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Solving the Tolerance Problem for Interval Linear Systems

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For the interval linear system $\mathbf{A}x = \mathbf{b}$, we consider the *linear tolerance problem*, requiring inner evaluation of the *tolerable solution set* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \,|\, (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}$ formed by all point vectors x such that the product Ax remains within $\mathbf{b}$ for all possible $A \in \mathbf{A}$. We develop solvability theory for the linear tolerance problem that not only settles whether $\Sigma_{\forall\exists}$ is empty or not, but also enables modification of the problem to ensure its desired properties. The paper concludes with a survey of new methods for construction of an interval solution to the linear tolerance problem around a given center.

Решение задачи о допусках для интервальных линейных систем

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Для интервальной линейной системы $\mathbf{A}x = \mathbf{b}$ в работе рассматривается *линейная задача о допусках*, требующая внутреннего оценивания *допустимого множестваа решений* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \,|\, (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}$ — множества всех таких точек x, что произведение Ax попадает в $\mathbf{b}$ при любых возможных $A \in \mathbf{A}$. Мы развиваем теорию разрешимости линейной задачи о допусках, которая не только решает вопрос о пустоте или непустоте $\Sigma_{\forall\exists}$, но и позволяет скорректировать постановку так, чтобы обеспечить ее желаемые свойства. Работа завершается обзором новых методов для построения интервального решения линейной задачи о допусках вокруг известного центра.

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1 Introduction

Let the interval system of linear algebraic equations

$$\mathbf{A}x = \mathbf{b} \tag{1}$$

be given with an interval $m \times n$-matrix $\mathbf{A}$ and an interval right-hand side m-vector $\mathbf{b}$. The solution set to (1) has been defined in a variety of ways: aside from the (united) solution set

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\exists\, A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b)\,\}$$

commonly used in applications, there exists, for example, the *controlled solution set*

$$\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall\, b \in \mathbf{b})(\exists\, A \in \mathbf{A})(Ax = b)\,\}$$

(see [17]) among many others. But the subject matter of our paper will be the *tolerable solution set*, formed by all point vectors x such that the product Ax falls into $\mathbf{b}$ for any $A \in \mathbf{A}$, i.e., the set

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists\, b \in \mathbf{b})(Ax = b)\} \tag{2}$$

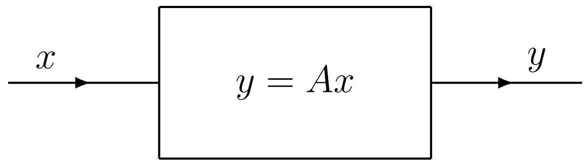
or, to put it otherwise, the set

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{\, x \in \mathbb{R}^n \mid \mathbf{A} \cdot x \subseteq \mathbf{b}\,\}$$

where " $\cdot$ " is the common interval martix multiplication.

Neumaier introduced the term *restricted solution set* in [7, 8], and other authors followed; they denote the set by $\Sigma_0(\mathbf{A}, \mathbf{b})$. Some researchers speak of "inner solutions," but I prefer a more explicit term, *tolerable*, the one used in Russian works. The history of the set (2) and of some related problems was described comprehensively in the papers by Neumaier [7] and by Kelling and Oelschlägel [4].

It is very instructive to consider the practical interpretation of the tolerable solution set. Let "the black box" be given with the input subjection vector $x \in \mathbb{R}^n$ and the output response vector $y \in \mathbb{R}^m$, where the input-output relationship is linear, i.e., $y = Ax$ with a real $m \times n$-matrix A.



Suppose that the parameters of the black box are not precisely known, but are given only by intervals $\mathbf{a}_{ij}$, $a_{ij} \in \mathbf{a}_{ij}$. For example, these parameters may vary in an unpredictable way (drift) within $\mathbf{a}_{ij}$, or interval uncertainty may be intrinsic to the very description of the mathematical model. Also assume that the set of the black box output states is specified as an interval vector $\mathbf{y}$ and we must ensure that we can arrive at y irrespective of the specific values of a_{ij} from $\mathbf{a}_{ij}$. Our interest is in finding input signals $\tilde{x}$ such that for any values of the parameters a_{ij} from $\mathbf{a}_{ij}$ we always get an output response within the required tolerances $\mathbf{y}$. The tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{y})$ is precisely the set of all such $\tilde{x}$'s. The above general scheme is known to be successfully applied to concrete problems in mathematical economics by Rohn [12], in automatic control by Khlebalin [5] and so on.

In general, $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is easily proved (in Section 3, for instance) to be a convex polyhedral set in $\mathbb{R}^n$. Nevertheless, if the dimension of the interval system is large, then the direct description of its tolerable solution set becomes laborious and practically useless (its complexity is proportional to $m \cdot 2^n$). For this reason it is expedient to confine ourselves to finding some simple subsets $\Pi \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, since for any $x \in \Pi$ the condition

$$(\forall A \in \mathbf{A})(Ax \in \mathbf{b})$$

remains valid. In other words, we replace $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ by its inner approximation, usually formulating the problem to be solved in the following form:

Find an interval vector that is contained in the tolerable solution set (if nonempty) of the interval linear system. (3)

This linear tolerance problem is the subject of the present paper.

Notice that the tolerable solution set may turn out to be empty even for "good" interval data, as, for instance, it does in the one-dimensional case $\mathbf{A} = [2, 3]$, $\mathbf{b} = [1, 2]$. The two-dimensional system

$$\begin{pmatrix} [1,2] & [-1,1] \\ [-1,1] & [1,2] \end{pmatrix} x = \begin{pmatrix} [1,3] \\ [1,3] \end{pmatrix} \tag{4}$$

gives a more complex example with an empty tolerable solution set. In such cases we shall say that the linear tolerance problem is *unsolvable* (incompatible).

