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Some Interpolation Problems Involving Interval Data

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We consider interpolation of families of functions depending on a parameter by families of interpolation polynomials. Inner and outer inclusions for the interpolating families are constructed in terms of interval and extended interval arithmetic. We achieve tighter inclusions under certain monotonicity assumptions with respect to the parameters involved. Some interpolation polynomials involving directed intervals are studied.

Некоторые интерполяционные задачи, использующие интервальные данные

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Рассматривается интерполяция семейств функций, зависящих от некоторого параметра, семействами интерполяционных многочленов. Внутреннее и внешнее включения для этих интерполяционных семейств строятся с использованием интервальной и расширенной интервальной арифметик. При некоторых предположениях о монотонности входящих в задачу параметров достигаются более тесные включения. Исследуются некоторые интерполяционные многочлены, в которые входят направленные интервалы.

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1 Introduction

Throughout the paper, we consider interpolation involving algebraic polynomials. However, the results obtained can be easily generalized to include interpolation using other classes of interpolating functions such as trigonometric polynomials, exponential functions, etc.

Let $p_n(x, y)$ be the interpolation polynomial of degree $n-1$ taking prescribed values $y = \{y_i\}_{i=1}^n \in R^n$ at a given mesh $x = \{x_i\}_{i=1}^n \in R^n$, $x_1 < x_2 < \cdots < x_n$. Using the Lagrange interpolation formula, we have for $\xi \in R$

$$p_n(x, y; \xi) = \sum_{i=1}^{n} l_i(\xi) y_i, \quad l_i(\xi) = \prod_{j=1,\, j\neq i}^{n} \big((\xi - x_j)/(x_i - x_j)\big). \tag{1}$$

Assume now that we are given intervals $Y_i = [y_i^-, y_i^+] \in IR$ for the values y_i. This means that $y_i \in Y_i$, for $i = 1, \ldots, n$, denoted by $y \in Y = \{Y_i\}_{i=1}^n$. Geometrically, the set x of mesh points and the set Y of intervals define a set of interval segments in the plane denoted by (x, Y). Denote the family of all interpolation polynomials taking at x_i all possible values in the corresponding intervals Y_i, $i = 1, \ldots, n$, by

$$p_n(x, Y) = \{p_n(x, y) \mid y \in Y\}. \tag{2}$$

From a continuity argument, for any fixed $\xi \in R$, the set

$$p_n(x, Y; \xi) = \{p_n(x, y; \xi) | y \in Y\} = \left\{ \sum_{i=1}^{n} l_i(\xi) y_i | y_i \in Y_i,\ i = 1, \ldots, n \right\} \tag{3}$$

with $l_i(\xi)$ defined by (1) is an interval. Thus (3) defines an interval polynomial $p_n(x, Y; \cdot)$ on R with boundary functions p_n^- and p_n^+, such that $p_n(x, Y; \xi) = \big[p_n^-(x, Y; \xi), p_n^+(x, Y; \xi)\big]$. We use the notation $p_n(x, Y)$ for the interval polynomial $p_n(x, Y; \cdot)$. It will be clear from the context whether $p_n(x, Y)$ means a family of the form (2) or an interval function that is the envelope of a family.

To our knowledge, the boundary functions p_n^- and p_n^+ have been first investigated in [4] by means of standard polynomial techniques. It has been noticed [10] (see also [7, 8]) that the interval polynomial $p_n(x, Y)$ can be evaluated (that is computed for given arguments) by means of the interval-arithmetic operations "addition of two intervals" and "multiplication of an interval by a real number" defined for $[a^-, a^+], [b^-, b^+] \in IR$, $\alpha \in R$ by

$$\begin{aligned} [a^-, a^+] + [b^-, b^+] &= [a^- + b^-, a^+ + b^+] && (4) \\ \alpha[a^-, a^+] &= \left\{[\alpha a^{-\mathrm{sign}(\alpha)}, \alpha a^{\mathrm{sign}(\alpha)}],\ \alpha \neq 0;\ 0,\ \alpha = 0\right\} && (5) \end{aligned}$$

where for $\alpha \neq 0$, $\mathrm{sign}(\alpha) = \{-, \alpha < 0; +, \alpha > 0\}$ and $a^{-+} = a^-$, $a^{--} = a^+$ (see [1, 20, 21, 26]). Using the interval-arithmetic operations (4) and (5), we can represent a set of the form $\left\{\sum_{i=1}^n \alpha_i y_i \mid y \in Y\right\}$ as

$$\begin{aligned} \left\{\sum_{i=1}^n \alpha_i y_i \mid y_i \in Y_i, i = 1,\ \ldots,\ n\right\} &= \sum_{i=1}^n \alpha_i Y_i \\ &= \left[\sum_{i=1}^n \alpha_i y_i^{-\mathrm{sign}(\alpha_i)}, \sum_{i=1}^n \alpha_i y_i^{\mathrm{sign}(\alpha_i)}\right] && (6) \end{aligned}$$

