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On Validated Newton Type Method for Nonlinear Equations

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Considered is an iterative procedure of Newton type for a nonlinear equation $f(x) = 0$ in a given interval X_0. Global quadratic convergence of the method is proved assuming that f' is Lipschitzian. An algorithm with result verification is constructed using computer interval arithmetic and some numerical experiments are reported.

Метод ньютоновского типа с верификацией для нелинейных уравнений

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Рассматривается итерационная процедура ньютоновского типа для нелинейного уравнения $f(x) = 0$ на заданном интервале X_0. Доказана глобальная квадратическая сходимость метода в предположении, что f' липшицева. Построен алгоритм с верификацией результата, использующий компьютерную интервальную арифметику, и представлены результаты численных экспериментов.

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1 Introduction

Let X_0 be a real compact interval and $f \in C^1[X_0]$. Denote by $x_1^*, x_2^*, \ldots, x_p^*$ the set of all real zeroes of $f(x)$ in X_0, i.e. $x_i^* \in X_0$, $i = 1, 2, \ldots, p$, and let $X^* \subset X_0$ be the shortest interval enclosing the set of all real zeroes x_i^*, $i = 1, 2, \ldots, p$. R. Krawczyk [5] formulates the following Newton type method for finding X^*:

$$\begin{aligned} \overline{x}_{k+1} &= \overline{x}_k - f(\overline{x}_k)/\overline{y}_k, \quad \overline{y}_k = \begin{cases} \sup\limits_{x\in X_k} \big(f'(x)\big) & \text{if } f(\overline{x}_k) \geq 0, \\ \inf\limits_{x\in X_k} \big(f'(x)\big) & \text{if } f(\overline{x}_k) < 0; \end{cases} \\ \underline{x}_{k+1} &= \underline{x}_k - f(\underline{x}_k)/\underline{y}_k, \quad \underline{y}_k = \begin{cases} \inf\limits_{x\in X_k} \big(f'(x)\big) & \text{if } f(\underline{x}_k) \geq 0, \\ \sup\limits_{x\in X_k} \big(f'(x)\big) & \text{if } f(\underline{x}_k) < 0 \end{cases} \end{aligned} \tag{1}$$

where $k = 0, 1, 2, \ldots$. The iteration process terminates if for some integer $k = m$ one of the following five conditions is fulfilled:

$$\begin{array}{llll} (i) & f(\overline{x}_m) > 0 & \text{and} & \overline{y}_m \leq 0; \\ (ii) & f(\overline{x}_m) < 0 & \text{and} & \overline{y}_m \geq 0; \\ (iii) & f(\underline{x}_m) > 0 & \text{and} & \underline{y}_m \geq 0; \\ (iv) & f(\underline{x}_m) < 0 & \text{and} & \underline{y}_m \leq 0; \\ (v) & \underline{x}_m \leq \overline{x}_m & \text{and} & \underline{x}_{m+1} > \overline{x}_{m+1}. \end{array} \tag{2}$$

The first four conditions (i)–(iv) mean that f is monotone on X_m and the range $f(X_m) = \{f(x) : x \in X_m\}$ of f on X_m does not contain zero.

The iteration scheme (1) will be further briefly denoted by $X_{k+1} = \mathbf{n}(X_k)$ and the interval operator $\mathbf{n}$ will be refered as Newton-Krawczyk operator. The iterations (1) generate a (finite or infinite) interval sequence $\{X_k\}$ which is inclusion isotone $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$. If the process terminates according to (2) after m steps, then the delivered interval X_m (and therefore X_0) does not contain any zero of $f(x)$. In the case when (1) generates an infinite sequence of intervals $\{X_k\}$, the latter converges to the interval $X^* = [\underline{x}^*, \overline{x}^*]$ such that $x_i^* \in X^*$, $i = 1, 2, \ldots, p$, and $\underline{x}^* = \min_k x_k^*$, $\overline{x}^* = \max_k x_k^*$. Krawczyk notices also that in the case of one simple zero $x^* \in X_0$ the convergence toward x^* is quadratic whenever f'' exists and is bounded in X_0. A correspondimg algorithm with result verification has been formulated in Triplex-ALGOL 60 form (see [5], pp. 361–362).

In this work we further investigate method (1) and the Newton-Krawczyk operator $\mathbf{n}$. We show that in the case when f is monotone the operator $\mathbf{n}(X)$ can be presented in extended interval arithmetic (see [4, 8]) by the simple expression $\mathbf{n}(X) = X -^- f(X) \ /^- f'(X)$, wherein $-^-$, $/^-$ are the alternative (nonstandard) interval operations for subtraction and division. We show various properties of the operator $\mathbf{n}(X)$ keeping in our investigations much to interval algebraic notations and computations. It is shown in [8] that under certain conditions on f and f' the interval operator $\mathbf{n}(X)$ is the range of the real Newton's operator $\mathbf{n}(x) = x - f(x)/f'(x)$ for $x \in X$, i.e. $\mathbf{n}(X) = \{\mathbf{n}(x) : x \in X\}$. This presentation clearly shows that method (1) does not involve intersection as most interval Newton-like methods do (see e.g. [1]). In Section 2 we consider some properties of the Newton-Krawczyk operator for monotone functions. In Section 3 a new method (10) for enclosing the set of all real zeroes of the equation $f(x) = 0$ in a given interval is proposed, which is a modification of Krawczyk's method (1). On the other side our method (10) is a generalization of a method of the form $X_{k+1} = \mathbf{n}(X_k)$ which has been studied in [3, 8]. Global convergence of (10) is proved and global quadratic convergence of (10) in the sense of [2] is shown in the special case when f is monotone and f' is Lipschitzian. In Section 4 we formulate an algorithm with result verification for enlosing the set of all real zeroes in an initial interval, using computer arithmetic operations. Some numerical experiments are given in Section 5.

2 The Newton-Krawczyk operator for monotone functions

Let $f : D \to R$, be a real valued function defined in $D \subseteq R$. Denote $ID = \{X : X \in IR, X \subseteq D\}$. The function f generates an interval function $f_R : ID \to IR$, defined for $X \in ID$ by $f_R(X) = \{f(x) : x \in X\}$, called the range of f. If no confusion occurs f_R will be again denoted by f.

Definition [10]. *An interval function $F : ID \to IR$ is called an (inclusion isotone) interval extension of f if $f(x) = F([x, x])$ for $x \in X$, $X \in ID$ and $F(X) \subseteq F(Y)$ whenever $X \subseteq Y$, $X, Y \in ID$.*

It follows from the inclusion isotonicity of F that $f(X) \subseteq F(X)$ for $X \in ID$ (see [10]).

Throughout this section we assume that f possesses a continuous derivative f' in D which has a constant sign in D, i.e. $f'(x) \neq 0$ for all $x \in D$. Since f is assumed monotone on D, we have $f(X) = [f(\underline{x}) \vee f(\overline{x})]$ for $X = [\underline{x}, \overline{x}]$. Similarly, $f'(X) = \{f'(x) : x \in X\}$ will denote the range of the derivative f' on X. Denote by $\mathbf{n} : ID \to IR$ the interval-arithmetic operator [8]

$$\mathbf{n}(X) = X \ -^{-} \ f(X) \ /^{-} \ f'(X). \tag{3}$$

Theorem 1. *If f is monotone then (3) is equivalent to the Newton-Krawczyk operator defined by* (1).

The proof follows from the definitions of the nonstandard interval-arithmetic operation $-^{-}$ and $/^{-}$ (see Appendix). □

In what follows we make use of the functionals ω and χ as usually defined in interval analysis [1, 8, 10, 12] (see also Appendix). We also make use of five simple Propositions (Proposition 1 to Proposition 5) given in the Appendix.