The main mathematical results of our work are new techniques for the investigation of solvability of the linear tolerance problem as well as methods for inner approximation of the tolerable solution set. We designate intervals and other interval objects by bold typeface, int X means topological interior of the set X and in other respects our notation follows that of Neumaier [8]. Also, throughout this paper, all arithmetic operations with intervals and interval objects are those of the classical interval arithmetic (see e.g. [1, 3, 8]).

2 Rough solvability examination

First, note that if the i-th row of $\mathbf{A}$ contains only zero elements, it is necessary that $\mathbf{b}_i = 0$ for the tolerable solution set to be nonempty. If this condition holds, then the property of $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ being empty or nonempty depends upon the other, not the i-th, rows of $\mathbf{A}$ and components of $\mathbf{b}$. Thus, without loss of generality, we may assume in general (and in the rest of this paper) that $\mathbf{A}$ does not have zero rows.

To characterize "relative narrowness" of nonzero intervals, Ratschek [9] introduced the functional

$$\chi(\mathbf{p}) = \begin{cases} \underline{\mathbf{p}}/\overline{\mathbf{p}}, & \text{if } \ |\underline{\mathbf{p}}| \le |\overline{\mathbf{p}}|, \\ \overline{\mathbf{p}}/\underline{\mathbf{p}}, & \text{otherwise.} \end{cases}$$

Clearly, $-1 \le \chi(\mathbf{p}) \le 1$, and $\chi(\mathbf{p}) = 1$ if and only if $\mathbf{p} \in \mathbb{R}$. Moreover, it is proved in [9] that

$$\chi(\mathbf{p}) = \chi(\mathbf{q}) \ \text{ if and only if } \ \mathbf{p} = t\,\mathbf{q}, \ t \in \mathbb{R}, \ t \ne 0, \tag{5}$$

$$\text{if } \ \mathbf{p}+\mathbf{q} \ne 0, \ \text{then } \ \chi(\mathbf{p}+\mathbf{q}) \le \max\{\,\chi(\mathbf{p}), \chi(\mathbf{q})\,\}, \tag{6}$$

$$\text{if } \ \mathbf{p} \supseteq \mathbf{q} \ \text{ and } \ \chi(\mathbf{q}) \ge 0, \ \text{then } \ \chi(\mathbf{p}) \le \chi(\mathbf{q}). \tag{7}$$

Now we are able to formulate and prove

Theorem 1 [15]. *Let the interval $m \times n$-matrix $\mathbf{A}$ and interval m-vector $\mathbf{b}$ be such that for some $k \in \{1, 2, \ldots, m\}$ the following conditions hold:*

(i) $0 \notin \mathbf{b}_k$,

(ii) $\max\{\ \chi(\mathbf{a}_{kj}) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} \neq 0\,\} < \chi(\mathbf{b}_k)$.

Then the tolerable solution set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *is empty.*

For example, using this criterion, one can verify that the above mentioned one-dimensional system with $\mathbf{A} = [2, 3]$, $\mathbf{b} = [1, 2]$ has empty tolerable solution set.

Proof of the Theorem will be conducted *ad absurdum* employing a technique similar to that developed in [10]. Let us assume that the tolerance problem nonetheless has a solution $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, that is, $\mathbf{A}t \subseteq \mathbf{b}$, the condition (i) making it impossible for the interval $(\mathbf{A}t)_k$ to equal zero. Then the following inequalities are true:

$$
\begin{aligned}
\chi\big(\,(\mathbf{A}t)_k\big) &= \chi\left(\,\sum_{j=1}^{n} \mathbf{a}_{kj} t_j\right) \\
&\leq \max\{\ \chi(\mathbf{a}_{kj} t_j) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} t_j \neq 0\ \} \qquad \text{by (6)} \\
&= \max\{\ \chi(\mathbf{a}_{kj}) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} t_j \neq 0\,\} \qquad \text{by (5)} \\
&\leq \max\{\ \chi(\mathbf{a}_{kj}) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} \neq 0\,\}.
\end{aligned}
$$

We have found

$$
\chi\big(\,(\mathbf{A}t)_k\big) \leq \max\{\,\chi(\mathbf{a}_{kj}) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} \neq 0\,\}. \tag{8}
$$

On the other hand, by virtue of our assumption, $(\mathbf{A}t)_k \subseteq \mathbf{b}_k$ which because of (7) implies $\chi\big(\,(\mathbf{A}t)_k\,\big) \geq \chi(\mathbf{b}_k)$. Combining this with (8) now gives

$$
\max\{\ \chi(\mathbf{a}_{kj}) \mid 1 \leq j \leq n,\ \mathbf{a}_{kj} \neq 0\,\} \geq \chi(\mathbf{b}_k)
$$

which is contrary to (ii). □

The importance of all the conditions of the Theorem 1 may be exhibited on the one-dimensional example with $\mathbf{A} = [-1, 2]$, $\mathbf{b} = [-2, 6]$. Here $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = [-1, 2] \neq \emptyset$, though $\chi(\mathbf{A}) = -\frac{1}{2} < -\frac{1}{3} = \chi(\mathbf{b})$. At the same time, if the conditions of the Theorem 1 fail, this does not necessarily mean compatibility of the linear tolerance problem. For instance, (ii) is not true for the system (4), but even so its tolerable solution set is empty.

