Reliable Computing 1 (4) (1995), pp. 403–410

Verification methods for inclusion disks

LJILJANA D. PETKOVIĆ and MIROSLAV TRAJKOVIĆ

In circular complex arithmetic a problem of finding an including circular approximation of some complex-valued range often arises. In this paper different approaches to verify the enclosure are considered. For each method an example is incorporated.

Методы верификации для дисков включения

Л. Д. ПЕТКОВИЧ, М. ТРАЙКОВИЧ

В круговой комплексной арифметике часто возникает проблема нахождения включающей круговой аппроксимации некоторой комплексно-значной области. Обсуждаются различные подходы к верификации таких включений. Для каждого из методов приведен пример.

Introduction

Let $Z = \{z : |z - \zeta| \le r\}$ be a disk in the complex plane with the center $\zeta = \operatorname{mid} Z \in \mathbb{C}$ and the radius $r = \operatorname{rad} Z > 0$, denoted shorter by parametric notation $Z = \{\zeta; r\}$. If f is a closed complex function then the complex-valued set $f(Z) = \{f(z) : z \in Z\}$ is closed. In general, the range $f(Z)$ is not a disk, which is quite impractical in calculations. In order to remain in the realm of disks, it is convenient to introduce a circular including approximation, denoted by $I\big(f(Z)\big)$, which completely includes the range $f(Z)$, that is, $I\big(f(Z)\big) \supseteq f(Z)$. The disk $I\big(f(Z)\big)$ is called a *circular including approximation*, or shorter, *I-approximation*. The practical point is to find an as good as possible I-approximation for given f and Z.

Many authors have studied I-approximations, especially for elementary functions, polynomials, rational functions and analytic functions. For a given disk $Z = \{\zeta; r\}$ and a function f let us assume that we have found (using some useful technique or an assumption based on geometrical construction) a disk $\{c; R\}$. Then the following important question arises:

Does disk $\{c; R\}$ completely contain the complex-valued range $f(Z)$, that is, is the enclosing condition

$$|f(z) - c| \le R \quad (z \in Z) \tag{1}$$

valid?

The checking of the inequality (1) is often very difficult. In some cases a suitable approach or technique can solve this problem. This paper is devoted to some methods for the verification of the enclosing condition (1). We present several such methods, together with illustrative examples (for demonstration).

In the sequel, an inclusion disk of the form

$$I_c\big(f(Z)\big) = \{f(\zeta); R\} \supseteq f(Z)$$

whose center is the image of the center of the domain Z, will be called the *centered form* for $f(Z)$. Obviously, among all inclusion disks with the centered form, the best enclosure is attained by the disk with the radius

$$R = R_0 = \max_{z \in Z} |f(z) - f(\zeta)| = \max_{z \in \Gamma} |f(z) - f(\zeta)|$$

where Γ is the contour of disk Z. This is the so-called *optimal centered form*, denoted by $I_o\big(f(Z)\big)$.

Evidently, the best I-approximation to the closed range $f(Z)$ would be a disk with the diameter equal to the diameter

$$d = \operatorname{diam}\{f(Z)\} = \max_{z_1, z_2 \in Z} |f(z_1) - f(z_2)| \tag{2}$$

of the range $f(Z)$ under the condition that this disk contains completely $f(Z)$. As it was proved in [5], if such a disk exists then it is unique and its center c (say) is the mean of the diametrical segment lines. This disk is called the *diametrical including approximation* or *D-form* for $f(Z)$, denoted by $I_d\big(f(Z)\big) = \{c; d/2\}$ (see [4]). The enclosing condition is given by the inequality

$$|f(z) - c| \leq \frac{d}{2} \quad (z \in Z). \tag{3}$$

Inclusion disks $I_c\big(f(Z)\big)$ and $I_d\big(f(Z)\big)$ with centered and diametrical form are shown in Figure 1.

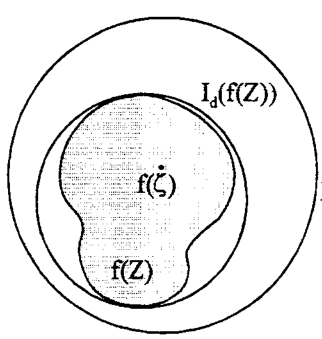

Figure 1. Centered and diametrical disks for the range $f(Z)$

1. Method of extremum

We find the maximum of the function $|f(z) - c|$, $z \in Z$, and show (according to (1)) that it is not greater than the radius R of the possible inclusion disk $\{c; R\}$.

