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Interval Valued Fuzzy Sets and Fuzzy Connectives

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The generalization of two-valued (crisp) sets to fuzzy sets gave us a rich and powerful representation scheme within the domain of semantic uncertainty. A parallel generalization of conjunctions and disjunction is needed from t-norms and t-conorms to what we might appropriately call linguistic (fuzzy) connectives, “AND,” “OR,” etc.

In fact a closer look at the canonical forms of the combination of concepts, known as the Disjunctive and Conjunctive Normal Forms, DNF and CNF, respectively, suggest that linguistic (fuzzy) connectives “AND,” “OR,” etc., should be represented with interval-valued fuzzy sets as particular realizations of type II fuzziness that could represent the semantic uncertainty contained in linguistic (fuzzy) connectives which exist in everyday natural language expressions of human use. In fact, the experimentally discovered “Compensatory ‘AND’ ” can be grounded within the boundaries of the interval-valued fuzzy sets so suggested and proposed. This leads to the possibility of defining other fuzzy relations such as linguistic (fuzzy) IMPLICATION to enrich its representation by a new definition which we may call “Compensatory ‘IMPLICATION’,” etc.

Интервальнозначные нечеткие множества и нечеткие связки

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Обобщение множеств с двузначной принадлежностью на случай нечетких множеств дало богатые и мощные средства представления в области семантической неопределенности. В этой связи потребовалось обобщение понятий коньюнкции и дизъюнкции от t-норм и t-конорм к тому, что можно было бы назвать лингвистическими (нечеткими) связками <И>, <ИЛИ> и т. д.

Фактически, внимательное рассмотрение канонических форм соответствующих комбинаций, известных как конъюнктивная и дизъюнктивная нормальные формы (КНФ/ДНФ), заставляет прийти в выводу, что лингвистические (нечеткие) связки <И>, <ИЛИ> и т. д. лучше всего могут

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быть представлены с помощью интервальнозначных нечетких множеств как особых реализаций нечеткости типа II. Такое представление позволяет отразить семантическую неопределенность, содержащуюся в повседневном естественно-языковом употреблении этих лингвистических связок. В частности, открытое экспериментально компенсаторное <И> может быть обосновано в рамках данного подхода. Оказывается возможным определить и другие нечеткие отношения, например, лингвистическую (нечеткую) импликацию, что расширяет возможности соответствующего представления путем определения того, что мы можем назвать компенсаторной импликацией, и т. д.

1 Introduction

A closer look into history reveals that a heated debate took place between Paramenides of Elea who proposed the notion of “Being" and hence “Nothingness" and Heraclitus of Euphesus who proposed the notion of “Becoming" and hence “Gradual," i.e., “Graded," notion of our knowledge about real world conditions.

The debate was won by the supporters of “Being" and “Nothingness" which culminated in Aristotelian two-valued logic $\{0,1\}$ which was later canonized by Descartes and hence became the foundation of Western science. Thus the principle of dichotomy better known as the “law of excluded middle," and its dual the “law of contradiction," dominated Western scientific thinking and education. It is now known that logicians such as Lukasiewicz and others realized the limitations of two-valued logic and introduced multi-valued logic while retaining the two major quantifiers known as the “universal quantifier," for all, i.e., $\forall$, and the “existential quantifier," there exist, i.e., $\exists$.

With his first seminal paper, Lotfi A. Zadeh [17] initially proposed fuzzy sets which are generalizations of multi-valued sets to infinite valued sets and later with his second seminal paper, Zadeh [18] proposed foundations of fuzzy logic with his “Compositional Rule of Inference," CRI, which is a generalization of modus ponens. In his third seminal paper, Zadeh [19] proposed a generalization of syllogisms introducing fuzzy quantifiers such as “most,” “few,” etc., to the generalization of classical syllogistic models. Furthermore, in all of his proposals, Zadeh presented examples that showed application of a class of operators known as “max-min and complement" De Morgan Triple.

Again in this historical perspective, we find that there was a parallel development that brought about t-norms and t-conorms. Noticing certain anomalies in the analysis of data, Schweitzer and Sklar [6, 7, 8] proposed the formation of t-norms and t-conorms. The introduction of t-norms and t-conorms caused further generalizations in the development of fuzzy theory. Many different fuzzy set and logic theories were formed by each De Morgan Triple of t-norm, t-conorm and complement. Confronted with certain anomalies, in CRI or generalized Modus Ponens, Dubois and Prade [5] introduced us pseudo-conjunction, and pseudo-disjunction to recapture certain regularity conditions displayed by classical modus ponens known as the four desirable properties of inference.

It should be noticed, however, in all these developments, linguistic connectives "AND," "OR," etc., were mostly interpreted crisply, meaning that they corresponded to a t-norm, t-conorm, respectively, or to a pseudo-conjunction or pseudo-disjunction, respectively.

It appears that very few researchers realized that certain types of intervals are generated by combinations of concepts when various axioms of classical set and logic theories are relaxed as it is the case when we consider fuzzy sets and t-norm and t-conorms. The exceptions are in the works of: (i) Turksen [9–16] where the Interval-Valued Fuzzy Sets, IVFS, are introduced as a consequence of natural separation of the disjunctive and conjunctive normal forms of any combination of concepts; and (ii) Bandler and Kohout [1–3] based on their "checklist paradigm" where they have identified a different set of intervals for the representation of connectives. In this context, it should be recalled that there is the work of Zimmermann and Zysno [20] under the heading of latent connectives which are also known as "Compensatory 'AND'," where they have experimentally found that human use of linguistic connectives do not necessarily correspond to a t-norm or t-conorm but to a weighted compensation between them.