3 Detailed examination of solvability

The basis of the solvability theory developed below for the linear tolerance problem is a new analytical characterization of the tolerable solution set. Along these lines, the most important result was obtained by Rohn, who has shown in [13] that $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is equivalent to

$$|\ \mathrm{mid}\ \mathbf{A} \cdot x - \mathrm{mid}\ \mathbf{b}\ | \le \mathrm{rad}\ \mathbf{b} - \mathrm{rad}\ \mathbf{A} \cdot |x|$$

(analogue of the Oettli-Prager criterion for the united solution set). But the starting point of our considerations is

Lemma 1. *Let an interval* $m\times n$*-matrix* $\mathbf{A}$ *and an interval right-hand side* m*-vector* $\mathbf{b}$ *be given, so the expression*

$$\mathrm{Tol}\,(x) \;=\; \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b}) \;=\; \min_{1\le i\le m} \left\{ \mathrm{rad}\ \mathbf{b}_i - \left|\ \mathrm{mid}\ \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\}$$

defines a functional $\mathrm{Tol} : \mathbb{R}^n \to \mathbb{R}$*. Then the inclusion* $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ *is equivalent to* $\mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b}) \ge 0$*, i.e., the tolerable solution set of the relevant interval system is the Lebesgue set* $\{\, x \in \mathbb{R}^n \mid \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b}) \ge 0 \,\}$ *of the functional* Tol*.*

Proof: $x \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ is equivalent to $\mathbf{A}x \subseteq \mathbf{b}$. We rewrite the latter in the following form

$$\mathrm{mid}\ \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \subseteq [\,-\mathrm{rad}\ \mathbf{b}_i,\ \mathrm{rad}\ \mathbf{b}_i\,], \qquad i = 1, 2, \ldots, m$$

which is equivalent to

$$\left|\ \mathrm{mid}\ \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \le \mathrm{rad}\ \mathbf{b}_i, \qquad i = 1, 2, \ldots, m.$$

Therefore, x actually belongs to $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ if and only if

$$\mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b}) \;=\; \min_{1\le i\le m} \left\{ \mathrm{rad}\ \mathbf{b}_i - \left|\ \mathrm{mid}\ \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right\} \;\ge\; 0. \qquad \square$$

Lemma 2. *The functional* Tol(x) *is concave.*

Proof: The functional Tol(x) is the lower envelope of the functionals

$$\xi_i(x) = \text{rad}\;\mathbf{b}_i - \left|\;\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j\;\right|$$

and we need only to establish the concavity of each $\xi_i(x)$.

Let $x,\ y \in \mathbb{R}^n$, $\lambda \in [0,1]$. The subdistributivity of the interval arithmetic then implies

$$\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij}\bigl(\lambda x_j + (1-\lambda) y_j\bigr) \subseteq \lambda\left(\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j\right) + (1-\lambda)\left(\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} y_j\right).$$

The magnitude $|\cdot|$ is isotonic with respect to the inclusion ordering of intervals and the standard linear order on $\mathbb{R}$ [8]. Hence,

$$\begin{aligned} &\left|\;\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij}\bigl(\lambda x_j + (1-\lambda) y_j\bigr)\;\right| \\ &\quad\le \left|\lambda\left(\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j\right) + (1-\lambda)\left(\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} y_j\right)\right| \\ &\quad\le \lambda\left|\;\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j\;\right| + (1-\lambda)\left|\;\text{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} y_j\;\right| \end{aligned}$$

and the assertion of the Lemma follows. □

Thus, the ordinate set

$$\text{hyp Tol} = \{\ (x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n,\ z \in \mathbb{R},\ \text{Tol}\,(x) \le z\ \}$$

of the map Tol $: \mathbb{R}^n \to \mathbb{R}$ is a convex set. We shall show that hyp Tol is intersection of finite number of the half-spaces of $\mathbb{R}^{n+1}$, i.e., it is a *convex polyhedral set* according to the terminology by Rockafellar [11]. Indeed, expressing the absolute value in terms of maximum, we get for each $i = 1, 2, \ldots, m$

$$\begin{aligned}
&\text{rad}\ \mathbf{b}_i - \left| \text{ mid } \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \\
&= \text{rad}\ \mathbf{b}_i - \max_{\hat{a}_{ij}} \left| \text{ mid } \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right| \\
&= \text{rad}\ \mathbf{b}_i - \max_{\hat{a}_{ij}} \left\{ \max \left\{ \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j,\ \sum_{j=1}^{n} \hat{a}_{ij} x_j - \text{mid } \mathbf{b}_i \right\} \right\} \\
&= \min_{\hat{a}_{ij}} \left\{ \min \left\{ \text{rad } \mathbf{b}_i - \text{mid } \mathbf{b}_i + \sum_{j=1}^{n} \hat{a}_{ij} x_j,\ \text{rad } \mathbf{b}_i + \text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right\} \right\}
\end{aligned}$$

where the n-tuple $(\hat{a}_{i1}, \hat{a}_{i2}, \ldots, \hat{a}_{in})$ runs over the finite set $\text{vert}(\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in})$, that is, over all vertices of the ith row of the interval matrix $\mathbf{A}$. Owing to this, the functional Tol is the lower envelope of at most $m \cdot 2^{n+1}$ affine functionals of the form

$$\text{rad } \mathbf{b}_i \pm \left(\text{mid } \mathbf{b}_i - \sum_{j=1}^{n} \hat{a}_{ij} x_j \right), \quad i = 1, 2, \ldots, m$$

the set hyp Tol being intersection of these functionals' ordinate sets.