Lemma 1. *If $0 \in f(X)$ then $\omega(X) \geq \omega\big(f(X) \ /^{-} \ F'(X)\big)$ holds true, where F' is an interval extension of f' satisfying*

$$0 \notin F'(X) \text{ for } X \in ID. \tag{4}$$

Proof. From the definition of the nonstandard division $/^{-}$ we obtain

$$\begin{aligned} f(X) \ /^{-} \ F'(X) &= [f(\underline{x}) \vee f(\overline{x})]/^{-}[F'^{+0}(X) \vee F'^{-0}(X)] \\ &= [f(\underline{x})/F'^{-0}(X), \ f(\overline{x})/F'^{-0}(X)], \\ \omega\big(f(X) \ /^{-} \ F'(X)\big) &= |f(\overline{x}) - f(\underline{x})| \ / \ |F'^{-0}(X)| \\ &= \big(|f'(\xi)| \ / \ |F'^{-0}(X)|\big)\omega(X) \text{ for } \xi \in (\underline{x}, \overline{x}). \end{aligned}$$

Since $f'(\xi) \in F'(X)$, $|f'(\xi)| \leq |F'^{-0}(X)|$ holds true and the above relation implies $\omega\big(f(X) \ /^{-} \ F'(X)\big) \leq \omega(X)$ which completes the proof. □

Let F' be any interval extension of f', satisfying (4) and let $\mathcal{N} : ID \to IR$ be the interval-arithmetic operator

$$\mathcal{N}(X) = X \ -^{-} \ f(X) \ /^{-} \ F'(X). \tag{5}$$

Corollary 1. *For $X \in ID$ the following inclusions hold*:

(a) $\mathcal{N}(X) \supseteq \mathbf{n}(X)$ if

$$\chi\big(f(X)\big) \leq \min\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\},\ \omega(X) \geq \omega\big(f(X) \,/^-\, f'(X)\big)$$

or $\chi\big(f(X)\big) \geq \max\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\},\ \omega(X) \leq \omega\big(f(X) \,/^-\, f'(X)\big)$;

(b) $\mathcal{N}(X) \subseteq \mathbf{n}(X)$ if

$$\chi\big(f(X)\big) \leq \min\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\},\ \omega(X) \leq \omega\big(f(X) \,/^-\, f'(X)\big)$$

or $\chi\big(f(X)\big) \geq \max\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\},\ \omega(X) \geq \omega\big(f(X) \,/^-\, f'(X)\big)$.

Proof. Using Proposition 2 (for all Propositions referred below see Appendix) we obtain

$$\begin{aligned} f(X) \,/^-\, f'(X) &\subseteq f(X) \,/^-\, F'(X) \\ &\quad \text{if } \chi\big(f(X)\big) \geq \max\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\}, \\ f(X) \,/^-\, f'(X) &\supseteq f(X) \,/^-\, F'(X) \\ &\quad \text{if } \chi\big(f(X)\big) \leq \min\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\}. \end{aligned}$$

From Proposition 1 we obtain the proof. In particular when $0 \in f(X)$, $\chi\big(f(X)\big) \leq 0 < \min\Big\{\chi\big(f'(X)\big), \chi\big(F'(X)\big)\Big\}$ holds, Lemma 1 implies $\omega(X) \geq \omega\big(f(X) \,/^-\, F'(X)\big)$ thus we have in this case $\mathbf{n}(X) \subseteq \mathcal{N}(X)$. □

Theorem 2. *Let $f : D \to R$, $D \subseteq R$, be a continuously differentiable function on D. Let $f(X)$ be the range of f on X and F' be an interval extension of the derivative f', which satisfies* (4). *Then for any $X \in ID$ the relation $\mathcal{N}(X) \not\supset X$ holds.*

Proof. Let $X \in ID$ be such that $0 \in f(X)$. From the definitions of the operations $-^-$, $/^-$, and Lemma 1 we obtain

$$\begin{aligned} \mathcal{N}(X) &= X \;-^-\; f(X) \,/^-\, F'(X) \\ &= [\underline{x}, \overline{x}] \;-^-\; [f(\underline{x})/F'^{-0}(X)\ ,\ f(\overline{x})/F'^{-0}(X)] \\ &= [\underline{x} - f(\underline{x})/F'^{-0}(X)\ ,\ \overline{x} - f(\overline{x})/F'^{-0}(X)]. \end{aligned}$$

Since $0 \in f(X)$ then $0 \in f(X) \,/^-\, F'(X)$, i.e. $f(\underline{x})/F'^{-0}(X) \leq 0$, $f(\overline{x})/F'^{-0}(X) \geq 0$ which implies $\mathcal{N}(X) \subseteq X$.

Let $X \in ID$ be such that $0 \notin f(X)$. Applying Proposition 5 (c) with $A = X$, $B = f(X)/^-F'(X)$ we obtain $\mathcal{N}(X) = X \;-^-\; f(X) \,/^-\, F'(X) \not\asymp X$, wherein $A \not\asymp B$ means either $A \not\subseteq B$ or $A \not\supseteq B$. □

Theorem 3. *Let the assumptions of Theorem 2 be fulfilled. Then* $\mathcal{N}(X) \subseteq X$ *is a necessary and sufficient condition for existence of an unique solution of* $f(x) = 0$ *in the interval* X*, i.e.* $N(X) \subseteq X$ *is equivalent to* $0 \in f(X)$*.*

Proof. If $0 \in f(X)$ then the inclusion $N(X) \subseteq X$ follows from the proof of Theorem 2.

Let $\mathcal{N}(X) \subseteq X$. Using Proposition 5 (a) with $A := X$ and $B := f(X)/^- F'(X)$ we obtain $0 \in f(X)/^- F'(X)$, i.e. $0 \in f(X)$. □

Corollary 2. *Under the assumptions of Theorem 2* $\mathcal{N}(X) \not\subseteq X$ *is a necessary and sufficient condition for nonexistence of a solution of* $f(x) = 0$ *in the interval* X*, i.e.* $\mathcal{N}(X) \not\subseteq X$ *is equivalent to* $0 \notin f(X)$*.*

Proof. It follows from the proof of Theorem 2 that $0 \notin f(X)$ implies $\mathcal{N}(X) \not\subseteq X$.

Let $\mathcal{N}(X) \not\subseteq X$ $\big($or equivalently $0 \notin \mathcal{N}(X) \; -^- \; X = \big(X -^- f(X)/^- F'(X)\big) -^- X\big)$. According to Proposition 5 (d) with $A = X$, $B = f(X)/^- F'(X)$ we have either $0 \notin f(X)$ or $\big(0 \in f(X)$ and $\omega(X) < \omega\big(f(X)/^- F'(X)\big)\big)$. If we assume $0 \in f(X)$ from Lemma 1 we obtain $\omega(X) \geq \omega\big(f(X)/^- F'(X)\big)$. This contradiction implies $0 \notin f(X)$. □

Theorem 4. *Let the assumptions of Theorem 2 hold true.*
(a) If $f(x^*) = 0$ *and* $x^* \in X$ *then* $x^* \in \mathcal{N}(X)$*;*
(b) If $f(x^*) = 0$ *and* $x^* \in X$ *then* $\mathcal{N}\big(\mathcal{N}(X)\big) \subseteq \mathcal{N}(X)$*;*
(c) $\mathcal{N}(X) = X$ *iff* $X = [x^*, x^*] = x^*$ *and* $f(x^*) = 0$*.*

Proof. (*a*) Let $f(x^*) = 0$ and $x^* \in X$, that is $0 \in f(X)$; Lemma 1 implies $\omega(X) \geq \omega\big(f(X)/^- F'(X)\big)$. Furthermore

$$\begin{aligned}\mathcal{N}(X) &= X \; -^- \; f(X) \; /^- \; F'(X) \\ &= [\underline{x} - f(\underline{x})/F'^{-0}(X), \overline{x} - f(\overline{x})/F'^{-0}(X)] \\ &= [\underline{\mathcal{N}}(X), \overline{\mathcal{N}}(X)].\end{aligned}$$

We have

$$\begin{aligned}\underline{\mathcal{N}}(X) - x^* &= \underline{x} - x^* - \big(f(\underline{x}) - f(x^*)\big)/F'^{-0}(X) \\ &= (\underline{x} - x^*) - (\underline{x} - x^*)f'(\xi)/F'^{-0}(X) \\ &= (\underline{x} - x^*)\big(1 - f'(\xi)/F'^{-0}(X)\big)\end{aligned}$$

wherein $\xi \in (\underline{x}, \overline{x})$. The inequalities $\underline{x} \leq x^*$ and $1 - f'(\xi)/F'^{-0}(X) \geq 0$ imply $\underline{\mathcal{N}}(X) \leq x^*$. Similarly, the inequality $\overline{\mathcal{N}}(X) \geq x^*$ can be proved. Therefore $x^* \in \mathcal{N}(X)$.