In the rest of this paper, first the essential concepts are reviewed for a basic understanding of the interval-valued representation of the linguistic (fuzzy) "AND," "OR," etc. [14] (in Section 2), together with its comparison to crisp representations. In Section 3, a skeletal discussion of the "Compensatory 'AND' " [20] is presented together with its grounding in Interval-Valued Fuzzy Sets [10, 14]. In Section 4, representation of the linguistic (fuzzy) "IF ... THEN" is presented together with its relationship to some of the well known crisp "IF ... THEN" expressions. In Section 5, we propose “new compensatory connectives" [9], for future research, as extensions of basic IVFS concepts.

2 Interval-valued fuzzy sets

IVFS representations of the combination of concepts with linguistic connectives “AND," “OR," “IF … THEN," etc., are particular generalizations of Boolean normal forms. For any two concepts, represented by sets A and B, it is well known that we can form sixteen new concepts as the combination of the original concepts A and B. It is also well known that their Boolean Normal Forms are known as Disjunctive Normal Form and Conjunctive Normal Form, DNF and CNF, respectively. These normal forms are constructed as the disjunctions of applicable primary conjunctions in the case of DNF and conjunctions of applicable primary disjunctions in the case of CNF.

It is well known that DNF$(\cdot)$ = CNF$(\cdot)$, if $a, b \in \{0, 1\}$ for $a \in A, b \in B$, and the connectives “AND," “OR," etc., are represented with the De Morgan Triple of min-max, and complement, i.e., when we are operating under the assumptions of Boolean algebra complete with its base axioms which include the law of “Excluded Middle," $A \cup \overline{A} = I$, and its dual, the law of “Contradiction," $A \cup \overline{A} = \emptyset$, distributivity, idempotency, absorbtion, etc. When the laws of “Excluded Middle” and “Contradiction" are relaxed (and $a, b \in [0, 1]$, as Zadeh proposed in his seminal paper [17]), it has been shown that DNF$(\cdot) \subseteq$ CNF$(\cdot)$ (see [14], for the special case of the De Morgan Triple of max-min and complement). Further it has been shown, that, in general, DNF$(\cdot) \subseteq$ CNF$(\cdot)$, [13] for a certain class of t-norms T, and t-conorms S (in the sense of Schweizer and Sklar [6, 7, 8]), that satisfy the following conditions:

$$a \leq \ T\left[S(a,b), S(a,\overline{b})\right],$$

$$a \geq \ S\left[T(a,b), T(a,\overline{b})\right],$$

$$b \geq \ S\left[T(a,b), T(\overline{a},b)\right]$$

where $a, b \in [0, 1]$ for $a \in A, b \in B$, $\overline{a} = 1 - a$, and $\overline{b} = 1 - b$, i.e., when we are operating under the four assumptions of boundary, associativity, commutativity, and monotonicity of t-norms T and t-conorms S. It should be noted that the three explicit conditions stated above in reality reduces to one: Indeed, since T and S are duals of each other, condition one and two are also dual to each other. Condition three is nothing more than a positional variable change due to symmetry of t-norms and t-conorms. With the realization that DNF $\subseteq$ CNF, Turksen [13] proposed that IVFS's defined by the DNF and CNF boundaries be approximate models to represent the combined concepts formed by the linguistic connectives "AND," "OR," i.e.,

$$A \text{ AND } B \overset{\triangle}{=} \big[\text{DNF}(A \text{ AND } B),\ \text{CNF}(A \text{ AND } B)\big],$$

$$A \text{ OR } B \overset{\triangle}{=} \big[\text{DNF}(A \text{ OR } B),\ \text{CNF}(A \text{ OR } B)\big].$$

In this proposed representation, specific DNF and CNF expressions are first to be identified for each combined linguistic concept, and hence the corresponding linguistic connective, in Tables 1 and 2. Secondly, they are to be evaluated in the membership domain with a selected De Morgan Triple of t-norm T, t-conorm S, and the complement that correspond to $\cap$, $\cup$, and $^-$. Thus, for example, for the linguistic connectives "AND," "OR," we have:

$$\mu_{\text{DNF}(A \text{ AND } B)} = T(a, b)$$

$$\mu_{\text{CNF}(A \text{ AND } B)} = T\Big[T\left[S(a, b), S(a, \overline{b})\right], S(\overline{a}, b)\Big]$$

$$\mu_{\text{DNF}(A \text{ OR } B)} = S\Big[S\left[T(a, b), T(a, \overline{b})\right], T(\overline{a}, b)\Big]$$

$$\mu_{\text{CNF}(A \text{ OR } B)} = S(a, b)$$

where $a, b \in [0, 1],\ \overline{a} = 1 - a,\ \overline{b} = 1 - b,\ a \in A$, and $b \in B$.

