U_k(x) - T_k(x) = xU_{k-1}(x).$$

Then, for $k+1$, we get

$$\begin{aligned}U_{k+1} - T_{k+1} &= 2xU_k(x) - U_{k-1}(x) - \left(2xT_k(x) - T_{k-1}(x)\right) \\ &= 2x\left(U_k(x) - T_k(x)\right) - \left(U_{k-1}(x) - T_{k-1}(x)\right) \\ &= x\left(2xU_{k-1}(x) - U_{k-2}(x)\right) = xU_k(x).\end{aligned}$$

The proof is thus completed. □

Equipped with the notations above, we are now in a position to find a Pythagoras form for $r_{d_3}^2(x)$.

Theorem 2. *The vectors* $\mathbf{h}$ *and* $\mathbf{g}$ *defined in (20) form an orthogonal basis for* $\mathcal{X}_N$. *Hence,*

$$r_{d_3}^2(x) = \left(\frac{\langle \mathbf{h}, \mathbf{a}\rangle}{\|\mathbf{h}\|}\right)^2 + \left(\frac{\langle \mathbf{g}, \mathbf{a}\rangle}{\|\mathbf{g}\|}\right)^2.$$

Proof. We prove only the first part of the theorem, since the second part then follows directly.

Let us first complete the proof for $n =$ odd. In this case, we get from (17), (19), and (20)

$$\begin{aligned}[\mathbf{h}\ \ \mathbf{g}] &= \left[\mathbf{h}_{\frac{n+1}{2}}\ \ \mathbf{h}_{\frac{n+1}{2}+1}\right] V_1 \\ &= \left[-\mathbf{x}_{n+1}\ \ \mathbf{x}_n\right] U_{\frac{n+1}{2},\frac{n+1}{2}+1} V_1\end{aligned} \tag{23}$$

where

$$V_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

From Corollary 2, we get $\det\left[U_{\frac{n+1}{2},\frac{n+1}{2}+1}\right] = U_0(x) \equiv 1$. Hence, the matrix $U_{\frac{n+1}{2},\frac{n+1}{2}+1}V_1$ is always nonsingular, and $[\mathbf{h}\ \ \mathbf{g}]$ forms a basis for $\mathcal{X}_N$. It remains then only to show that $\mathbf{h}$ and $\mathbf{g}$ are orthogonal. From Corollary 1, we get

$$P\left[\mathbf{h}_i\ \ \mathbf{h}_{n+1-(i-1)}\right] = -\left[\mathbf{h}_{n+1-(i-1)}\ \ \mathbf{h}_i\right] \tag{24}$$

for $j = 1, 2, \ldots, \frac{n+1}{2}$. Hence, $\mathbf{h}_{\frac{n+1}{2}+1} = -P\mathbf{h}_{\frac{n+1}{2}}$. From (23) we get further

$$\mathbf{h} = \frac{I+P}{2}\mathbf{h}_{\frac{n+1}{2}}, \qquad \mathbf{g} = \frac{I-P}{2}\mathbf{h}_{\frac{n+1}{2}}.$$

Since $(I+P)(I-P) = 0$, $\mathbf{h}^T\mathbf{g} = \frac{1}{4}\mathbf{h}^T_{\frac{n+1}{2}}(I+P)(I-P)\mathbf{h}_{\frac{n+1}{2}} = 0$. $\mathbf{h}$ and $\mathbf{g}$ are orthogonal.

The proof for $n =$ even can be completed in the same way, except that we have to show that the matrix $U_{\frac{n}{2},\frac{n}{2}+2}V_2$, where

$$V_2 = \frac{1}{2}\begin{pmatrix} -1 & \frac{1}{x} \\ 1 & \frac{1}{x} \end{pmatrix}$$

is nonsingular for all x since

$$\begin{aligned}[\mathbf{h}\ \ \mathbf{g}] &= \left[\mathbf{h}_{\frac{n}{2}}\ \ \mathbf{h}_{\frac{n}{2}+2}\right] V_2 \\ &= \left[-\mathbf{x}_{n+1}\ \ \mathbf{x}_n\right] U_{\frac{n}{2},\frac{n}{2}+2} V_2.\end{aligned}$$

From (21) and the recursive form of the Chebyshev polynomials, we get

$$\begin{aligned}U_{\frac{n}{2},\frac{n}{2}+2}V_2 &= \frac{1}{2}\begin{bmatrix} U_{n-1-(\frac{n}{2}-1)}(x) & U_{n-1-(\frac{n}{2}+1)}(x) \\ -U_{n-2-(\frac{n}{2}-1)}(x) & -U_{n-2-(\frac{n}{2}+1)}(x)\end{bmatrix}\begin{bmatrix} -1 & \frac{1}{x} \\ 1 & \frac{1}{x}\end{bmatrix} \\ &= \begin{bmatrix} -T_{\frac{n}{2}}(x) & U_{\frac{n}{2}-1}(x) \\ T_{\frac{n}{2}-1}(x) & -U_{\frac{n}{2}-2}(x)\end{bmatrix}.\end{aligned}$$

Hence, the matrix $U_{\frac{n}{2},\frac{n}{2}+2}V_2$ is well-defined for all x. To show that this matrix is also nonsingular for all x, we just recall Corollary 2. Then, from $\det(V_2) = -\frac{1}{2x}$, we get $\det\left[U_{\frac{n}{2},\frac{n}{2}+2}V_2\right] = -1 \neq 0$. This completes the proof. □

Remark. For $n =$ odd, we have chosen $V(x) = P_1 U_{\frac{n+1}{2},\frac{n+1}{2}+1}V_1$, while for $n =$ even, $V(x) = P_1 U_{\frac{n}{2},\frac{n}{2}+2}V_2$. In both cases, $V(x)$ are unimodular polynomial matrices. Hence, $V(x)$ is nonsingular for all x.

4. Conclusion

In this paper, we have done the following. First, we have determined the degrees of the polynomials $p(x)$ and $q(x)$. This result is useful in numerically computating the minimum of $r_{d_3}^2(x)$. Further, using the basis vectors $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ given in [5] and a 2×2 nonsingular matrix $V(x)$, we have found an orthogonal basis for $\mathcal{X}_N$ and hence a Pythagoras form for $r_{d_3}^2(x)$. This is the counterpart of the result in [1] for discrete-time systems.

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