(b) Theorem 2 implies $\mathcal{N}(X) \subseteq X$. According to ($a$) we have $x^* \in \mathcal{N}(X)$, i.e. $0 \in f\big(\mathcal{N}(X)\big)$. Theorem 2 implies again $\mathcal{N}\big(\mathcal{N}(X)\big) \subseteq \mathcal{N}(X)$.

(c) Let $X = [x^*, x^*] = x^*$. We have $\mathcal{N}(X) = \mathcal{N}(x^*) = x^* - f(x^*)/f'(x^*) = x^* = X$. Suppose that $\mathcal{N}(X) = X$, i.e. $\mathcal{N}(X) -^- X = 0$. Proposition 4 implies $0 = \mathcal{N}(X) -^- X = -f(X) \;/^-\; F'(X)$, i.e. $f(X)/^- F'(X) = [0, 0] = 0$, $\omega\big(f(X)/^- F'(X)\big) = 0$, which means $\omega(X) = 0$ and $X = [x^*, x^*] = x^*$. □

3 A method for enclosing real zeroes

Let $f : D \to R$, $D \subseteq R$, be a continuously differentiable function. Denote by F' an interval extension of f' on D. In this section we first consider the method $X_{n+1} = \mathcal{N}(X_n)$ for enclosing one simple zero x^* in a given interval X_0. Global quadratic convergence toward x^* of the latter is proved in Theorem 5. Using the end-point presentation of $X_{n+1} = \mathcal{N}(X_n)$ we formulate a method (see (10)) for enclosing the set X^* of all zeroes in X_0. Global onvergence toward X^* of the last one is proved in Theorem 6.

Assume first that the derivative f' has a constant sign in D. Denote by $f(X)$ and $f'(X)$ the ranges of f and f' resp. Let the interval extension F' satisfies (4).

As Theorem 1 shows under the above assumptions method (1) can be written as

$$\begin{cases} X_0 \in ID, \\ X_{n+1} = \mathbf{n}(X_n), \quad n = 0, 1, \ldots \end{cases} \tag{6}$$

wherein $\mathbf{n}(X) = X -^- f(X)/^- f'(X)$. Method (6) has been studied in some detail in [3] and [8] and will not be discussed here.

Using the interval-arithmetic operator $\mathcal{N}$ defined by (5) we formulate the following generalization of (6):

$$\begin{cases} X_0 \in ID, \\ X_{n+1} = \mathcal{N}(X_n), \quad n = 0, 1, \ldots . \end{cases} \tag{7}$$

Theorem 5. *Let $f : D \to R$, $D \subseteq R$, be a continuously differentiable function on D, whose derivative f' has an interval extension F' satisfying* (4). *Then:*
(*a*) *If $\mathcal{N}(X_0) \nsubseteq X_0$, the equation does not possess any solution in X_0 and the iteration procedure* (7) *terminates after the first step*;
(*b*) *If $\mathcal{N}(X_0) \subseteq X_0$, the iteration procedure* (7) *produces a sequence of intervals $\{X_n\}$ with the following properties*:
(*i*) $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$;
(*ii*) $x^* \in X_n$ *for* $n = 1, 2, \ldots$ *and* $\lim_{n\to\infty} X_n = x^*$;
(*iii*) *If F' satisfies a Lipschitz condition in the sense of* [10] *with a constant $L > 0$, that is $\omega\big(F'(X)\big) \le L\omega(X)$ for all $X \in ID$ then $\omega(X_{n+1}) \le c\omega^2(X_n)$, $c > 0$ holds.*

Proof. As mentioned above the first part (a) of our statement follows from Corollary 2.

(b) Assume now that $\mathcal{N}(X_0) \subseteq X_0$, i.e. there is a solution $x^* \in X_0$ of $f(x) = 0$. We shall prove simultaneously (i) and the first part of (ii) by induction. By assumption $x^* \in X_0$. Theorem 3 implies $X_1 = \mathcal{N}(X_0) \subseteq X_0$. From Theorem 4($a$) it follows that $x^* \in \mathcal{N}(X_0) = X_1$.

Supposing $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_k$ and $x^* \in X_k$, we shall show that $X_k \supseteq X_{k+1}$ and $x^* \in X_{k+1}$. Since $X_{k+1} = \mathcal{N}(X_k)$ and $x^* \in X_k$, it follows from Theorem 4(a) $x^* \in X_{k+1}$. By assumption $X_{k-1} \supseteq X_k = \mathcal{N}(X_{k-1})$. From Theorem 4($b$) it follows that $\mathcal{N}\big(\mathcal{N}(X_{k-1})\big) \subseteq \mathcal{N}(X_{k-1})$, which is equivalent to $X_{k+1} \subseteq X_k$.

We have further

$$\begin{aligned}\omega(X_{n+1}) &= \omega(X_n) - \omega\big(f(X_n)/^{-}F'(X_n)\big) \\ &= \omega(X_n) - |f(\underline{x}_n) - f(\overline{x}_n)|/|F'^{-0}(X_n)| \\ &= \omega(X_n) - \big(|f'(\xi)|/|F'^{-0}(X_n)|\big)\omega(X_n) \\ &= \omega(X_n)\big(1 - |f'(\xi)|/|F'^{-0}(X_n)|\big) \qquad (8)\end{aligned}$$

wherein $\underline{x}_0 \le \underline{x}_n < \xi < \overline{x}_n \le \overline{x}_0$. Since $X_n \subseteq X_0$ and $F'(X_n) \subseteq F'(X_0)$ we have $|F'^{-0}(X_n)| \le |F'^{-0}(X_0)|$ and $|f'(\xi)| \ge |F'^{+0}(X_0)|$. It follows from (8)

$$\begin{aligned}\omega(X_{n+1}) &\le \big(1 - |F'^{+0}(X_0)|/|F'^{-0}(X_0)|\big)\omega(X_n) \qquad (9) \\ &= q\omega(X_n)\end{aligned}$$

where $q = 1 - |F'^{+0}(X_0)|/|F'^{-0}(X_0)|$, $0 < q < 1$. The inequality $\omega(X_{n+1}) \le q\omega(X_n)$ means $\lim_{n\to\infty} X_n = x^*$.

(*iii*) The quadratic convergence of the sequence $\{X_n\}$ remains to be shown. We have from (9)

$$\begin{aligned}\omega(X_{n+1}) &\leq \omega(X_n)\bigl(1 - |F'^{+0}(X_n)|/|F'^{-0}(X_n)|\bigr)\\ &= \omega(X_n)\bigl(|F'^{-0}(X_n)| - |F'^{+0}(X_n)|\bigr)/|F'^{-0}(X_n)|.\end{aligned}$$

Since $0 \notin F'(X_n)$ it follows $\omega\bigl(F'(X_n)\bigr) = |F'^{-0}(X_n)| - |F'^{+0}(X_n)|$ and therefore $\omega(X_{n+1}) \leq \omega(X_n)\omega\bigl(F'(X_n)\bigr)/|F'^{-0}(X_n)|$. According to our assumption, there is a constant $L > 0$, independent on n, such that $\omega\bigl(F'(X_n)\bigr) < L\omega(X_n)$ and

$$\begin{aligned}\omega(X_{n+1}) &\leq \bigl(L/|F'^{-0}(X_n)|\bigr)\omega^2(X_n)\\ &\leq \bigl(L/|F'^{+0}(X_0)|\bigr)\omega^2(X_n)\\ &= c\omega^2(X_n)\end{aligned}$$

wherein $c = L/|F'^{+0}(X_0)| > 0$. □

Assuming that the computational costs for $f(X)$ and $F'(X)$ are about the same, we obtain for the efficiency index of (7) in the sense of Ostrowski [11] *eff* $\{(7)\} = \sqrt{2} \approx 1.42$.