It is to be observed that $\mu_{\text{DNF}(A \text{ AND } B)} = T(a, b)$ corresponds to the usual crisp representations of "AND" in most current fuzzy literature and its applications. But this is only the lower bound of the IVFS (A AND B), where IVFS (A AND B) approximately but somewhat adequately represents the semantic vagueness of "AND" that should be associated with its natural language use. In the same vain, it is to be observed that,

$$\mu_{\text{CNF}(A \text{ OR } B)} = S(a, b)$$

corresponds to the usual crisp representations of "OR," in most current fuzzy literature and its applications. But these crisp representations do not effectively reflect the vagueness that should be associated with the natural language use of these linguistic connectives, and their uncertain semantic meanings. Whereas, as it will be explained in the next section, these IVFS

No.	Concept	Combination
1	Complete Affirmation	True
2	Complete Negation	False
3	Disjunction	A OR B
4	Conjunctive Negation	NOT A AND NOT B
5	Incompatibility	NOT A OR NOT B
6	Conjunction	A AND B
7	Implication (IF ... THEN)	NOT A OR B
8	Non-Implication	A AND NOT B
9	Inverse Implication	A OR NOT B
10	Non-inverse Implication	NOT A AND B
11	Equivalence	A IFF B
12	Exclusion	A XOR B
13	Affirmation	A
14	Negation	NOT A
15	Affirmation	B
16	Negation	NOT B

Table 1: List of combined concepts for A and B

No.	DNF	CNF
1	$(A\cap B)\cup(A\cap\bar{B})\cup(\bar{A}\cap B)\cup(\bar{A}\cap\bar{B})$	U
2	ϕ	$(A\cup B)\cap(A\cup\bar{B})\cap(\bar{A}\cup B)\cap(\bar{A}\cup\bar{B})$
3	$(A\cap B)\cup(A\cap\bar{B})\cup(\bar{A}\cap B)$	$(A\cup B)$
4	$(\bar{A}\cap\bar{B})$	$(A\cup\bar{B})\cap(\bar{A}\cup B)\cap(\bar{A}\cup\bar{B})$
5	$(A\cap\bar{B})\cup(\bar{A}\cap B)\cup(\bar{A}\cap\bar{B})$	$(\bar{A}\cup\bar{B})$
6	$(A\cap B)$	$(A\cup B)\cap(A\cup\bar{B})\cap(\bar{A}\cup B)$
7	$(A\cap B)\cup$ $(\bar{A}\cap B)\cup(\bar{A}\cap\bar{B})$	$(\bar{A}\cup B)$
8	$(A\cap\bar{B})$	$(A\cup B)\cap(A\cup\bar{B})\cap$ $(\bar{A}\cup\bar{B})$
9	$(A\cap B)\cup(A\cap\bar{B})\cup$ $(\bar{A}\cap\bar{B})$	$(A\cup\bar{B})$
10	$(\bar{A}\cap B)$	$(A\cup B)\cap$ $(\bar{A}\cup B)\cap(\bar{A}\cup\bar{B})$
11	$(A\cap B)\cup$ $(\bar{A}\cap\bar{B})$	$(A\cup\bar{B})\cap(\bar{A}\cup B)$
12	$(A\cap\bar{B})\cup(\bar{A}\cap B)$	$(A\cup B)\cap$ $(\bar{A}\cup\bar{B})$
13	$(A\cap B)\cup(A\cap\bar{B})$	$(A\cup B)\cap(A\cup\bar{B})$
14	$(\bar{A}\cap B)\cup(\bar{A}\cap\bar{B})$	$(\bar{A}\cup B)\cap(\bar{A}\cup\bar{B})$
15	$(A\cap B)\cup$ $(\bar{A}\cap B)$	$(A\cup B)\cap$ $(\bar{A}\cup B)$
16	$(A\cap\bar{B})\cup$ $(\bar{A}\cap\bar{B})$	$(A\cup\bar{B})$ $(\bar{A}\cup\bar{B})$

Table 2: List of DNF and CNF corresponding to Table 1

representations of linguistic connectives have an experimentally verified effective representativeness associated with them, at least, for the linguistic connectives "AND" and "OR."

3 Compensatory 'AND'

It was experimentally discovered by Zimmermann and Zysno [20], that natural language use of "AND," "OR" by human experts do not correspond to t-norms T and t-conorms S. They proposed latent connectives in human decision making, known as "Compensatory 'AND' " which they defined either as:

$$\mu_{A\theta B} = \mu_{A\cap B}^{1-\gamma}\ \mu_{A\cup B}^{\gamma} \tag{1}$$

or as:

$$\mu_{A\theta B} = (1-\gamma)\mu_{A\cap B} + \gamma \cdot \mu_{A\cup B} \tag{2}$$

where θ is the "Compensatory 'AND' " and A and B are fuzzy sets. The first expression is known as the "Exponential Compensatory 'AND' ," and the second expression is known as the "Convex-Linear Compensatory 'AND'." In a recent investigation [10], it is shown that either:

$$\mu_{\mathrm{DNF}(A\ \mathrm{AND}\ B)} \leq \mu_{A\theta B} \leq \mu_{\mathrm{CNF}(A\ \mathrm{AND}\ B)}$$

or:

$$\mu_{\mathrm{DNF}(A\ \mathrm{OR}\ B)} \leq \mu_{A\theta B} \leq \mu_{\mathrm{CNF}(A\ \mathrm{OR}\ B)}$$

or:

$$\mu_{\mathrm{CNF}(A\ \mathrm{AND}\ B)} \leq \mu_{A\theta B} \leq \mu_{\mathrm{DNF}(A\ \mathrm{OR}\ B)}$$