As a consequence we get the following well known result: *tolerable solution set is a convex polyhedral set.*

Lemma 3. *The functional* Tol(x) *attains a finite maximum on all of* $\mathbb{R}^n$.

Proof: Being a convex polyhedral set, the ordinate set hyp Tol is the convex hull of a finite set of points (c_k, γ_k), $k = 1, 2, \ldots, p$, and directions (c_k, γ_k), $k = p+1, \ldots, q$, of $\mathbb{R}^{n+1}$ (excluding the direction $(0, \ldots, 0, 1)$ since Tol(x) is defined everywhere) [11]. More precisely,

$$\text{hyp Tol } = \left\{ \sum_{k=1}^{q} \lambda_k (c_k, \gamma_k) \;\middle|\; c_k \in \mathbb{R}^n,\ \gamma_k,\ \lambda_k \in \mathbb{R},\ \lambda_k \geq 0,\ \sum_{k=1}^{p} \lambda_k = 1 \right\}.$$

Insofar as $\mathrm{Tol}\,(x) \leq \min_{1\leq i\leq m} \mathrm{rad}\;\mathbf{b}_i$ we have $\gamma_k \leq 0$, $k = p+1,\ldots,q$, since otherwise the functional Tol would be unbounded from above. For this reason,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \mathrm{Tol}\,(x) &= \sup\{\, z \mid (x,z) \in \mathrm{hyp}\;\mathrm{Tol}\,,\; x\in\mathbb{R}^n,\; z\in\mathbb{R}\,\} \\
&= \sup\left\{\, \sum_{k=1}^{q} \lambda_k\gamma_k \;\middle|\; \lambda_k \geq 0,\; \sum_{k=1}^{p} \lambda_k = 1 \,\right\} \\
&= \sup\left\{\, \sum_{k=1}^{p} \lambda_k\gamma_k \;\middle|\; \lambda_k \geq 0,\; \sum_{k=1}^{p} \lambda_k = 1 \,\right\} \\
&= \max_{1\leq k\leq p} \gamma_k. \qquad \square
\end{aligned}$$

Lemma 4. *If the interval matrix* $\mathbf{A}$ *does not have zero rows, then* $y \in \mathrm{int}\;\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$ *implies* $\mathrm{Tol}\,(y;\mathbf{A},\mathbf{b}) > 0$.

Proof: Let $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) \neq \emptyset$, and assume $\max\;\mathrm{Tol}\,(x)$ is reached at some point $m \in \Sigma_{\forall\exists}$. If $y \in \mathrm{int}\;\Sigma_{\forall\exists}$, then y is an interior point of some segment $[m,z] \subset \Sigma_{\forall\exists}$, i.e. $y = \lambda m + (1-\lambda)z$ for some $\lambda\in(0,1)$, $z\in\Sigma_{\forall\exists}$. Therefore

$$\mathrm{Tol}\,(y) \geq \lambda\,\mathrm{Tol}\,(m) + (1-\lambda)\,\mathrm{Tol}\,(z)$$

because the functional Tol is concave.

Suppose $\mathrm{Tol}\,(y) = 0$. Then the above inequality holds only when $\mathrm{Tol}\,(m) = \mathrm{Tol}\,(z) = 0$ and the functional Tol must equal zero on the entire set $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$. Furthermore, let

$$\mathbb{R}^n = \bigcup_{1\leq i\leq m} \Xi_i$$

with

$$\mathrm{Tol}\,(x) = \mathrm{rad}\;\mathbf{b}_i - \left|\; \mathrm{mid}\;\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij}x_j \right|$$

for $x \in \Xi_i$. It is fairly simple to see that

$$\Sigma_{\forall\exists} = \bigcup_{1\leq i\leq m} (\Sigma_{\forall\exists}\cap\,\Xi_i)$$

all the sets $\Sigma_{\forall\exists}\cap\Xi_i$, $i = 1, 2, \ldots, m$, being closed. Hence, $\operatorname{int}(\Sigma_{\forall\exists}\cap\Xi_k) \neq \emptyset$ for at least one $k \in \{1, 2, \ldots, m\}$ and we have

$$\operatorname{rad}\mathbf{b}_k - \left| \operatorname{mid}\mathbf{b}_k - \sum_{j=1}^{n} \mathbf{a}_{kj}x_j \right| = 0 = \text{const}$$

for all $x \in \operatorname{int}(\Sigma_{\forall\exists}\cap\Xi_k)$. The latter may occur only when all $\mathbf{a}_{k1}, \ldots, \mathbf{a}_{kn}$ are zeros, which contradicts the assertion of the Lemma. □

Lemma 5. *If* $\operatorname{Tol}(y; \mathbf{A}, \mathbf{b}) > 0$, *then* $y \in \operatorname{int}\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$.

Proof: The map $\operatorname{Tol} : \mathbb{R}^n \to \mathbb{R}$ is continuous, so the set $Z = \{\, z \in \mathbb{R}^n \mid \operatorname{Tol}(z) > 0 \,\}$ is open. Also, it is nonempty: $y \in Z \subseteq \Sigma_{\forall\exists}$. Hence $\operatorname{int}\Sigma_{\forall\exists} \neq \emptyset$ and $y \in \operatorname{int}\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. □

To summarize, we come to the following technique to investigate solvability of the linear tolerance problem, i.e., to the criterion for the tolerable solution set to be nonempty.