Under the above assumption on f and in the situation when $0 \in f(X_0)$, method (7) can be written end-point wise in the following manner:

$$\begin{cases} X_0 = [\underline{x}_0, \overline{x}_0] \in ID;\\ \underline{x}_{n+1} = \underline{x}_n - f(\underline{x}_n)/F'^{-0}(X_n),\\ \overline{x}_{n+1} = \overline{x}_n - f(\overline{x}_n)/F'^{-0}(X_n);\\ n = 0, 1, \ldots \end{cases}$$

or equivalently $\bigl($since $f(\underline{x}_n)f(\overline{x}_n) \leq 0\bigr)$

$$\begin{cases} X_0 = [\underline{x}_0, \overline{x}_0] \in ID;\\ \underline{x}_{n+1} = \underline{x}_n + |f(\underline{x}_n)|/|F'(X_n)|,\\ \overline{x}_{n+1} = \overline{x}_n - |f(\overline{x}_n)|/|F'(X_n)|;\\ n = 0, 1, \ldots . \end{cases} \tag{10}$$

Formulae (10) are defined independently on the sign of the product $f(\underline{x}_n)f(\overline{x}_n)$; they are also defined even if the condition $0 \notin F'(X_0)$ is violated. The process (10) is not defined if and only if $F'(X_0) = [0, 0]$ holds.

Obviously, (10) generates a sequence of intervals $\{X_n\}$ with $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$. Let $x_1^*, x_2^*, \ldots, x_p^*$ be the real zeroes of $f(x) = 0$ in the initial interval X_0. W.l.g. we can assume that $x_1^* < x_2^* < \cdots < x_p^*$. Denote $X^* = [x_1^*, x_p^*]$. The iteration sheme (10) can be considered as a generalization of (7) and also as a modification of method (1) for enclosing the set X^*.

Theorem 6. *Let $f : D \to R$, $D \subseteq R$, be a continuously differentable function in D and F' be an interval extension of f'. Let $X_0 \in ID$. Then:*
(a) If $f(x) = 0$ has (at least one) solution in the initial interval X_0, the iteration procedure (10) generates an infinite interval sequence $\{X_n\}$ with $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$; $X^* \subseteq X_n$ for all $n = 1, 2, \ldots$ and $lim_{n\to\infty} X_n = X^*$.
(b) If there is an index m such that $\underline{x}_m \le \overline{x}_m$ but $\underline{x}_{m+1} > \overline{x}_{m+1}$ holds then the equation $f(x) = 0$ does not possess any solution in X_0.

Proof. (a) We shall show that $X^* \subseteq X_n$ for all $n = 1, 2, \ldots$. Using the mean value theorem we have for $\xi \in (\underline{x}_0, \overline{x}_0)$

$$\begin{aligned} \underline{x}_1 &= \underline{x}_0 + |f(\underline{x}_0)|/|F'(X_0)| \\ &= \underline{x}_0 + |f(\underline{x}_0) - f(x_1^*)|/|F'(X_0)| \\ &= \underline{x}_0 + \big(|f'(\xi)|/|F'(X_0)|\big)|\underline{x}_0 - x_1^*| \\ &\le \underline{x}_0 + |\underline{x}_0 - x_1^*| \\ &= \underline{x}_0 - \underline{x}_0 + x_1^* = x_1^*. \end{aligned}$$

Similarly, $\overline{x}_1 \ge x_p^*$ can be proved. By induction we prove that $X^* \subseteq X_n$ for all $n = 1, 2, \ldots$. Obviously, the interval sequence $\{X_n\}$ converges to the interval X^*.

(b) Let there is an index m such that $\underline{x}_m \le \overline{x}_m$ but $\underline{x}_{m+1} > \overline{x}_{m+1}$ holds. From (10) for $n = m$ we obtain

$$0 < \underline{x}_{m+1} - \overline{x}_{m+1} = -\omega(X_m) + \big(|f(\underline{x}_m)| + |f(\overline{x}_m)|\big)/|F'(X_m)|$$

or equivalently

$$\omega(X_m) < \big(|f(\underline{x}_m)| + |f(\overline{x}_m)|\big)/|F'(X_m)|.$$

Suppose that there is an $x^* \in X_0$ such that $f(x^*) = 0$; then $x^* \in X_m$ and

$$\omega(X_m) \;<\; \big(|f(\underline{x}_m) - f(x^*)| + |f(\overline{x}_m) - f(x^*)|\big)/|F'(X_m)|$$

$$\begin{aligned} &= \big(|f'(\xi_1)|(x^* - \underline{x}_m) + |f'(\xi_2)|(\overline{x}_m - x^*)\big)/|F'(X_m)| \\ &\le \big(\max\{|f'(\xi_1)|, |f'(\xi_2)|\}/|F'(X_m)|\big)\omega(X_m) \\ &\le \omega(X_m) \end{aligned}$$

wherein $\xi_1 \in (\underline{x}_m, x^*)$, $\xi_2 \in (x^*, \overline{x}_m)$. The obtained contradiction proves the theorem. □

A computer implementation of the iteration procedure (10) will be considered in the next section.

4 An algorithm with result verification for enclosing the set of all real zeroes in a given interval

Let $f : D \to R$, $D \subseteq R$, be a continuously differentiable function on D and F' is an interval extension of f' on ID.

Let D_S be the set of all machine numbers contained in the domain D of f. Denote by $ID_S = \{I \in IS : I \subseteq D_S\}$ the set of all computer intervals, contained in ID. For $x \in D_S$ let $\Diamond f(x) = [\bigtriangledown f(x), \triangle f(x)]$ be the interval obtained by the computation of $f(x)$. For the sake of brevity we shall use the notation $\Diamond f(x) = [f^{+0}(x) \vee f^{-0}(x)]$. For $X \in ID_S$ let $\Diamond F'(X)$ be the computed interval for $F'(X)$.

Consider the following computer arithmetic procedure based on (10):

$$\begin{cases} X_0 = [\underline{x}_0, \overline{x}_0] \in ID_S; \\ \underline{x}_{n+1} = \underline{x}_n \,\overline{\bigtriangledown}\big(|f^{+0}(\underline{x}_n)| \bigtriangledown |\Diamond F'(X_n)|\big), \\ \overline{x}_{n+1} = \overline{x}_n \,\underline{\triangle}\big(|f^{+0}(\overline{x}_n)| \bigtriangledown |\Diamond F'(X_n)|\big); \\ n = 0, 1, 2, \ldots \ \textbf{until} \ (\underline{x}_{n+1} > \overline{x}_{n+1} \ \textbf{or} \ X_{n+1} = [\underline{x}_{n+1}, \overline{x}_{n+1}] \not\subset X_n). \end{cases} \tag{11}$$

Assume now that f is monotone on $X \in ID_S$, $X = [\underline{x}, \overline{x}]$. Let $f(X) = \{f(x) : x \in X\} = [f(\underline{x}) \vee f(\overline{x})] = [\underline{f}, \overline{f}]$ be the range of f on X. According to the definitions of $\Diamond$ and $\bigcirc$ (see Appendix) we have:

$$\Diamond f(X) = [\bigtriangledown \underline{f}, \triangle \overline{f}]; \quad \bigcirc f(X) = \begin{cases} [\triangle \underline{f}, \bigtriangledown \overline{f}] & \text{if } \triangle \underline{f} \le \bigtriangledown \overline{f}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Obviously $\bigcirc f(X) \subseteq f(X) \subseteq \Diamond f(X)$, $X \in ID_S$, holds true. Also the presentation $\Diamond f(X) = [\Diamond f(\underline{x}) \vee \Diamond f(\overline{x})]$, $\bigcirc f(X) = [\Diamond f(\underline{x}) \wedge \Diamond f(\overline{x})]$ is valid.

To the real interval-arithmetic operator $\mathcal{N}$ we connect a computer interval-arithmetic operator $\hat{\mathcal{N}} : ID_S \to IS$, defined by

$$\hat{\mathcal{N}}(X) = X\langle -^- \rangle \big(\bigcirc f(X)(/^-)\Diamond F'(X)\big).$$

According to relations (13) (see Appendix) the inclusion $\mathcal{N}(X) \subseteq \hat{\mathcal{N}}(X)$ holds for $X \in ID_S$. Using $\hat{\mathcal{N}}$ we formulate the following computer interval-arithmetic iteration method related to (7) for enclosing an unique real zero in X_0:

$$\begin{cases} X_0 \in ID_S; \\ X_{n+1} = \hat{\mathcal{N}}(X_n), \\ n = 0, 1, 2, \ldots \ \textbf{until} \ X_{n+1} \not\subset X_n. \end{cases} \tag{12}$$

Due to the finite convergence principle [10] the iteration procedure (11) or (12) produces a finite sequence $\{X_n\}$, such that for some k, $X_k = X_{k+l}$, $l = 1, 2, \ldots$, and $X_k \supset X^*$ holds.