for certain values of $\gamma \in [0,1]$ and for both the "Exponential" and "Convex-linear" "Compensatory 'AND' ." As shown in Figure 1, a human experts meaning of the linguistic "AND" or "OR" should and could be identified and hence represented either within "OR" interval region, i.e., between

$$\left[\mu_{\mathrm{DNF}(A\ \mathrm{OR}\ B)}, \mu_{\mathrm{CNF}(A\ \mathrm{OR}\ B)}\right]$$

where an experts meaning, i.e., his degree of "OR"ness would be represented with a value within "OR" boundaries, or "AND" interval region, i.e., between,

$$\left[\mu_{\mathrm{DNF}(A\ \mathrm{AND}\ B)}, \mu_{\mathrm{CNF}(A\ \mathrm{AND}\ B)}\right]$$

where his/her degree of "AND"ness would be represented within "AND" boundaries or as an appropriate mixture of "AND" or "OR" in the middle interval region between,

$$\left[\mu_{\mathrm{CNF}(A \text{ AND } B)}, \mu_{\mathrm{DNF}(A \text{ OR } B)}\right]$$

depending on an experts emphasis placed on his/her meaning of "AND"ness and "OR"ness. Only when an expert means definitely "AND," should we use

$$\mu_{\mathrm{DNF}(\mathrm{IF}\ A \text{ AND } B)} = T(a, b)$$

and only when a expert means definitely "OR" should we use

$$\mu_{\mathrm{CNF}(\mathrm{IF}\ A \text{ OR } B)} = S(a, b).$$

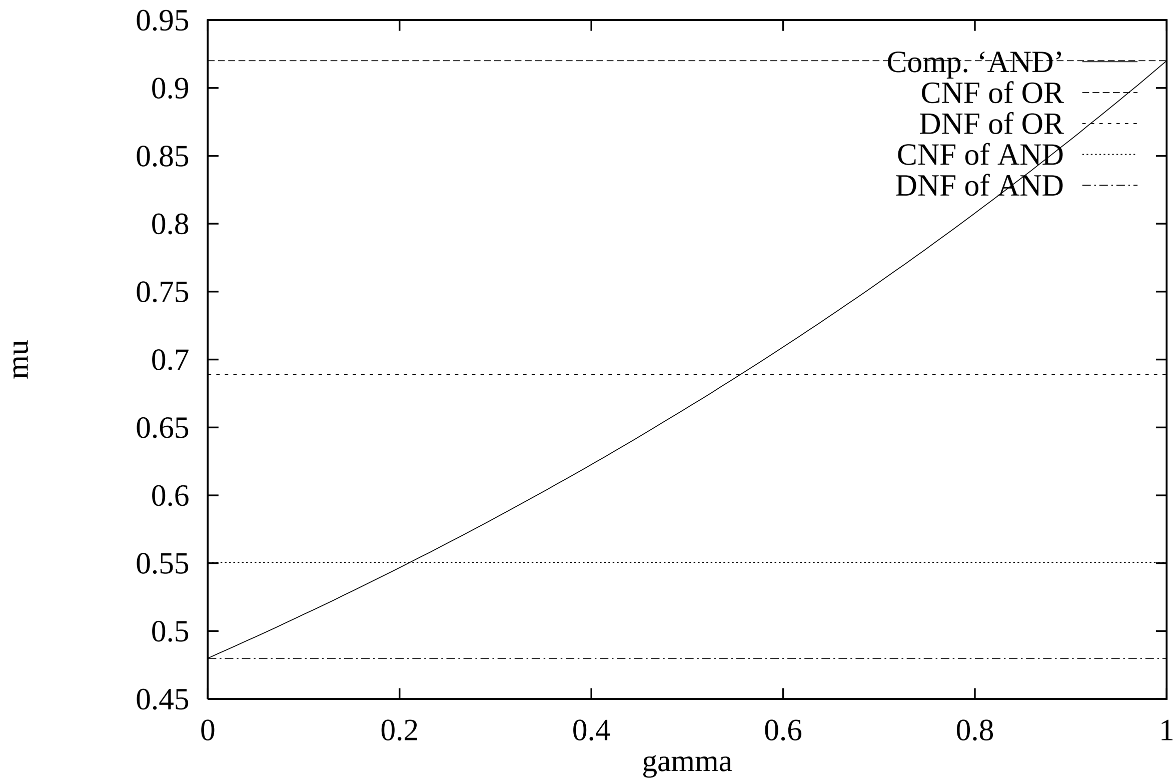

Figure 1: IVFS "A AND B," "A OR B," expressed in DNF and CNF and "Exponential Compensatory AND" using algebraic sum-product for realization of t-conorm and t-norm for $\mu_A = 0.8, \mu_B = 0.6$, and $\gamma \in [0, 1]$.

Otherwise we should express them with appropriate IVFS's as explained above. After the computation is carried out in IVFS to reflect the semantic vagueness associated with these linguistic connectives, then a point-valued representative may be chosen such as mean with the clear indication of the type II spread around it.