Solve the unconstrained maximization problem for the functional

$$\operatorname{Tol}(x) \;=\; \min_{1\le i\le m} \left\{ \operatorname{rad}\mathbf{b}_i - \left| \operatorname{mid}\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij}x_j \right| \right\}.$$

Let $T = \max_{x\in\mathbb{R}^n} \operatorname{Tol}(x; \mathbf{A}, \mathbf{b})$, and let T be reached at a point t. We have

- if $T \geq 0$, then $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$, i.e., the linear tolerance problem is compatible, and if $T > 0$, then $t \in \operatorname{int}\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$;
- if $T < 0$, then $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \emptyset$, i.e., the linear tolerance problem is incompatible.

It is worth noting that the Lemmae 1–4 as well as the above solvability criterion would remain valid if the functional Tol was defined by the expression

$$\min_{1\le i\le m} \left\{ \eta_i \left(\operatorname{rad}\mathbf{b}_i - \left| \operatorname{mid}\ \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij}x_j \right| \right) \right\}$$

where η_i, $i = 1, 2, \ldots, m$, are positive reals. Below are some examples in which such functionals naturally come into existence and then are employed fruitfully.

Maximization of nonsmooth concave functions has been much studied during the last few decades. A good many numerical methods have been proposed to solve this problem (see [6, 18] et al.) and this is reason to hope that the solvability criterion developed above is quite practical.

4 Correction of the linear tolerance problem

Imagine solving an actual practical problem. Usually, the effort does not terminate even after we reach the conclusion that the problem has no solutions (unsolvable). A client is very likely to be interested in information about

- how unsolvable the problem is,
- how one must change the input data to make the problem solvable,
- and so on.

Alternately, if the original problem proves to be solvable, then, frequently, the region of variations of input data within which the problem remains solvable is to be outlined. We are able to give quite expanded answers to some of these questions.

If $\mathbf{A}$ and mid $\mathbf{b}$ are unchanged, increasing the radii of all the components of $\mathbf{b}$ by the same value K is easily seen to lead to adding the constant K to the functional $\mathrm{Tol}\,(x)$. Therefore,

$$\max_{x\in\mathbb{R}^n} \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b} + K\mathbf{e}) = K + \max_{x\in\mathbb{R}^n} \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b})$$

where $\mathbf{e} = ([-1, 1], \ldots, [-1, 1])^\top$. If the linear tolerance problem is unsolvable and

$$\max_{x\in\mathbb{R}^n} \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b}) = T < 0$$

we can make it solvable with the same matrix $\mathbf{A}$ by widening the right-hand vector by $K\mathbf{e}$, $K \geq 0$, and the points $t \in \mathrm{Arg\,max}\ \mathrm{Tol}\,(x; \mathbf{A}, \mathbf{b})$ will certainly belong to the nonempty set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b} + K\mathbf{e})$. Conversely, if

$$\max_{x \in \mathbb{R}^n} \operatorname{Tol}(x; \mathbf{A}, \mathbf{b}) = T \geq 0$$

that is, the linear tolerance problem is solvable, it will remain so even after we decrease the radii of all right-hand side components by K, $K \leq T$.

Sometimes, such uniform widening of all the components of $\mathbf{b}$ may prove unacceptable in practice. So, let us assume that a vector $(v_1, v_2, \ldots, v_m)$, $v_i \geq 0$, is given such that the increase of the width of $\mathbf{b}_i$ is to be proportional to v_i. Now, calculate

$$T_v = \max_{x \in \mathbb{R}^n} \operatorname{Tol}_v(x; \mathbf{A}, \mathbf{b})$$

where

$$\operatorname{Tol}_v(x) \;=\; \min_{1 \leq i \leq m} \left\{ {v_i}^{-1} \left(\operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right) \right\}. \tag{9}$$

If, for instance, initially, the linear tolerance problem with the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ had no solutions, then the problem with the same matrix $\mathbf{A}$ and the expanded vector $(\mathbf{b}_i + Kv_i[-1, 1])_{i=1}^{m}$ in the right-hand side becomes solvable for $K \geq |T_v|$.

The most important particular case of the above construction is that of ensuring equal relative (proportional to the absolute values) increases of the radii of the right-hand side components, when $v_i = |\mathbf{b}_i|$ for nonzero $\mathbf{b}_i$, $i = 1, 2, \ldots, m$. Denote

$$\operatorname{Tol}_0(x) \;=\; \min_{1 \leq i \leq m} \left\{ |\mathbf{b}_i|^{-1} \left(\operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} x_j \right| \right) \right\}$$

and let

$$T_0 = \max_{x \in \mathbb{R}^n} \operatorname{Tol}_0(x; \mathbf{A}, \mathbf{b}).$$

The magnitude of T_0 is a quantitative characteristic of compatibility of the linear tolerance problem. Judging by the absolute value of T_0, one can precisely estimate the degree of unsolvability in the case $T_0 < 0$ and the reserve of solvability (stability of the solvable state) in the case $T_0 \geq 0$. Naturally, all this is attained at the price of more laborious computation.

We have demonstrated some capabilities to correct the linear tolerance problem by modification of only the right-hand side vector $\mathbf{b}$. In fact, the linear tolerance problem can also be corrected by varying the elements of the matrix $\mathbf{A}$ as well, but these results fall outside the scope of the present short paper.

5 Construction of an interval solution

Once the compatibility of the linear tolerance problem is established and a point of the tolerable solution set has been found, we may turn to the actual construction of the interval solution. To do so, we follow the so-called "center" approach adopted by Khlebalin [5], Neumaier [8], Shaidurov [3] and others, in which the point of the tolerable solution set found earlier is taken to be the center of the interval solution under construction.