Example 1. Optimal centered form for the range Z^n, $n \in \mathbb{N}$.

Let Γ be the contour of a given disk $Z = \{\zeta; r\}$, $\zeta \neq 0$ (the case $\zeta = 0$ is trivial). Then an arbitrary point $z \in Z$ is given by

$$z = \zeta + re^{it} = \zeta(1 + pe^{i(t-\varphi)}) = \zeta(1 + pe^{i\theta})$$

where

$$p = \frac{r}{|\zeta|}, \quad \varphi = \arg(\zeta), \quad \theta \in [0, 2\pi).$$

Than for $f(z) = z^n$ we find the radius R_0 of the optimal centered form:

$$\begin{aligned} R_0 &= \max_{z\in\Gamma} |f(z) - f(\zeta)| = \max_{z\in\Gamma} |z^n - \zeta^n| = |\zeta^n| \max_{\theta} |(1 + pe^{i\theta})^n - 1| \\ &= |\zeta^n| \max_{\theta} \left| \sum_{k=0}^{n} \binom{n}{k} p^k e^{ik\theta} - 1 \right| = |\zeta^n| \max_{\theta} \left| \sum_{k=1}^{n} \binom{n}{k} p^k e^{ik\theta} \right| \\ &= |\zeta^n| \sum_{k=1}^{n} \binom{n}{k} p^k = |\zeta|^n [(1+p)^n - 1]. \end{aligned}$$

2. Circular arithmetic

Since the absolute value of a disk W is defined as $|W| := |\operatorname{mid} W| + \operatorname{rad} W$, we find I approximation $I\big(f(Z) - c\big)$ and reduce the enclosing condition (1) to the inequality

$$|f(z) - c|_{z\in Z} \le \max_{z\in Z} |f(z) - c| \le \left|\operatorname{mid} I\big(f(Z) - c\big)\right| + \operatorname{rad} I\big(f(Z) - c\big) \le R. \tag{4}$$

Example 2. Diametrical form for $\log Z$.

Let $0 \notin Z = \{\zeta; r\}$ and let $p = r/|\zeta| < 1$. Using circular arithmetic the following inequality has been derived in [8]:

$$|\log z_1 - \log z_2| \le \log \frac{1+p}{1-p} \tag{5}$$

for all $z_1, z_2 \in Z$. Therefore, the diameter of the range $\log Z$ is given by

$$d = \operatorname{diam}\{\log Z\} = \max_{z_1, z_2 \in Z} |\log z_1 - \log z_2| = \log \frac{1+p}{1-p} = \log \frac{|\zeta| + r}{|\zeta| - r}.$$

The boundary of the range $\log Z$ is centrally symmetrical in reference to the two mutually perpendicular axes which are parallel to the real and imaginary axes (see Börsken [2]). Hence, we can conclude that the equality in (5) appears for $z_1^* = \zeta + re^{i\arg\zeta}$ and $z_2^* = \zeta - re^{i\arg\zeta}$, which means that these points lie on the diametrical segment line. As mentioned previously, if a diametrical disk exists (that is, if the enclosing condition (3) is valid), then its center is the mean of the diametrical segment lines. Therefore, the possible center of the diametrical disk for $\log Z$ must be the point determined by

$$A = \frac{\log z_1^* + \log z_2^*}{2} = \log \sqrt{|\zeta|^2 - r^2} + i \arg \zeta.$$

Thus, the disk $\{A; d/2\}$ will be the diametrical disk for the *principal-value* range $\log Z$ if the enclosing condition

$$|\log v - A| =\le \frac{d}{2} = \frac{1}{2} \log \frac{1+p}{1-p} \qquad (p = r/|\zeta|) \tag{6}$$

is fulfilled for all $v \in Z$.