4 Interval-valued implication

Just as linguistic connectives "AND," "OR," the linguistic implication "IF ... THEN" can and should be expressed in an interval-valued fuzzy set representation to reflect the vagueness associated with the humans use of "IF ... THEN" in natural languages. In this regard, again it should be noted that "crisp implications," Bandler and Kohout [3], such as:

R6: Łukasiewicz	$a \to_5 b = \min(1, \overline{a} + b)$,
R5.5: Råchenbach	$a \to_{5.5} b = \overline{a} + ab$,
R6: Kleene-Dienes	$a \to_6 b = \overline{a} \vee b$,
R7: Early Zadeh	$a \to_7 b = (a \wedge b) \vee \overline{a}$,
R8: Willmott	$a \to_8 b = (a \to_7 b) \wedge (b \vee \overline{b})$,

are the ones that are researched and applied in most of the current literature. It has been shown [11, 13], that most of these "crisp implications" expressions are contained within the IVFS representation of the linguistic "IF ... THEN" as follows:

$$\begin{array}{rcl}
\mu_{\mathrm{DNF}(A \to B|\vee,\wedge,^-)} \leq & \{R_6, R_7, R_8\} & \leq \mu_{\mathrm{CNF}(A \to B|\vee,\wedge,^-)} \\
\mu_{\mathrm{DNF}(A \to B|\oplus,\cdot,^-)} \leq & \{R_{5\cdot5}\} & \leq \mu_{\mathrm{CNF}(A \to B|\oplus,\cdot,^-)} \\
\mu_{\mathrm{DNF}(A \to B|\vee_L,\wedge_L,^-)} \leq & \{R_5, R_{5.5}, R_6\} & \leq \mu_{\mathrm{CNF}(A \to B|\vee_L,\wedge_L,^-)}
\end{array}$$

where $\to$ represents "IF ... THEN" as a short hand notation, and

$$(\vee, \wedge, ^-),\ (\oplus,\ \cdot\ , ^-),\ (\vee_L, \wedge_L, ^-)$$

are the De Morgan Triples for max-min, algebraic sum-product, and bold union-intersection, respectively. Unfortunately, for the cases of "linguistic implications," we have no experimental results to validate the existence of "Compensatory 'IMPLICATION' " in natural language uses by humans.

5 New compensatory connectives

In this section, we define and investigate the four possible realization of "Compensatory 'AND' " operators when A AND B and A OR B are interpreted as linguistic connectives in terms of their DNF and CNF expressions of t-norms and t-conorms [9]. It should be noticed that compensatory connectives are basically interpreted as the combination of t-norms and t-conorms [20] either exponentially or convex linearly. Hence by replacing t-norms with DNF of AND and CNF of AND, and t-conorms with DNF of OR and CNF of OR, we can obtain four new "Compensatory 'AND' " operators for either exponential compensatory connectives or convex linear compensatory connectives.

5.1 Exponential "Compensatory 'AND' " Operators

We define four exponential "Compensatory 'AND' " based on the normal forms of fuzzy sets proposed by Turksen [14]. Obviously, $T(a,b)$ and $S(a,b)$ in the exponential "Compensatory 'AND' " can be extended to either $\mu_{\text{CNF}(A\text{ AND }B)}$ or $\mu_{\text{DNF}(A\text{ AND }B)}$ and to either $\mu_{\text{CNF}(A\text{ OR }B)}$ or $\mu_{\text{DNF}(A\text{ OR }B)}$, respectively. Thus four normal forms based exponential "Compensatory 'AND' " can be defined as follows:

Definition 1. *The normal form based exponential "Compensatory 'AND' " operators are defined on the membership domain as follows:*

$$I_1^{(\gamma)}(a,b) = \left[\mu_{CNF(A\ AND\ B)}\right]^{1-\gamma}\left[\mu_{CNF(A\ OR\ B)}\right]^{\gamma};$$

$$I_2^{(\gamma)}(a,b) = \left[\mu_{DNF(A\ AND\ B)}\right]^{1-\gamma}\left[\mu_{CNF(A\ OR\ B)}\right]^{\gamma};$$

$$I_3^{(\gamma)}(a,b) = \left[\mu_{CNF(A\ AND\ B)}\right]^{1-\gamma}\left[\mu_{DNF(A\ OR\ B)}\right]^{\gamma};$$

$$I_4^{(\gamma)}(a,b) = \left[\mu_{DNF(A\ AND\ B)}\right]^{1-\gamma}\left[\mu_{DNF(A\ OR\ B)}\right]^{\gamma}.$$

Obviously $\theta_1^{(\gamma)}(a,b) = I_2^{(\gamma)}(a,b) = \left[\mu_{\text{DNF}(A\text{ and }B)}\right]^{1-\gamma}\left[\mu_{\text{CNF}(A\text{ OR }B)}\right]^{\gamma}$ is the Zimmermann and Zysno [20] definition of the "Compensatory 'AND'." We derive the following properties:

Theorem 1. *For $\theta_1^{(\gamma)}(a,b)$, $I_1^{(\gamma)}(a,b)$, $I_2^{(\gamma)}(a,b)$, and $I_4^{(\gamma)}(a,b)$, $\forall\ a,b,\ \gamma\in[0,1]$, the following relations hold:*

1) $\theta_1^{(\gamma)}(a,b)=I_2^{(\gamma)}(a,b)=\left[\mu_{DNF(A\text{ AND }B)}\right]^{1-\gamma}\left[\mu_{DNF(A\text{ OR }B)}\right]^{\gamma}$;

2) $\theta_1^{(\gamma)}(a,b)\le I_1^{(\gamma)}(a,b)=\left[\mu_{CNF(A\text{ AND }B)}\right]^{1-\gamma}\left[\mu_{CNF(A\text{ OR }B)}\right]^{\gamma}$;

3) $\theta_1^{(\gamma)}(a,b)\ge I_4^{(\gamma)}(a,b)=\left[\mu_{DNF(A\text{ AND }B)}\right]^{1-\gamma}\left[\mu_{DNF(A\text{ OR }B)}\right]^{\gamma}$.