In applications, the statement of the linear tolerance problem is often more rigid than (3). In addition to (3), we take the ratio of the tolerances of the separate components of the solution to be determined by a real vector $w = (w_1, w_2, \ldots, w_n)$, $w_i > 0$, so that

$$\text{rad}\,\mathbf{U}_i/\text{rad}\,\mathbf{U}_j = w_i/w_j.$$

Through scaling by the diagonal matrix $\text{diag}\{w_1, w_2, \ldots, w_n\}$, all such cases are easily reduced to a standard one, when $w = (1, 1, \ldots, 1)$, and we are to inscribe a hypercube in the properly modified set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. Indeed, we introduce matrices $D = \text{diag}\{w_1, w_2, \ldots, w_n\}$ and $\tilde{\mathbf{A}} = \mathbf{A}D$. Let the interval vector $\tilde{\mathbf{U}}$, $\text{rad}\,\mathbf{U}_i = \text{const}$, be a solution to the linear tolerance problem with the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$. Then $\mathbf{U} = D\tilde{\mathbf{U}}$ is a solution to the original problem, since

$$\{\, \mathbf{A}x \mid x \in \mathbf{U} \,\} = \{\, \mathbf{A}DD^{-1}x \mid x \in \mathbf{U} \,\} = \{\, \tilde{\mathbf{A}}\tilde{x} \mid \tilde{x} \in \tilde{\mathbf{U}} \,\} \subseteq \mathbf{b}$$

and moreover $\text{rad}\,\mathbf{U}_i/\text{rad}\,\mathbf{U}_j = w_i/w_j$ as required. That is why from now on the linear tolerance problem will be refered to as a problem of finding an interval vector $\mathbf{U}$ with components of equal width and such that $\{\, \mathbf{A}x \mid x \in \mathbf{U} \,\} \subseteq \mathbf{b}$. The expedient described above (introducing weighting coefficients for tolerances) is due to Shaidurov [3, 14]. He is also the author of the fundamental Theorem 2, but here we have substantially reworked its proof, compared with the proof presented in [3, 14].

Theorem 2. *If $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ then for*

$$h = \min_{1 \le i \le m} \min_{A \in \mathbf{A}} \left\{ \frac{\operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{i=1}^{n} |\, a_{ij}|} \right\} \tag{10}$$

the interval vector $\mathbf{U} = (t + h\mathbf{e})$ is also entirely contained in $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$.

Proof: First assume that in the linear tolerance problem the matrix $\mathbf{A}$ is thin, i.e., $\mathbf{A} = A$. We represent each $x \in \mathbf{U}$ in the form $x = t + y$, where $\max_{1 \le k \le n} |\, y_k| \le h_A$ and

$$h_A = \min_{1 \le i \le m} \left\{ \frac{\operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{i=1}^{n} |\, a_{ij}|} \right\} \tag{11}$$

so that the following holds for $i = 1, 2, \ldots, m$:

$$\begin{aligned} |\,(Ay)_i| &= \left| \sum_{j=1}^{n} a_{ij} y_j \right| \le \sum_{j=1}^{n} |\, a_{ij}|\,|\, y_j| \\ &\le h_A \cdot \sum_{j=1}^{n} |\, a_{ij}| \\ &\le \operatorname{rad} \mathbf{b}_i - \left| \operatorname{mid} \mathbf{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|. \end{aligned}$$

Therefore, since $Ax = At + Ay$, we obtain

$$(At)_i - \operatorname{rad} \mathbf{b}_i + |\operatorname{mid} \mathbf{b}_i - (At)_i\,|$$

$$\le (Ax)_i \le$$

$$(At)_i + \operatorname{rad} \mathbf{b}_i - |\operatorname{mid} \mathbf{b}_i - (At)_i\,|$$

or, equivalently,

$$\begin{aligned} \underline{\mathbf{b}}_i - (\,\mathrm{mid}\,\mathbf{b}_i - (At)_i) + |\,\mathrm{mid}\,\mathbf{b}_i - (At)_i\,| \\ \leq (Ax)_i \leq \\ \overline{\mathbf{b}}_i - (\,\mathrm{mid}\,\mathbf{b}_i - (At)_i) - |\,\mathrm{mid}\,\mathbf{b}_i - (At)_i\,|. \end{aligned} \tag{12}$$

By virtue of the fact that

$$-z + |z| \geq 0 \quad \text{and} \quad -z - |z| \leq 0$$

for any real z, the inequality (12) implies

$$\underline{\mathbf{b}}_i \leq (Ax)_i \leq \overline{\mathbf{b}}_i$$

that is, $Ax \in \mathbf{b}$ as was expected.

Now, let the matrix $\mathbf{A}$ of the problem be a thick interval matrix, and let $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) \neq \emptyset$. We consider the totality of all linear tolerance problems for systems $Ax = \mathbf{b}$ with $A \in \mathbf{A}$. It is clear that

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \bigcap_{A \in \mathbf{A}} \Sigma_{\forall\exists}(A, \mathbf{b})$$

and if for each $A \in \mathbf{A}$ the corresponding interval solution vector is $\mathbf{U}_A$, $\mathbf{U}_A \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, then

$$\mathbf{U} = \bigcap_{A \in \mathbf{A}} \mathbf{U}_A \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}).$$

In particular, when all $\mathbf{U}_A$ have the common center t and their radii are defined by formula (11), we have

$$\mathbf{U} = t + h\mathbf{e}$$

where

$$h = \min_{A \in \mathbf{A}} r_A = \min_{1 \leq i \leq m} \min_{A \in \mathbf{A}} \left\{ \frac{\mathrm{rad}\,\mathbf{b}_i - \left|\,\mathrm{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} a_{ij} t_j \right|}{\sum_{i=1}^{n} |\,a_{ij}|} \right\}.$$