Let $v = \zeta + re^{i\theta}$, $\theta \in [0, 2\pi)$ be an arbitrary point belonging to the disk $Z = \{\zeta; r\}$. Then we have

$$\begin{aligned} |\log v - A| &= \left|\log(\zeta + re^{i\theta}) - \log\sqrt{|\zeta|^2 - r^2} - i\arg\zeta\right|_{\theta\in[0,2\pi)} \\ &= \left|\log(1 + pe^{i\varphi}) - \frac{1}{2}\log(1-p^2)\right|_{\varphi=\theta-\arg\zeta} \\ &= \left|\int_0^p \frac{e^{i\varphi}}{1+te^{i\varphi}}\,dt - \int_0^p \frac{t}{1-t^2}\,dt\right| \\ &\le \int_0^p \left|\frac{e^{i\varphi}}{1+te^{i\varphi}} + \frac{t}{1-t^2}\right| dt. \end{aligned}$$

By circular arithmetic and (4) we estimate

$$\begin{aligned} \left|\frac{e^{i\varphi}}{1+te^{i\varphi}} + \frac{t}{1-t^2}\right| &= \left|\frac{1}{t}\left(1 - \frac{1}{1+te^{i\varphi}}\right) + \frac{t}{1-t^2}\right| \\ &\le \left|\frac{1}{t}\left(1 - \frac{1}{\{1;t\}}\right) + \frac{t}{1-t^2}\right| \\ &= \left|\frac{1}{t(1-t^2)} - \frac{\{1;t\}}{t(1-t^2)}\right| = \left|\frac{\{0;t\}}{t(1-t^2)}\right| = \frac{1}{1-t^2} \end{aligned}$$

so that

$$|\log v - A| \le \int_0^p \frac{1}{1-t^2}\,dt = \frac{1}{2}\log\frac{1+p}{1-p}.$$

Therefore, the inequality (6) is valid so that the disk

$$\{A; d/2\} = I_d(\log Z) = \left\{\log\sqrt{|\zeta|^2 - r^2} + i\arg\zeta; \frac{1}{2}\log\frac{|\zeta| + r}{|\zeta| - r}\right\}$$

is the *diametrical disk* for the range $\log\{\zeta; r\}$, $|\zeta| > r$.

3. Method of normals

In some examples the following theorem, established by the authors (see [8]), can be useful:

Theorem 1. *A disk D will completely contain the region G with the boundary Γ_G if all points belonging to the boundary Γ_G whose normals pass through the center of the disk D lie inside the disk D.*

Example 3. Diametrical form for the range $Z^{1/m}$, $0 \notin Z$, $m \in \mathbb{N}$.

Let $Z = \{\zeta; r\}$ and let us assume that the disk Z does not contain the origin, that is, $p := r/|\zeta| < 1$. As it was shown in [6], the construction of the diametrical disk for the range $Z^{1/m}$ reduces to the following problem:

Let $D = \{u_0; d/2\}$ be the disk with the center

$$u_0 := \frac{(1+p)^{1/m} + (1-p)^{1/m}}{2}$$

and the radius

$$r_0 := \frac{d}{2} = \frac{(1+p)^{1/m} - (1-p)^{1/m}}{2}$$

and let G be the image (one of m) of the disk $\{1;p\}$, $p \in (0,1)$, under $w = z^{1/m}$ with $\arg z^{1/m} \in [-\frac{1}{m}\arcsin p, \frac{1}{m}\arcsin p]$. *The question arises whether G is completely contained inside the disk D.*

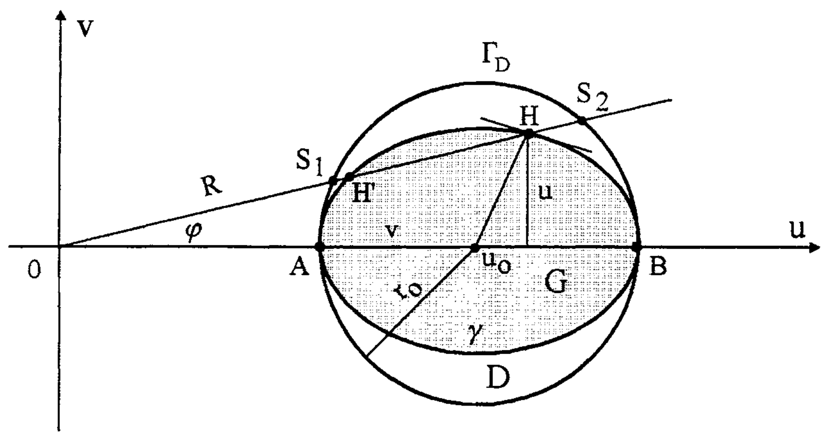

Figure 2.