It is clear that the relation between $I_3^{(\gamma)}(a,b)$ and $I_1^{(\gamma)}(a,b)$, and the relation between $I_3^{(\gamma)}(a,b)$ and $I_4^{(\gamma)}(a,b)$ are quite obvious. In fact,

$$I_3^{(\gamma)}(a,b)\ge I_4^{(\gamma)}(a,b)$$

and

$$I_3^{(\gamma)}(a,b)\le I_1^{(\gamma)}(a,b).$$

All these properties can be derived from the properties of the conjunctive and disjunctive normal forms of fuzzy sets and the monotonicity of the function $f(x,y)=x^{(1-\gamma)}y^{(\gamma)}$. However the relation between $I_3^{(\gamma)}(a,b)$ and $\theta_1^{(\gamma)}(a,b)$ is not as obvious as others. We have noticed that the relation is a conditional one as the relation largely depends on the ranges that it lies in. In order to study this relation, we construct a new function as follows:

$$h(\gamma)=\theta_1^{(\gamma)}(a,b)-I_3^{(\gamma)}(a,b)=\left[\mu_{\mathrm{DNF}(A\text{ AND }B)}\right]^{1-\gamma}\left[\mu_{\mathrm{CNF}(A\text{ OR }B)}\right]^{\gamma}-\left[\mu_{\mathrm{CNF}(A\text{ AND }B)}\right]^{1-\gamma}\left[\mu_{\mathrm{DNF}(A\text{ OR }B)}\right]^{\gamma}.$$

Let

$$\mu_{\mathrm{DNF}(A\text{ AND }B)}=r_1,\quad \mu_{\mathrm{CNF}(A\text{ AND }B)}=r_2,\quad \mu_{\mathrm{DNF}(A\text{ OR }B)}=p_1,$$

$$\mu_{\mathrm{CNF}(A\text{ OR }B)}=p_2.$$

Hence, the function $h(\gamma)$ can be rewritten as:

$$h(\gamma)=\theta_1^{(\gamma)}(a,b)-I_3^{(\gamma)}(a,b)=r_1^{(1-\gamma)}p_2^{\gamma}-r_2^{(1-\gamma)}p_1^{\gamma} \tag{3}$$

where $r_1\le r_2$ and $p_1\le p_2$.

The equation that defines $h(\gamma)$ is complicated. By studying the differential properties of $h(\gamma)$, we can conclude the following theorem:

Theorem 2. *For the function $h(\gamma)$ constructed as above in equation (3), if $h(\gamma) \neq 0$ then $h(\gamma)$ has one and only one $\gamma_0 \in [0,1]$, so that $h(\gamma_0) = 0$, $h(\gamma \leq \gamma_0) \leq 0$, and $h(\gamma > \gamma_0) \geq 0$.*

In order to find γ_0, we simply set:

$$h(\gamma) = r_1^{(1-\gamma)} p_1^{\gamma} \big[(p_2/p_1)^{\gamma} - (r_2/r_1)^{(1-\gamma)}\big] = h_1(\gamma) h_2(\gamma) = 0 \tag{4}$$

where $h_1(\gamma) = r_1^{(1-\gamma)} p_1^{\gamma}$; $h_2(\gamma) = (p_2/p_1)^{\gamma} - (r_2/r_1)^{(1-\gamma)}$, and the γ derived from (3) is γ_0. Since $h_1(\gamma) > 0$ for all $r_1 > 0$ and $p_1 > 0$, then $h(\gamma) = 0 \Leftrightarrow h_2(\gamma) = 0$. Hence, we have $(p_2/p_1)^{\gamma_0} - (r_2/r_1)^{(1-\gamma_0)} = 0$,

$$\gamma_0 = \ln \frac{r_2}{r_1} \Big/ \ln \frac{p_2 r_2}{p_1 r_1} \tag{5}$$

where $r_1 = \mu_{\mathrm{DNF}(A \text{ AND } B)}$, $r_2 = \mu_{\mathrm{CNF}(A \text{ AND } B)}$, $p_1 = \mu_{\mathrm{DNF}(A \text{ OR } B)}$, $p_2 = \mu_{\mathrm{CNF}(A \text{ OR } B)}$.

Since $r_1 \geq r_2$ and $p_1 \geq p_2$, then $0 \leq \gamma_0 \leq 1$ is derivable from formula (4). Based on this analysis, we can conclude the relation between $I_3^{(\gamma)}(a,b)$ and $\theta_1^{(\gamma)}(a,b)$ in the following theorem:

Theorem 3. *The normal form based exponential "Compensatory 'AND'" operator $I_3^{(\gamma)}(a,b)$ has the following properties with $\theta_1^{(\gamma)}(a,b)$:*

1) when $\gamma \leq \gamma_0$, $I_3^{(\gamma)}(a,b) \geq \theta_1^{(\gamma)}(a,b)$, i.e., $h(\gamma) \leq 0$;

2) when $\gamma > \gamma_0$, $I_3^{(\gamma)}(a,b) < \theta_1^{(\gamma)}(a,b)$, i.e., $h(\gamma) > 0$,

where γ_0 is given by formula (5) and $\gamma_0 \in [0,1]$.