The Theorem is completely proved. □

Thus, for known $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, construction of an interval solution to the linear tolerance problem amounts to finding the value (10) or an estimate for it from below. Since taking the minimum on $i \in \{1, 2, \dots, m\}$ involves no difficulties, the central problem is the computation of $\min_{A\in\mathbf{A}}$. The simplest way to estimate this minimum is to take the left endpoint of the natural interval extension of the expression in the braces of (10), as in the following algorithm of Shaidurov [3, 14]:

For a given $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ calculate the intervals

$$\mathbf{h}_i = \frac{\text{rad}\,\mathbf{b}_i - \left|\ \text{mid}\,\mathbf{b}_i - \sum_{j=1}^{n} \mathbf{a}_{ij} t_j\ \right|}{\sum_{i=1}^{n} |\,\mathbf{a}_{ij}|} \tag{13}$$

$i = 1, 2, \dots, m$, and then put $h = \min_{1\le i\le m} \underline{\mathbf{h}}_i$. The interval vector $(t + h\mathbf{e})$ is a solution to the linear tolerance problem.

The other important result on the linear tolerance problem is due to Neumaier [7], who has proposed the following simple method to construct an interval solution of the linear tolerance problem around a given center. If $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, we find the largest nonnegative η such that

$$\eta \cdot \mathbf{A}\mathbf{e} \subseteq \mathbf{b} \ominus \mathbf{A}t \tag{14}$$

where "$\ominus$" is the endwise interval subtraction

$$[\,\underline{\mathbf{p}}, \overline{\mathbf{p}}\,] \ominus [\,\underline{\mathbf{q}}, \overline{\mathbf{q}}\,] = [\,\underline{\mathbf{p}} - \underline{\mathbf{q}}, \overline{\mathbf{p}} - \overline{\mathbf{q}}\,].$$

Then the interval vector $(t+\eta\mathbf{e})$ is the desired solution of the linear tolerance problem, since

$$\mathbf{A}x \subseteq \mathbf{A}(t+\eta\mathbf{e}) \subseteq \mathbf{A}t + \mathbf{A}(\eta\mathbf{e}) \subseteq \mathbf{A}t + \mathbf{b} \ominus \mathbf{A}t = \mathbf{b}$$

for each $x \in t + \eta\mathbf{e}$.

The results obtained by this method turn out to be completely identical to those given by Shaidurov's algorithm. In fact, condition (14) means

$$\eta \cdot (\underline{\mathbf{A}\mathbf{e}})_i \ge (\underline{\mathbf{b} \ominus \mathbf{A}t})_i \quad \text{and} \quad \eta \cdot (\overline{\mathbf{A}\mathbf{e}})_i \le (\overline{\mathbf{b} \ominus \mathbf{A}t})_i,$$

$$i = 1, 2, \dots, m$$

where $\mathbf{Ae}$ is a symmetrical interval vector in which

$$-(\underline{\mathbf{Ae}})_i = (\overline{\mathbf{Ae}})_i = |\,(\mathbf{Ae})_i|, \qquad i = 1, 2, \dots, m.$$

Moreover, for $t \in \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$,

$$(\underline{\mathbf{b} \ominus \mathbf{A}t})_i \leq 0 \leq (\overline{\mathbf{b} \ominus \mathbf{A}t})_i$$

and thus the following chain of transformations is valid for each i:

$$\begin{aligned}
\eta \;&\leq\; \min\left\{ \frac{(\,\underline{\mathbf{b} \ominus \;\mathbf{A}t}\,)_i}{(\,\underline{\mathbf{Ae}}\,)_i}, \frac{(\,\overline{\mathbf{b} \ominus \;\mathbf{A}t}\,)_i}{(\,\overline{\mathbf{Ae}}\,)_i} \right\} \\
&= \frac{\min\{\,-(\,\mathrm{mid}\;\mathbf{b}_i - \mathrm{rad}\;\mathbf{b}_i) + (\underline{\mathbf{A}t})_i,\; (\,\mathrm{mid}\;\mathbf{b}_i + \mathrm{rad}\;\mathbf{b}_i) - (\overline{\mathbf{A}t})_i\}}{|\,(\mathbf{Ae})_i|} \\
&= \frac{\min\{\,\mathrm{rad}\;\mathbf{b}_i - (\,\mathrm{mid}\;\mathbf{b}_i - (\underline{\mathbf{A}t})_i),\; \mathrm{rad}\;\mathbf{b}_i - ((\overline{\mathbf{A}t})_i - \mathrm{mid}\;\mathbf{b}_i)\}}{|\,(\mathbf{Ae})_i|} \\
&= \frac{\mathrm{rad}\;\mathbf{b}_i - \max\{\,\mathrm{mid}\;\mathbf{b}_i - (\underline{\mathbf{A}t})_i,\; (\overline{\mathbf{A}t})_i - \mathrm{mid}\;\mathbf{b}_i\}}{|\,(\mathbf{Ae})_i|} \\
&= \frac{\mathrm{rad}\;\mathbf{b}_i - |\,\mathrm{mid}\;\mathbf{b}_i - (\mathbf{A}t)_i|}{|\,(\mathbf{Ae})_i|}.
\end{aligned}$$

For $i = 1, 2, \dots, m$, the last expression coincides with the lower bounds of the corresponding intervals (13), so taking the minimum on all i leads to the equality $\eta = h$.