The algorithm with result verification for enclosing the set of all real zeroes in a given interval X_0 based on (11) and (12) is presented below in a PASCAL-like form.

Algorithm ManyZeroes1

begin
  Compute $\Diamond F'(X_0)$;
  If $0 \notin \Diamond F'(X_0)$ **then goto** OneZero(X_0);
  else goto MoreZeroes;
OneZero(X):
  Compute $\bigcirc f(X) = [\Diamond f(\underline{x}) \wedge \Diamond f(\overline{x})]$;
  If $0 \notin \bigcirc f(X)$ **then**
    write(Message) and stop;
  else
    $X_1 := X\langle -^- \rangle \bigcirc f(X)\ (/^-)\ \Diamond F'(X)$;
    repeat
      $X := X_1$;

Compute $\bigcirc f(X)$ and $\Diamond F'(X)$;
$X_1 := X \langle -^{-} \rangle \bigcirc f(X)\ (/^{-})\ \Diamond F'(X)$;
until $X_1 \not\subset X$;
write(X) and stop;
MoreZeroes:
Compute $\Diamond f(\underline{x}_0)$, $\Diamond f(\overline{x}_0)$;
$\underline{x}_1 := \underline{x}_0 \mathbin{\triangledown_{+}} \big(|f^{+0}(\underline{x}_0)| \mathbin{\triangledown_{/}} |\Diamond F'(X_0)|\big)$,
$\overline{x}_1 := \overline{x}_0 \mathbin{\vartriangle_{-}} \big(|f^{+0}(\overline{x}_0)| \mathbin{\triangledown_{/}} |\Diamond F'(X_0)|\big)$;
If $\underline{x}_1 > \overline{x}_1$ **then write**(Message) and stop;
else
$X_1 := [\underline{x}_1, \overline{x}_1]$;
repeat
$X := X_1$;
Compute $\Diamond F'(X)$;
If $0 \notin \Diamond F'(X)$ **then goto** OneZero(X);
else
Compute $\Diamond f(\underline{x})$, $\Diamond f(\overline{x})$;
$\underline{x}_1 := \underline{x} \mathbin{\triangledown_{+}} \big(|f^{+0}(\underline{x})| \mathbin{\triangledown_{/}} |F'(X)|\big)$,
$\overline{x}_1 := \overline{x} \mathbin{\vartriangle_{-}} \big(|f^{+0}(\overline{x})| \mathbin{\triangledown_{/}} |F'(X)|\big)$;
If $\underline{x}_1 > \overline{x}_1$ **then write** (Message) and stop;
else $X_1 := [\underline{x}_1, \overline{x}_1]$;
until $X_1 \not\subset X$;
write(X);
end.
{ The resulting interval for the solution set is X. }
Message = '`The equation has no solution in the initial interval`'.

The interval with optimal roundings $\Diamond f(x) = [\triangledown f(x), \vartriangle f(x)]$, $x \in D_S$, is difficult to be computed in practice. With a slight modification the algorithm **ManyZeroes1** works with any (rough) roundings $\triangledown$, $\vartriangle$. In what follows by $\triangledown a$ resp. $\vartriangle a$ we mean any two numbers such that $\triangledown a \le a$ resp. $\vartriangle a \ge a$. We now have to do several checks. First we have to check whether $0 \in \Diamond f(\underline{x}_0)$, $0 \in \Diamond f(\overline{x}_0)$ hold. If both relations are obtained, then the algorithm can not determine existence/nonexistence of a solution in the initial interval. If both relations are obtained on some step n, then X_n is displayed as a resulting interval, containing the solution set. If at step n one of the above relations is valid, say $0 \in f(\underline{x}_n)$, but $0 \notin f(\overline{x}_n)$ holds, then we can expect improvement only at the endpoint $\overline{x}_n$ of the current iteration X_n

or maybe after several steps $\overline{x}_{n+m} > \underline{x}_{n+m}$ happens, i.e. the equation does not possess solutions in the initial interval.

For monotone functions the situation $0 \in \Diamond f(\underline{x})$ and/or $0 \in \Diamond f(\overline{x})$ is equivalent to $0 \notin \bigcirc f(X)$ or $\bigcirc f(X) = \emptyset$. If X is the initial interval, i.e. $X = X_0$, this does not necessarily mean that $f(x) = 0$ has no solution in X_0. The equation possesses no solution in X_0 if $0 \notin \Diamond f(X_0)$ and it has an unique root in X_0 if $0 \in \bigcirc f(X_0)$ holds. But if $0 \notin \bigcirc f(X_0)$ and $0 \in \Diamond f(X_0)$ are simultaneously true then one can not claim existence/nonexistence of a solution in the initial interval X_0. In this situation either another X_0 should be chosen or we should compute using higher precision. The same situation may occur on some step n, that is $0 \notin \bigcirc f(X_n)$ or $\bigcirc f(X_n) = \emptyset$. Further iterations are then useless even if X_n is not sufficiently small.

Using the "rough" roundings $\bigtriangledown a \le a$, $\bigtriangleup a \ge a$ we obtain a modified algorithm with result verification, presented below under the name **ManyZeroes2**.

Algorithm ManyZeroes2

begin
  Compute $\Diamond F'(X_0)$, $\Diamond f(\underline{x}_0)$, $\Diamond f(\overline{x}_0)$;
  If $0 \notin \Diamond F'(X_0)$ **then goto** OneZeroInitTest;
  If $0 \in \Diamond F'(X_0)$ **then**
    If $0 \in \Diamond f(\underline{x}_0)$, $0 \in \Diamond f(\overline{x}_0)$ **then write**(message 3) and stop;
    If $0 \in \Diamond f(\underline{x}_0)$, $0 \notin \Diamond f(\overline{x}_0)$ **then goto** RightEP(X_0);
    If $0 \notin \Diamond f(\underline{x}_0)$, $0 \in \Diamond f(\overline{x}_0)$ **then goto** LeftEP(X_0);
    If $0 \notin \Diamond f(\underline{x}_0)$, $0 \notin \Diamond f(\overline{x}_0)$ **then goto** LeftRightEP(X_0);
OneZeroInitTest:
  Compose $\bigcirc f(X_0)$, $\Diamond f(X_0)$;
  If $\bigcirc f(X_0) = \emptyset$ **then write** (message 1) and stop;
  If $0 \notin \Diamond f(X_0)$ **then**
    write (message 2) and stop;
  else
    If $0 \notin \bigcirc f(X_0)$ **then write** (message 3) and stop;
    else goto OneZero(X_0);
OneZero(X):
  $X_1 := X \langle -^- \rangle \big(\bigcirc f(X) \ (/^-) \ \Diamond F'(X) \big)$;
  repeat

$X := X_1$;
Compute $\bigcirc f(X)$;
If $\bigcirc f(X) = \emptyset$ **or** $0 \notin \bigcirc f(X)$ **then**
write(X + message 4) and stop;
else
Compute $\Diamond F'(X)$;
$X_1 := X \langle -^- \rangle \big(\bigcirc f(X) \ (/^-) \ \Diamond F'(X) \big)$;
until $X_1 \not\subset X$;
write(X) and stop;

RightEP(X):
$\overline{x}_1 := \overline{x} \triangle \big(|f^{+0}(\overline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;
If $\overline{x}_1 < \underline{x}$ **then write** (message 2) and stop;
else
repeat
$X := [\underline{x}, \overline{x}_1]$;
Compute $\Diamond F'(X)$, $\Diamond f(\overline{x})$;
If $0 \in \Diamond f(\overline{x})$ **then write**(X + message 4) and stop;
else
$\overline{x}_1 := \overline{x} \triangle \big(|f^{+0}(\overline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;
If $\overline{x}_1 < \underline{x}$ **then write**(message 2) and stop;
until $\overline{x}_1 \geq \overline{x}$;
write(X + message 4) and stop;