Let $H(u,v)$ be an arbitrary point on the contour γ whose normal passes through the center u_0 of the disk D. From Figure 2 we can find the modulus of the radius vectors $\overrightarrow{OH'}$ and $\overrightarrow{OH}$,

$$R_{H,H'} = u_0 \cos\varphi \pm u_0 \frac{\sin\varphi}{\sin m\varphi}\sqrt{p^2 - \sin^2 m\varphi}. \tag{7}$$

Using the polar coordinate system (R,φ) the contour Γ_D of the disk D can be written as

$$R^2 - 2Ru_0\cos\varphi + (1-p^2)^{1/m} = 0. \tag{8}$$

Let S_1 and S_2 be the points of intersection of the circle Γ_D and the straight line $\overline{OH}$. Then from (8) we find the distances R_{S_1} and R_{S_2} of the intersection points S_1 and S_2 (see Figure 2) from the origin,

$$R_{S_2,S_1} = u_0\cos\varphi \pm \sqrt{u_0^2\cos^2\varphi - (1-p^2)^{1/m}}. \tag{9}$$

To prove that disk D contains the region G, according to the Theorem 1 it is necessary and sufficient that the point H belong to the interior of Γ_G, denoted by $\operatorname{int}\Gamma_D$. From Figure 2 we see that $H \in \operatorname{int}\Gamma_D$ if and only if the inequalities $R_{S_1} \le R_{H'}$ and $R_H \le R_{S_2}$ hold. Taking into account (7) and (9), the two last inequalities reduce to the inequality

$$u_0\frac{\sin\varphi}{\sin m\varphi}\sqrt{p^2 - \sin^2 m\varphi} \le \sqrt{u_0^2\cos^2\varphi - (1-p^2)^{1/m}}.$$

After a short rearrangement and some manipulations presented in [8], this inequality becomes

$$(1+p)^{1/m} - (1-p)^{1/m} \ge \left[(1+p)^{1/m} + (1-p)^{1/m}\right]\sin\left(\frac{\arcsin p}{m}\right).$$

This inequality was proved in [8]; therefore, the disk D completely contains the closed region G.

4. Analytical inequalities

In some suitable cases the inequality (1) can be proved directly using some particular inequalities in the complex plane.

Example 4. Optimal centered form for the range e^Z.

As mentioned above

$$z = \zeta + re^{it} = \zeta + z_1, \quad z_1 = re^{it} \in \Omega := \{v : |v| = r\}.$$

We have

$$\begin{aligned}
\max_{z\in\Gamma} |f(z) - f(\zeta)| &= \max_{z\in\Gamma} |e^z - e^\zeta| = \max_{z_1\in\Omega} |e^{\zeta+z_1} - e^\zeta| \\
&= |e^\zeta| \max_{z_1\in\Omega} |e^{z_1} - 1| = |e^\zeta| \max_{z_1\in\Omega} \left|\sum_{k=0}^{\infty} \frac{1}{k!} z_1^k - 1\right| \\
&= |e^\zeta| \max_{z_1\in\Omega} \left|\sum_{k=1}^{\infty} \frac{1}{k!} z_1^k\right| \leq |e^\zeta| \sum_{k=1}^{\infty} \frac{1}{k!} |z_1|^k \\
&= |e^\zeta| \left(\sum_{k=0}^{\infty} \frac{1}{k!} r^k - 1\right) = |e^\zeta|(e^r - 1).
\end{aligned}$$

Since $|f(0) - f(\zeta)| = |e^\zeta|(e^r - 1)$, the above maximum is always reached at the point $z = 0$ so that we have *optimal* centered form $\{e^\zeta; |e^\zeta|(e^r - 1)\}$.

5. Method of curvature

We use the following Blaschke's result [2]:

Theorem 2. *If the curvature of the simple closed smooth boundary $w(\theta)$ of a region G is strictly positive and has exactly 2λ extreme points, then the contour $w(\theta)$ has at most 2λ intersections with any circle. Tangential intersections are counted as double intersections.*

Sometimes, Theorem 2 enables us to check (1) in an elegant and simple way proving that the curvature of the curve $w(\theta)$ is greater than the curvature of a possible inclusion disk.

Example 5. Diametrical centered form for $\log Z$:

Let $Z = \{1; p\}$ be a disk in z-plane, and let G be the image of disk Z under $w = \log z$. Obviously, boundary Γ_G of G is given by

$$w(\theta) = \log(1 + pe^{i\theta}), \quad \theta \in [0, 2\pi).$$

Let D denote the disk $|w - c| \leq R$ with

$$c = \frac{1}{2}\log(1 - p^2), \quad R = \frac{1}{2}\log\frac{1+p}{1-p}.$$

The mapping $w = \log z$ sends points $z = 1 \pm p$ to the points $w = \log(1 \pm p) = c \pm R$, so that G can not have diameter less than $2R$. We shall prove in the same manner as in [3], that disk D is diametrical form for $\log Z$.