Both these algorithms, by Shaidurov and by Neumaier, are simple and easy to implement, but at the price of considerable coarsening of the final result, especially for wide $\mathbf{A}$. We demonstrate how the size of the interval solution defined by the formula (10) can be computed exactly. The basis of the corresponding algorithm is quasiconcavity of the functions in the braces in (10). Let S be a convex set. Recall that a function $f : S \to \mathbb{R}$ is said to be *quasiconcave* if

$$f\big(\lambda x + (1-\lambda) y\big) \geq \min\{f(x), f(y)\}$$

for all x, $y \in S$ [2].

Theorem 3 [16]. *If R, M, $t_1, \dots, t_n$ are real constants and $S \subset \mathbb{R}^n$ is a convex set that does not contain the origin of the coordinate system, then the function $\Phi : S \to \mathbb{R}$ defined by the formula*

$$\Phi(x) = \frac{R - \left|\, M - \sum_{j=1}^{n} x_j\, t_j \,\right|}{\sum_{j=1}^{n} |\, x_j|}$$

is quasiconcave.

Proof of the Theorem is based on the fact that quasiconcavity of a function f is equivalent to convexity of all the Lebesgue sets $S_\alpha = \{\, x \mid f(x) \geq \alpha \,\}$.

Denote for brevity

$$\Psi(x) = R - \left|\, M - \sum_{j=1}^{n} x_j\, t_j \,\right| \qquad \text{and} \qquad \Theta(x) = \sum_{j=1}^{n} |\, x_j|$$

so that $\Phi(x) = \Psi(x)/\Theta(x)$. For any x, $y \in \mathbb{R}^n$, $\lambda \in (0,1)$, the following inequalities are obvious

$$\begin{aligned}
\Psi\bigl(\lambda x + (1-\lambda) y\bigr) \;&\geq\; R - \lambda \left|\, M - \sum_{j=1}^{n} x_j\, t_j \,\right| - (1-\lambda) \left|\, M - \sum_{j=1}^{n} y_j\, t_j \,\right| \\
&=\; \lambda\, \Psi(x) + (1-\lambda)\, \Psi(y), \\
\Theta\bigl(\lambda x + (1-\lambda) y\bigr) \;&\leq\; \lambda \sum_{j=1}^{n} |\, x_j| + (1-\lambda) \sum_{j=1}^{n} |\, y_j| \\
&=\; \lambda\, \Theta(x) + (1-\lambda)\, \Theta(y).
\end{aligned}$$

Therefore,

$$\Phi\bigl(\lambda x + (1-\lambda) y\bigr) = \frac{\Psi\bigl(\lambda x + (1-\lambda) y\bigr)}{\Theta\bigl(\lambda x + (1-\lambda) y\bigr)} \;\geq\; \frac{\lambda\, \Psi(x) + (1-\lambda)\, \Psi(y)}{\lambda\, \Theta(x) + (1-\lambda)\, \Theta(y)} \tag{15}$$

for any x, $y \in S$. Now, let us suppose that S_α is a nonempty Lebesgue set of the function Φ and x, $y \in S_\alpha$, i.e., $\Phi(x) \geq \alpha$ and $\Phi(y) \geq \alpha$. Then $\Psi(x) \geq \alpha\, \Theta(x)$ and $\Psi(y) \geq \alpha\, \Theta(y)$. Summing these inequalities with the weights λ and $1-\lambda$ we obtain

$$\lambda\, \Psi(x) + (1-\lambda)\, \Psi(y) \;\geq\; \alpha \left(\lambda\, \Theta(x) + (1-\lambda)\, \Theta(y)\right)$$

which is equivalent to

$$\frac{\lambda\,\Psi(x) + (1-\lambda)\,\Psi(y)}{\lambda\,\Theta(x) + (1-\lambda)\,\Theta(y)} \;\geq\; \alpha.$$

Together with (15) this implies $\Phi\big(\lambda x + (1-\lambda)y\big) \geq \alpha$, that is, under our assumptions $\lambda\,x + (1-\lambda)\,y \in S_\alpha$ too. □

A quasiconcave function is known to reach its minimum at extreme points of its convex domain of definition [2]. Thus, for each $i = 1, 2, \ldots, m$, the expressions in the braces in (10) attain their minimal values on $A \in \mathbf{A}$ at vertices of the interval vectors $(\,\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{in})$, and they can be found by exhaustive search. Then we take the minimum over i. The complexity of this algorithm, which is proportional to $m\cdot 2^n$, can be considerably dicreased if the item-by-item examination of the vertices is carried out in a special manner, passing at each step to an adjacent vertex and recalculating the sums mid $\mathbf{b}_i - \sum_{j=1}^n a_{ij}x_j$ and $\sum_{i=1}^n |a_{ij}|$ reccurently. In such an algorithm [16] the reduction in complexity is larger, the larger the dimension of the problem, but the exponentiality is still not overcome. For this reason the practical significance of these exhaustive algorithms is limited only to the problems of moderate dimension.

As is seen, for computing the value (10), we need a more advanced algorithm having precision better than that of (13), but with complexity less than that of the exhaustive algorithm rested on the Theorem 3. Such an algorithm was developed in [16]. With the well known “branch and bound method” as a basis, it occupies an intermediate position between the simplest algorithm (13) and the exhaustive algorithms. Its running time is exponential with respect to the dimension only in the worst case (as in all methods of this kind), but, due to its flexible computational scheme, it can be successfully applied to the problems of any size.

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