LeftEP(X):
$\underline{x}_1 := \underline{x} \bigtriangledown\!\!\!\!\!- \big(|f^{+0}(\underline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;
If $\underline{x}_1 > \overline{x}$ **then write** (message 2) and stop;
else
repeat
$X := [\underline{x}_1, \overline{x}]$;
Compute $\Diamond F'(X)$, $\Diamond f(\underline{x})$;
If $0 \in \Diamond f(\underline{x})$ **then write**(X + message 4) and stop;
else
$\underline{x}_1 := \underline{x} \bigtriangledown\!\!\!\!\!- \big(|f^{+0}(\underline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;
If $\underline{x}_1 > \overline{x}$ **then write**(message 2) and stop;
until $\underline{x}_1 \leq \underline{x}$.
write(X + message 4) and stop;

LeftRightEP(X):
$\underline{x}_1 := \underline{x} \bigtriangledown\!\!\!\!\!- \big(|f^{+0}(\underline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;
$\overline{x}_1 := \overline{x} \triangle \big(|f^{+0}(\overline{x})| \bigtriangledown |\Diamond F'(X)| \big)$;

If $\underline{x}_1 > \overline{x}_1$ **then write** (message 2) and stop;
else
  repeat
    $X := [\underline{x}_1, \overline{x}_1]$;
    Compute $\Diamond F'(X)$, $\Diamond f(\underline{x})$, $\Diamond f(\overline{x})$;
    If $0 \notin \Diamond F'(X)$ **then**
      Compose $\bigcirc f(X)$;
      If $\bigcirc f(X) = \emptyset$ **or** $0 \notin \bigcirc f(X)$ **then**
        write(X + message 4) and stop;
      else goto OneZero(X);
    else
      If $0 \in \Diamond f(\underline{x})$ and $0 \notin \Diamond f(\overline{x})$ **then goto** RightEP(X);
      If $0 \notin \Diamond f(\underline{x})$ and $0 \in \Diamond f(\overline{x})$ **then goto** LeftEP(X);
      If $0 \notin \Diamond f(\underline{x})$ and $0 \notin \Diamond f(\overline{x})$ **then**
        $\underline{x}_1 := \underline{x} \overline{\triangledown} \big(|f^{+0}(\underline{x})| \triangledown |\Diamond F'(X)|\big)$;
        $\overline{x}_1 := \overline{x} \underline{\triangle} \big(|f^{+0}(\overline{x})| \triangledown |\Diamond F'(X)|\big)$;
        If $\underline{x}_1 > \overline{x}_1$ **then write**(message 2) and stop;
        else $X_1 := [\underline{x}_1, \overline{x}_1]$;
  until $X_1 \not\subset X$;
write(X);
end.
{ The resulting interval for the solution set is X. }

Messages:

message 1 = '$\bigcirc f(X_0) = \emptyset$.
`The algorithm can not determine existence/nonexistence of a solution in the initial interval. Restart the algorithm with another initial interval.`'

message 2 = '`The equation has no solution in the initial interval.`'

message 3 = '`The algorithm can not determine existence/nonexistence of a solution in the initial interval. Restart the algorithm with another initial interval.`'

message 4 = '`The enclosing interval can not be made smaller in this precision.`'

5 Numerical experiments

The algorithm **ManyZeroes2** was applied to an example communicated to us by Prof. G. Corliss. A program was written in PASCAL–SC, where the operations of the extended interval arithmtic were simulated using the operator concept facilities of the language.

Example:

$$f(x) = a - xe^x$$

where a is a real parameter. For $a < -1/e$ the equation $f(x) = 0$ has no solution; for $a = -1/e$ it possesses one solution; if $-1/e < a < 0$ the equation has two solutions and it possesses one solution if $a \geq 0$.

Since the computations in PASCAL–SC are performed with 12 decimal digits we take the following interval for the constant $-1/e$:

$$-1/e \in [-0.367879441172,\ -0.367879441171].$$

(i) $a = -0.36$; $X_0 = [-2,\ -0.6]$.

The program displays

$$\begin{aligned} \diamondsuit F'(X_0) &= [-2.19524654438E-01,\ 5.48811636095E-01], \\ \diamondsuit f(\underline{x}_0) &= [-8.93294335280E-02,\ -8.93294335260E-02], \\ \diamondsuit f(\overline{x}_0) &= [-3.07130183436E-02,\ -3.07130183430E-02] \end{aligned}$$

and further

$$\begin{aligned} X_1 &= [-1.83723115975E+00,\ -6.55962768139E-01], \\ &\dots \\ X_{18} &= [-1.22277035031E+00,\ -8.06084315968E-01]. \end{aligned}$$

On this iteration we obtain

$$\begin{aligned} \diamondsuit f(\underline{x}_{18}) &= [-1.41888687880E-08,\ -1.41876460176E-08], \\ \diamondsuit f(\overline{x}_{18}) &= [-5.23274694912E-13,\ 2.82809621056E-13]. \end{aligned}$$

Since $0 \in \Diamond f(\overline{x}_{18})$, improvements only at the left end-point on the next steps are expected, thus

$$X_{19} = [-1.22277020771E+00,\ -8.06084315968E-01].$$

The final result is

$$X_{28} = [-1.22277013399E+00,\ -8.06084315968E-01]$$

with the message that it can not be made smaller in this precision.

(ii) $a = -0.36$; $X_0 = [-0.9,\ -0.6]$.

On the initial interval we obtain

$$\Diamond F'(X_0) = [-2.19524654438E-01,\ -4.0656969740E-02]$$

which means that the equation has at most one zero in X_0. Further,

$$\bigcirc f(X_0) = [-3.07130183430E-02,\ 5.91269376600E-03]$$

that is $0 \in \bigcirc f(X_0)$ and therefore the equation possesses an unique root in the initial interval. After five iterations we obtain

$$X_5 = [-8.06084328220E-01,\ -8.06084315964E-01].$$

On this iterate,

$$\bigcirc f(X_5) = [1.08564634116E-13,\ 1.06036819586E-09]$$

i.e. it does not contain zero. The final result is then X_5 with the message that it can not be made smaller in this precision.

(iii) $a = -0.36$; $X_0 = [-2,\ -1.1]$.

The following results are displayed:

$$\begin{aligned} \Diamond F'(X_0) &= [3.32871083699E-02,\ 1.353352283236E-02]; \\ \bigcirc f(X_0) &= [-8.93294335260E-02,\ 6.15819206760E-03] \end{aligned}$$

which means that the equation has an unique root in the initial interval. The enclosing interval for the solution is

$$X_6 = [-1.22277013398E+00,\ 1.22277013397E+00].$$

(*iv*) $a = -0.4$; $X_0 = [-2, 0]$.

For this initial interval we obtain

$$\begin{array}{lcl} \Diamond F'(X_0) & = & [-1.00000000000E+00,\ 1.00000000000E+00], \\ \dots & & \\ X_4 & = & [-1.14077776185E+00,\ -1.00935558364E+00], \\ \Diamond F'(X_4) & = & [3.40967767091E-03,\ 4.49884022148E-02], \\ \Diamond f(X_4) & = & [-3.54412222973E-02,\ -3.21365584470E-02] \end{array}$$

which means that the equation possesses no solutions in the initial interval.

(*v*) $a = -0.36787944117$; $X_0 = [-1.1,\ -0.9]$.

We obtain

$$\Diamond F'(X_0) = [-4.06569659741E-02,\ 4.06569659741E-02]$$

and further

$$X_{17} = [-1.00000299962E+00,\ -9.99996688607E-01].$$

On this iteration the following intervals are delivered:

$$\begin{array}{lcl} \Diamond f(\underline{x}_{17}) & = & [-7.80746316120E-13,\ 2.19256683500E-13], \\ \Diamond f(\overline{x}_{17}) & = & [-1.440259955684E-12,\ -4.40263268231E-13]. \end{array}$$

Since $0 \in \Diamond f(\underline{x}_{17})$, after two steps we obtain

$$\begin{array}{lcl} X_{19} & = & [-1.00000299962,\ -9.99997175387E-01], \\ & & \\ \Diamond f(\underline{x}_{19}) & = & [-7.80746316120E-13,\ 2.19256683500E-13], \\ \Diamond f(\overline{x}_{19}) & = & [-9.87070553157E-13,\ 1.29262223000E-13] \end{array}$$

i.e. $0 \in \Diamond f(\underline{x}_{19})$, $0 \in \Diamond f(\overline{x}_{19})$, so that the final result is X_{19}. It can not be done better in this precision.

(*vi*) $a = -0.367879441171$; $X_0 = [-1.1,\ -1.0000000001]$.