The overall properties of the four normal form based exponential "Compensatory 'AND'" operators are summarized in the following theorem:

Theorem 4. *Normal form based exponential "Compensatory 'AND'" operators defined in accordance with Definition 1 satisfy the following relation:*

i) when $\gamma > \gamma_0$:

$$I_4^{(\gamma)}(a,b) \leq I_3^{(\gamma)}(a,b) < \theta_1^{(\gamma)}(a,b) = I_2^{(\gamma)}(a,b) \leq I_1^{(\gamma)}(a,b);$$

ii) when $\gamma \leq \gamma_0$*:*

$$I_4^{(\gamma)}(a,b) \leq \theta_1^{(\gamma)}(a,b) = I_2^{(\gamma)}(a,b) \leq I_3^{(\gamma)}(a,b) \leq I_1^{(\gamma)}(a,b)$$

where γ_0 *is given by the formula (5).*

5.2 Convex linear "Compensatory 'AND' " operators

Similarly we also define four convex linear "Compensatory 'AND' " operators based on the normal forms of fuzzy sets:

Definition 2. *The four normal form based fuzzy convex linear "Compensatory 'AND' " operators are defined on the membership domain as follows:*

$$J_1^{(\gamma)}(a,b) = (1-\gamma)\mu_{CNF(A\text{ AND }B)} + \gamma \cdot \mu_{CNF(A\text{ OR }B)};$$

$$J_2^{(\gamma)}(a,b) = (1-\gamma)\mu_{DNF(A\text{ AND }B)} + \gamma \cdot \mu_{CNF(A\text{ OR }B)};$$

$$J_3^{(\gamma)}(a,b) = (1-\gamma)\mu_{CNF(A\text{ AND }B)} + \gamma \cdot \mu_{DNF(A\text{ OR }B)};$$

$$J_4^{(\gamma)}(a,b) = (1-\gamma)\mu_{DNF(A\text{ AND }B)} + \gamma \cdot \mu_{DNF(A\text{ OR }B)}.$$

It should be noticed that

$$\theta_2^{(\gamma)}(a,b) = J_2^{(\gamma)}(a,b) = (1-\gamma)\mu_{\mathrm{DNF}(A\text{ AND }B)} + \gamma \cdot \mu_{\mathrm{CNF}(A\text{ OR }B)}$$

is the Zimmermann and Zysno [20] definition of the "Compensatory 'AND' ." Similar to Section 5.1, we derive the following properties based on Definition 2 and the properties of t-norms and t-conorms.

Theorem 5. *For* $\theta_2^{(\gamma)}(a,b)$*,* $J_1^{(\gamma)}(a,b)$*,* $J_2^{(\gamma)}(a,b)$*, and* $J_4^{(\gamma)}(a,b)$*,* $\forall a,b,\gamma \in [0,1]$*, the following relations hold:*

1) $\theta_2^{(\gamma)}(a,b) = J_2^{(\gamma)}(a,b) = (1-\gamma)\mu_{DNF(A\text{ AND }B)} + \gamma \cdot \mu_{CNF(A\text{ OR }B)};$

2) $\theta_2^{(\gamma)}(a,b) \leq J_1^{(\gamma)}(a,b) = (1-\gamma)\mu_{CNF(A\text{ AND }B)} + \gamma \cdot \mu_{CNF(A\text{ OR }B)};$

3) $\theta_2^{(\gamma)}(a,b) \geq J_4^{(\gamma)}(a,b) = (1-\gamma)\mu_{DNF(A\text{ AND }B)} + \gamma \cdot \mu_{DNF(A\text{ OR }B)}.$

The relation between $J_3^{(\gamma)}(a,b)$ and $J_1^{(\gamma)}(a,b)$, and the relation between $J_3^{(\gamma)}(a,b)$ and $J_4^{(\gamma)}(a,b)$ are quite obvious. In fact,

$$J_3^{(\gamma)}(a,b) \geq J_4^{(\gamma)}(a,b),$$

$$J_3^{(\gamma)}(a,b) \leq J_1^{(\gamma)}(a,b).$$

All these properties can be derived from the properties of the triangular norms. The relation between $J_3^{(\gamma)}(a,b)$ and $\theta_2^{(\gamma)}(a,b)$ is not so obvious as other operators. It should be noticed that this relation is also a conditional one as the relation largely depends on the range of values that it lies in. In order to study the relation, we construct a new function as follows:

$$H = \theta_2^{(\gamma)}(a,b) - J_3^{(\gamma)}(a,b).$$

Let $\mu_{\mathrm{DNF}(A \text{ AND } B)} = r_1$, $\mu_{\mathrm{CNF}(A \text{ AND } B)} = r_2$, $\mu_{\mathrm{DNF}(A \text{ OR } B)} = p_1$, $\mu_{\mathrm{CNF}(A \text{ OR } B)} = p_2$, hence the function $H(\gamma)$ can be rewritten as:

$$H(\gamma) = \theta_2^{(\gamma)}(a,b) - J_3^{(\gamma)}(a,b) = \gamma(p_2 - p_1 + r_2 - r_1) - (r_2 - r_1) \tag{6}$$

where $r_1 \leq r_2$ and $p_1 \leq p_2$.