Γ_G is tangential to the circle D in the points $c + R$. To prove that Γ_G lies inside D we compute its curvature.

The curvature k of the curve $w(\theta)$ in the complex plane is given by

$$k = \frac{Im(\bar{\dot{w}}\ddot{w})}{|\dot{w}|^3}$$

where dots denote differentiation with respect to θ. For $w(\theta) = \log z(\theta)$ with $z(\theta) = 1 + pe^{i\theta}$, we compute

$$\dot{w}(\theta) = \frac{ipe^{i\theta}}{z(\theta)} \quad \text{and} \quad \ddot{w}(\theta) = -\frac{pe^{i\theta}}{z(\theta)^2}.$$

Hence

$$k(\theta) = \frac{1 + p\cos\theta}{p|z(\theta)|}$$

wherefrom we see that the curvature is strictly positive; therefore the domain G is strictly convex. Further, we compute

$$\dot{k}(\theta) = \frac{-p\sin\theta(p + \cos\theta)}{|z(\theta)|^3}.$$

We see that $\dot{k}(\theta)$ has precisely four simple zeros in $[0, 2\pi)$, at $\theta = 0, \pi$ and $\pm\arccos(-p)$. Since circle D is tangential to $w(\theta)$ in two points there are no more points of intersection. Hence $w(\theta)$ lies either completely inside or completely outside D.

It remains to show that in the point $c + R$ the curvature of $w(\theta)$ is greater than the curvature $1/R$ of D. Thus we want to show that $k(0) > \frac{1}{R}$, or

$$\log\frac{1+p}{1-p} > 2p \tag{10}$$

for $0 < p < 1$. Let $h(p) = \log\frac{1+p}{1-p} - 2p$. Since $h(0) = 0$ and $h'(p) = 2p^2/(1-p^2) > 0$, for $0 < p < 1$, we conclude that the inequality (10) holds and the proof is completed.

Regarding the domain $Z = \{\zeta; r\}$ with $p = r/|\zeta|$ it is easy to construct the diametrical disk for the range $\log\{\zeta; r\}$. Of course, the result is the same with that established in Example 2.

6. Conclusion

In this paper we have considered the problem of the complete covering of the exact range $f(Z)$ by a circular region (assumed or calculated) which leads to checking the enclosing condition (1). This is often a difficult problem and there is no a unified approach to solving it. For that reason we were forced to develope various methods for checking procedure. In this way, we have formed a base (certainly not complete) for solving problems of this kind. Which procedure should be employed depends on the form of the considered function f. One should also have in mind that for a given range $f(Z)$, inclusion disks are not all the same being of the centered form, diametrical form, etc (see Figure 1). So if one procedure is not successful we can still try with another.

References

[1] Blaschke, W. *Kreis und Kugel*, 2. Auflage, De Gruyter, Berlin, 1956.

[2] Börsken, N. C. *Komplexe Kreis-Standardfunktionen*. Freiburger Intervall-Berichte **2** (1978).

[3] McCoy, T. L. and Kuijlaars, A. B. J. *Answer to a query concerning the mapping* $w = z^{1/m}$. Indagat. Math., N.S., **4** (4), pp. 479–481.

[4] Petković, Lj. D. *On some approximations by disks*. Doctoral dissertation, University of Kragujevac, 1984 (in Serbian).

[5] Petković, Lj. D. *On optimal including circular approximation for range of a complex exponential function*. ZAMM **72** (1992).

[6] Petković, Lj. D. and Petković, M. S. *On the k-th root in circular arithmetic*. Computing **33** (1984), pp. 27–35.

[7] Petković, Lj. D. and Petković, M. S. *On some applications of circula complex functions*. Computing **41** (1989), pp. 141–148.

[8] Petković, Lj. D. and Trajković, M. *On some optimal inclusion approximations by disks*. Interval Computations 1 (1993).

Received: March 9, 1994
Revised version: January 24, 1995

Lj. D. Petković
Faculty of Mechanical Engineering
University of Niš
18000 Niš
Yugoslavia

M. Trajković
Faculty of Electronic Engineering
University of Niš
18000 Niš
Yugoslavia