For this initial interval we obtain

$$\begin{array}{lcl} \Diamond F'(X_0) & = & [3.67879441135E-11,\ 3.32871083698E-02]; \\ \bigcirc f(X_0) & = & [-1.72124910210E-03,\ -2.12055886600E-13]; \\ \Diamond f(X_0) & = & [-1.72124910320E-03,\ 7.87944113500E-13] \end{array}$$

and the message

```
The algorithm can not determine existence/nonexistence
of a solution in the initial interval.  Restart the
algorithm with another initial interval.
```

(*vii*) $a = -0.367879441172$, $X_0 = [-2, 2]$.

We obtain

$$\begin{array}{lcl} X_{28} & = & [-1.00000111092E+00,\ -9.99999105122E-01], \\ \Diamond f(\underline{x}_{28}) & = & [-8.25229541960E-13,\ 1.74771568960E-13] \ni 0, \\ & & \\ X_{29} & = & [-1.00000111092E+00,\ -1.00000036075+00] \end{array}$$

but $\overline{x}_{30} = -1.00000168077E+00$, $\underline{x}_{30} = -1.00000111092E+00$, that is $\underline{x}_{30} > \overline{x}_{30}$, and the equation possesses no solutions in the initial interval.

(*viii*) $a = 3$; $X_0 = [-2, 2]$.

We obtain

$$\Diamond F'(X_0) = [-2.21671682969E+01,\ 7.38905609894E+00].$$

On the 4th step the following result is delivered:

$$\begin{array}{lcl} X_4 & = & [-4.89264623342E-01,\ 1.04995072006E+00], \\ \Diamond F'(X_4) & = & [-5.85775529022E+00,\ -3.13120148789E-01], \\ \bigcirc f(X_4) & = & [-2.44993546531E-04,\ 3.29995692223E+00]. \end{array}$$

This information means that the equation possesses one simple zero in the initial interval; the enclosing interval for the solution is

$$X_{11} = [1.04990889496E+00,\ 1.04990889497E+00].$$

Appendix. Basic concepts of extended interval arithmetic

Let IR be the set of all compact intervals on the real line R. Denote by $\underline{x}$ and $\overline{x}$, $\underline{x} \le \overline{x}$, the end-points of $X \in IR$, i.e. $X = [\underline{x}, \overline{x}]$. The width of X is defined by $\omega(X) = \overline{x} - \underline{x}$. The interval X with end-points x_1 and x_2 will be written as $X = [x_1 \vee x_2] = \{[x_1, x_2]$ if $x_1 \le x_2; [x_2, x_1]$ if $x_1 \ge x_2\}$. The notation $[x_1 \vee x_2]$ does not necessary require $x_1 \le x_2$. By x^{-0} and x^{+0} we denote the end-points

$$x^{+0} = \{\underline{x},\ \text{if } |\underline{x}| \le |\overline{x}|;\ \ \overline{x},\ \text{otherwise}\};$$
$$x^{-0} = \{\overline{x},\ \text{if } |\underline{x}| \le |\overline{x}|;\ \ \underline{x},\ \text{otherwise}\}$$

which satisfy $|x^{+0}| \le |x^{-0}|$. For $X = [x^{-0} \vee x^{+0}]$ the functional $\chi : IR \setminus [0,0] \to [-1,1]$ is defined as $\chi(X) = x^{+0}/x^{-0}$ (see [12]). For $X, Y \in IR$, $X = [\underline{x}, \overline{x}]$, $Y = [\underline{y}, \overline{y}]$ define the intervals

$$\begin{aligned} X \vee Y &= [\min\{\underline{x}, \underline{y}\}, \max\{\overline{x}, \overline{y}\}]; \\ X \wedge Y &= \begin{cases} [\min\{\overline{x}, \overline{y}\}, \max\{\underline{x}, \underline{y}\}] & \text{if } X \bigcap Y = \emptyset, \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

The interval-arithmetic operations in IR will be denoted by $+$, $-$, $\times$, $/$, $+^-$, $-^-$, $\times^-$, $/^-$, where the first four operations are the conventional ones [1, 2, 10] and the last four are the extended interval-arithmetic operations [3, 4, 7–9]. For $X, Y \in IR$, $X = [\underline{x}, \overline{x}] = [x^{+0} \vee x^{-0}]$, $Y = [\underline{y}, \overline{y}] = [y^{+0} \vee y^{-0}]$ we define:

$$\begin{aligned}
X + Y &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}]; \\
X - Y &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}]; \\
X \times Y &= \begin{cases} [x^{+0}y^{+0} \ \vee \ x^{-0}y^{-0}] & \text{if } 0 \notin X, Y, \\ y^{-0}X \ = \ [y^{-0}\underline{x} \ \vee \ y^{-0}\overline{x}] & \text{if } 0 \in X,\ 0 \notin Y; \end{cases} \\
X \,/\, Y &= \begin{cases} [x^{+0}/y^{-0} \ \vee \ x^{-0}/y^{+0}] & \text{if } 0 \notin X, Y, \\ X/y^{+0} \ = \ [\underline{x}/y^{+0} \ \vee \ \overline{x}/y^{+0}] & \text{if } 0 \in X,\ 0 \notin Y; \end{cases} \\
X +^- Y &= [\underline{x} + \overline{y} \ \vee \ \overline{x} + \underline{y}] \\
&= \begin{cases} [\underline{x} + \overline{y},\ \overline{x} + \underline{y}] & \text{if } \omega(X) \ge \omega(Y), \\ [\overline{x} + \underline{y},\ \underline{x} + \overline{y}] & \text{if } \omega(X) < \omega(Y); \end{cases} \\
X -^- Y &= [\underline{x} - \underline{y} \ \vee \ \overline{x} - \overline{y}] \\
&= \begin{cases} [\underline{x} - \underline{y},\ \overline{x} - \overline{y}] & \text{if } \omega(X) \ge \omega(Y), \\ [\overline{x} - \overline{y},\ \underline{x} - \underline{y}] & \text{if } \omega(X) < \omega(Y); \end{cases} \\
X \times^- Y &= \begin{cases} [x^{-0}y^{+0} \ \vee \ x^{+0}y^{-0}] & \text{if } 0 \notin X, Y, \\ y^{+0}X \ = \ [y^{+0}\underline{x} \ \vee \ y^{+0}\overline{x}] & \text{if } 0 \in X,\ 0 \notin Y; \end{cases} \\
X \,/^-\, Y &= \begin{cases} [x^{+0}/y^{+0} \ \vee \ x^{-0}/y^{-0}] & \text{if } 0 \notin X, Y, \\ X/y^{-0} \ = \ [\underline{x}/y^{-0} \ \vee \ \overline{x}/y^{-0}] & \text{if } 0 \in X,\ 0 \notin Y. \end{cases}
\end{aligned}$$

The conventional interval-arithmetic operations $+, -, \times, /$ ([1, 10]) are inclusion monotone in the sense that $X_1 \subseteq X$, $Y_1 \subseteq Y$ imply $X_1 * Y_1 \subseteq X * Y$ for any operation $* \in \{+, -, \times, /\}$. The nonstandard interval-arithmetic operations $+^-, -^-, \times^-, /^-$ are quasi-inclusion monotone in the sense of the following two propositions.

Proposition 1. *Let* $X, X_1, Y, Y_1 \in IR$, $X \supseteq X_1$, $Y \subseteq Y_1$, $* \in \{+^-, -^-\}$. *Then*
(a) $\max\{\omega(X), \omega(X_1)\} \le \min\{\omega(Y), \omega(Y_1)\}$ *implies* $X * Y \subseteq X_1 * Y_1$;
(b) $\min\{\omega(X), \omega(X_1)\} \ge \max\{\omega(Y), \omega(Y_1)\}$ *implies* $X * Y \supseteq X_1 * Y_1$.

Proposition 2. *Let* $X, X_1, Y, Y_1 \in IR$, $0 \notin Y, Y_1$, $X \supseteq X_1$, $Y \subseteq Y_1$, $* \in \{\times^-, /^-\}$. *Then*
(a) $\min\{\chi(X), \chi(X_1)\} \ge \max\{\chi(Y), \chi(Y_1)\}$ *implies* $X * Y \subseteq X_1 * Y_1$;
(b) $\max\{\chi(X), \chi(X_1)\} \le \min\{\chi(Y), \chi(Y_1)\}$ *implies* $X * Y \supseteq X_1 * Y_1$.