Theorem 6. *For the function $H(\gamma)$ constructed in (6), if $H(\gamma) \neq 0$, then $H(\gamma)$ is a monotonic increasing function of γ and has one and only one $\gamma_1 \in [0,1]$, so that $H(\gamma_1) = 0$, $H(\gamma \leq \gamma_1) \leq 0$ and $H(\gamma > \gamma_1) \geq 0$.*

In order to find γ_1, we simply set $H(\gamma) = 0$. Hence the solution:

$$\gamma_1 = \frac{r_2 - r_1}{(p_2 - p_1) + (r_2 - r_1)} \tag{7}$$

where $r_1 = \mu_{\mathrm{DNF}(A \text{ AND } B)}$, $r_2 = \mu_{\mathrm{CNF}(A \text{ AND } B)}$, $p_1 = \mu_{\mathrm{DNF}(A \text{ OR } B)}$, $p_2 = \mu_{\mathrm{CNF}(A \text{ OR } B)}$.

From (7), it is clear that $0 \leq \gamma_1 \leq 1$. Hence, based on this analysis, we summarize the relation between $J_3^{(\gamma)}(a,b)$ and $\theta_2^{(\gamma)}(a,b)$ and in the following theorem.

Theorem 7. *The normal form based convex linear "Compensatory 'AND' " operator $J_3^{(\gamma)}(a,b)$ has the following properties with $\theta(\gamma)$:*

1) *when* $\gamma \leq \gamma_1$, $J_3^{(\gamma)}(a,b) \geq \theta_2^{(\gamma)}(a,b)$, *i.e.,* $H(\gamma) \leq 0$*;*

2) *when* $\gamma > \gamma_1$, $J_3^{(\gamma)}(a,b) < \theta_2^{(\gamma)}(a,b)$, *i.e.,* $H(\gamma) > 0$,

where γ_1 *is given by formula (7) and* $\gamma_1 \in [0,1]$.

The overall properties of the four convex linear "Compensatory 'AND' " operators are summarized in the following theorem.

Theorem 8. *Normal form based convex linear "Compensatory 'AND' " operators defined in Definition 2 satisfy the following relation:*

i) *when* $\gamma > \gamma_1$*:*

$$J_4^{(\gamma)}(a,b) \leq J_3^{(\gamma)}(a,b) < \theta_2^{(\gamma)}(a,b) = J_2^{(\gamma)}(a,b) \leq J_1^{(\gamma)}(a,b);$$

ii) *when* $\gamma \leq \gamma_1$*:*

$$J_4^{(\gamma)}(a,b) \leq \theta_2^{(\gamma)}(a,b) = J_2^{(\gamma)}(a,b) \leq J_3^{(\gamma)}(a,b) \leq J_1^{(\gamma)}(a,b)$$

where γ_1 *is given by the formula (8).*

Example: Suppose $\mu_A = a = 0.7$, $\mu_B = b = 0.2$, $(T,S) = (\cdot, \oplus)$ (i.e., algebraic product, sum), then we have Figure 2 to illustrate the overall relationship among 4 convex linear "Compensatory 'AND'" operators.

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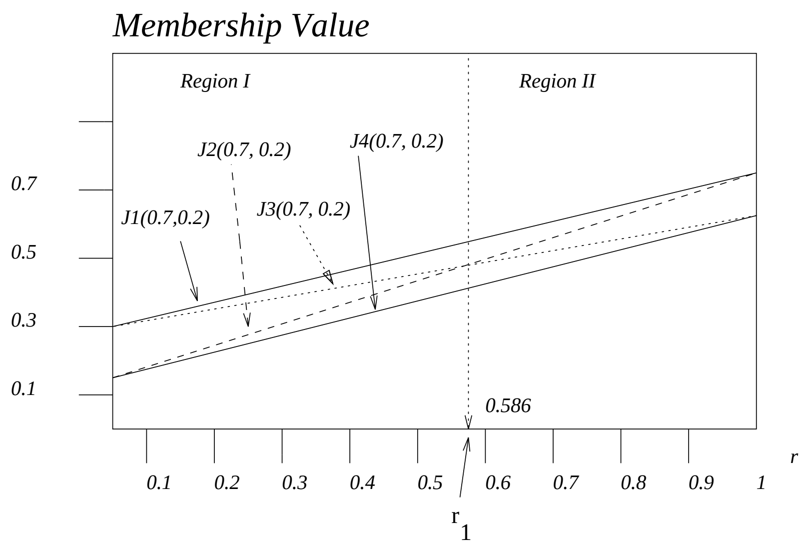

Figure 2: *The relationship, among Ji(r) (i = 1, 2, 3, 4) with respect to r within [0, 1], given that a = 0.7, b = 0.2, and (T, S)=(., +), "compensatory 'AND'" operators have a cross point at r1 = 0.586, where*

$J_1(A, B) = (1 - r)\ CNF(A\ AND\ B) + r\ CNF(A\ OR\ B);$

$J\ (A, B) = (1 - r)\ DNF(A\ AND\ B) + r\ CNF(A\ OR\ B);$

$J_3\ (A, B) = (1 - r)\ CNF(A\ AND\ B) + r\ DNF(A\ OR\ B);$

$J_4\ (A, B) = (1 - r)\ DNF(A\ AND\ B) + r\ DNF(A\ OR\ B).$

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