We omit the straightforward verification of the above two propositions.

Proposition 3 [9]. *For* $A, B, C, D \in IR$,

$$(A -^- B) -^- (C -^- D) = \begin{cases} (A -^- C) -^- (B -^- D) & \text{if } m_2 \ge 0, m_1 \ge 0; \\ (A -^- C) \;-\; (B -^- D) & \text{if } m_2 \ge 0, m_1 < 0; \\ (A \;-\; C) -^- (B \;-\; D) & \text{if } m_2 < 0 \end{cases}$$

where $m_1 = \big(\omega(A) - \omega(C)\big)\big(\omega(B) - \omega(D)\big)$, $m_2 = \big(\omega(A) - \omega(B)\big)\big(\omega(C) - \omega(D)\big)$.

For $A, B \in IR$ we write $A \asymp B$ if $A \subseteq B$ or $B \subseteq A$ holds true. In the opposite situation we shall write $A \not\asymp B$. The following two propositions show the connection between $\asymp$ and $-^-$. (Note that $0 \in A$, $0 \notin A$ are equivalent to $0 \asymp A$, $0 \not\asymp A$ resp.)

Proposition 4. *For* $A, B \in IR$, $A -^- B \asymp 0$ *if and only if* $A \asymp B$. *Alternatevely* $0 \not\asymp A -^- B$ *iff* $A \not\asymp B$.

Proposition 5. *Let* $A, B \in IR$.
(a) $A -^- B \asymp A$ *implies* $0 \in B$;
(b) *If* $0 \in B$ *and* $\omega(A) \ge \omega(B)$ *then* $A -^- B \asymp A$ *holds*;
(c) $0 \notin B$ *implies* $A -^- B \not\asymp A$;
(d) *If* $A -^- B \not\asymp A$ *then either* $0 \notin B$ *or* $\big(0 \in B$ *and* $\omega(A) < \omega(B)\big)$ *is fulfilled.*

Proof. According to Proposition 4 $A -^- B \asymp A$ is equivalent to $0 \in (A -^- B) -^- A$. Applying Proposition 3 to the difference $(A -^- B) -^- A$ with

$m_1 = \big(\omega(A) - \omega(A)\big)\omega(B) = 0$ and $m_2 = \big(\omega(A) - \omega(B)\big)\omega(A)$ we obtain

$$\begin{aligned}(A -^- B) -^- A &= \begin{cases} (A -^- A) -^- B & \text{if } m_2 \geq 0, \\ (A \; - \; A) -^- B & \text{otherwise;} \end{cases} \\ &= \begin{cases} -B & \text{if } \omega(A) \geq \omega(B), \\ [\,-\,\omega(A), \omega(A)] -^- B & \text{otherwise.} \end{cases}\end{aligned}$$

Let be first $\omega(A) \geq \omega(B)$. Then $0 \in (A -^- B) -^- A$ is equivalent to $0 \in B$ and $0 \notin B$ is equivalent to $0 \notin (A -^- B) -^- A$, that is $(A -^- B) \not\asymp A$, which proves (b).

Consider the case $\omega(A) < \omega(B)$. Then $(A -^- B) \asymp A$ is equivalent to $0 \in (A -^- B) -^- A = [-\omega(A), \omega(A)] -^- B$, that is to $[-\omega(A), \omega(A)] \asymp B$. There are two possibilities: (i) $[\,-\,\omega(A), \omega(A)] \subseteq B$, which leads to $0 \in B$; (ii) $[\,-\,\omega(A), \omega(A)] \supseteq B$, which together with the inequality $\omega(A) < \omega(B)$ implies $0 \in B$. This proves (a). Assume now that $0 \notin B = [\underline{b}, \overline{b}]$. This means $\underline{b}\overline{b} > 0$. We shall show that the product of the end-points of the interval $[-\omega(A), \omega(A)] -^- B \; = \; \Big[\big(-\,\omega(A) - \underline{b}\big) \;\vee\; \big(\omega(A) - \overline{b}\big)\Big]$ is positive. Indeed,

$$\begin{aligned}\big(-\,\omega(A) - \underline{b}\big)\big(\omega(A) - \overline{b}\big) &= -\omega^2(A) + \omega(A)\omega(B) + \underline{b}\overline{b} \\ &= \omega(A)\big(\omega(B) - \omega(A)\big) + \underline{b}\overline{b} > 0\end{aligned}$$

since $\omega(A) < \omega(B)$. The last inequality means $0 \notin [\,-\,\omega(A), \omega(A)] -^- B$, i.e. $0 \notin (A -^- B) -^- A$, which proves (c). Let $0 \notin (A -^- B) -^- A$. It follows then $[\,-\,\omega(A), \omega(A)] \not\asymp B$, which can mean $0 \in B$ or $0 \notin B$. □

Let S be a floating-point system [6] and IS be the set of intervals with end-points over S. The computer realization of algorithms written in interval-arithmetic form and using the operations of the extended interval arithmetic is discussed in detail in [3]. Two kinds of monotone roundings $\Diamond, \bigcirc : IR \to IS$ of intervals are used:

$$\Diamond A = [\,\triangledown\,\underline{a}, \triangle\overline{a}]; \;\; \bigcirc A = \begin{cases} [\,\triangle\,\underline{a}, \triangledown\overline{a}] & \text{if } \triangle\,\underline{a} \leq \triangledown\overline{a}, \\ \emptyset & \text{otherwise} \end{cases}$$

where $\triangledown a = \max\{x \in S : x \leq a\}$, $\triangle a = \min\{x \in S : x \geq a\}$. They generate the computer interval-arithmetic operations

$$A\langle * \rangle B = \Diamond(A * B), \quad A(*)B = \bigcirc(A * B)$$

where "$*$" can be any one of the interval-arithmetic operations defined above.

Using the quasi-inclusion properties of the operations $+^-$, $-^-$, $\times^-$ and $/^-$ (Propositions 1–2) we obtain the following inclusions for $A,\ B \in IR$ (see [3], Section 2):

$$\left\{\begin{array}{lllll}
\bigcirc A\,(+^-)\,\diamondsuit B & \subseteq\ A +^- B & \subseteq\ \diamondsuit A\langle +^-\rangle \bigcirc B & \text{if } \omega(A) \geq \omega(B), \\
\diamondsuit\, A\,(+^-) \bigcirc B & \subseteq\ A +^- B & \subseteq\ \bigcirc A\langle +^-\rangle\, \diamondsuit B & \text{if } \omega(A) < \omega(B); \\
\bigcirc A\,(-^-)\,\diamondsuit B & \subseteq\ A -^- B & \subseteq\ \diamondsuit A\langle -^-\rangle \bigcirc B & \text{if } \omega(A) \geq \omega(B), \\
\diamondsuit\, A\,(-^-) \bigcirc B & \subseteq\ A -^- B & \subseteq\ \bigcirc A\langle -^-\rangle\, \diamondsuit B & \text{if } \omega(A) < \omega(B); \\
\bigcirc A\,(\,\times^-)\,\diamondsuit B & \subseteq\ A\ \ \times^- B & \subseteq\ \diamondsuit A\,\langle\,\times^-\rangle \bigcirc B & \text{if } \chi(A) \leq \chi(B), \\
\diamondsuit\, A\,(\,\times^-) \bigcirc B & \subseteq\ A\ \ \times^- B & \subseteq\ \bigcirc A\,\langle\,\times^-\rangle\, \diamondsuit B & \text{if } \chi(A) > \chi(B); \\
\bigcirc A\,(\,/^-)\,\diamondsuit B & \subseteq\ A\ \ /^- B & \subseteq\ \diamondsuit A\,\langle\,/^-\rangle \bigcirc B & \text{if } \chi(A) \leq \chi(B), \\
\diamondsuit\, A\,(\,/^-) \bigcirc B & \subseteq\ A\ \ /^- B & \subseteq\ \bigcirc A\,\langle\,/^-\rangle\, \diamondsuit B & \text{if } \chi(A) > \chi(B).
\end{array}\right. \tag{13}$$

Aknowledgements

We are indepted to the referee who wanted us to discuss the relation between the Newton-Krawczyk operator defined by (1) and the operator (3). This relation is given by Theorem 1.

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