Applying Equation (6), we obtain from (3) for $\xi \in R$

$$p_n(x, Y; \xi) = \sum_{i=1}^n l_i(\xi) Y_i. \tag{7}$$

Formula (7) offers a simple and remarkable example of a nontrivial application of interval arithmetic. Indeed, without interval arithmetic, the interval interpolation polynomial (7) and its boundary functions cannot be represented in such a concise form. Let $p_n^-(x, Y)$ and $p_n^+(x, Y)$ denote the lower and the upper bounds, respectively, of (7). That is, $p_n^-(x, Y)$ and $p_n^+(x, Y)$ are the real-valued functions satisfying $p_n(x, Y; \xi) = \left[p_n^-(x, Y; \xi), p_n^+(x, Y; \xi)\right]$ for $\xi \in R$. The boundary functions of (7) are piecewise polynomial functions. More precisely, they are polynomials in each subinterval $[x_k, x_{k+1}]$, $k = 0,\ \ldots,\ n$, where $x_0 = -\infty$ and $x_{n+1} = \infty$. However, in different subintervals, the boundary functions are (generally) pieces of different polynomials. Indeed, using (6), relation (7) can be written in terms of end points as

$$\left[p_n^-(x, Y; \xi), p_n^+(x, Y; \xi)\right] = \left[\sum_{i=1}^n l_i(\xi) y_i^{-\mathrm{sign}(l_i(\xi))}, \sum_{i=1}^n l_i(\xi) y_i^{-\mathrm{sign}(l_i(\xi))}\right]$$

for $\xi \in R$. Considering the signs of $l_i(\xi)$ in the corresponding intervals, we see that the upper bound $p_n^+(x, Y)$ of (7) in the interval $[x_k, x_{k+1}]$ is (piece of) the interpolation polynomial $p^+_{n,k}$ determined by the points (x_{k-2j}, y^+_{k-2j}), $(x_{k+2j+1}, y^+_{k+2j+1})$, $(x_{k-2j-1}, y^-_{k-2j-1})$, (x_{k+2j}, y^-_{k+2j}), $j = 0, 1, 2, \ldots$, where all mesh points x_i, $i = 1, \ldots, n$, are involved. (This is a definition of $p^+_{n,k}$!) The lower bound $p^-_n(x, Y)$ in the interval $[x_k, x_{k+1}]$ is (piece of) the interpolation polynomial $p^-_{n,k}$ passing through the alternative end points of the vertical segments $(x_i, Y_i)_{i=1}^n$, the points having reversed ($\pm$)-signs as upper indices for their y-components, namely (x_{k-2j}, y^-_{k-2j}), $(x_{k+2j+1}, y^-_{k+2j+1})$, $(x_{k-2j-1}, y^+_{k-2j-1})$, (x_{k+2j}, y^+_{k+2j}), $j = 0, 1, 2, \ldots$. However, this is only valid in the interval $[x_k, x_{k+1}]$. In another interval $[x_l, x_{l+1}]$, $l \neq k$, the boundary functions are pieces of different interpolation polynomials $p^-_{n,l}, p^+_{n,l}$ in general. Thus we have

$$p_n(x, Y; \xi) = \left[p^-_n(\xi), p^+_n(\xi)\right] = \left[p^-_{n,k}(\xi), p^+_{n,k}(\xi)\right], \qquad \xi \in [x_k, x_{k+1}] \tag{8}$$

where the functions $p^-_{n,k}$ and $p^+_{n,k}$ are the interpolating polynomials on R defined by n points of their graphs as explained above. The boundary functions p^-_n and p^+_n are continuous, but not necessarily differentiable, at the mesh points and are polynomials in each subinterval.

For $\xi \in R$, we can write

$$\begin{aligned} p_n(x, Y; \xi) &= \bigcup_{k=0}^{n+1} \left[p^-_{n,k}(\xi), p^+_{n,k}(\xi)\right] \\ &= \left[\min_k\{p^-_{n,k}(\xi)\}, \max_k\{p^+_{n,k}(\xi)\}\right] \end{aligned} \tag{9}$$

where $x_0 = -\infty$ and $x_{n+1} = \infty$. Relation (9) represents $p_n(x, Y; \xi)$ on the whole line by the polynomials $p^-_{n,k}$ and $p^+_{n,k}$ (not by the splines p^-_n and p^+_n). The interval polynomial (7) comprises the set of all polynomials of degree $n - 1$ lying between p^- and p^+.

It is somewhat astonishing that the simple interval arithmetic expression (7) presents such a complex interval function whose boundaries are piecewise polynomial functions. Standard “real” arithmetic cannot provide a simple expression for the boundary functions over the whole real line. The “secret” is hidden in the fact that (7) actually comprises as many “real” expressions as is the number of subintervals generated by the mesh points. Indeed, the signs of the Lagrangian coefficients $l_i(\xi)$, $i = 1, \ldots, n$, have particular values in each subinterval, which, as seen from (7) and (5), leads to particular expressions for the boundary functions in each subinterval.

We point out that the interpolation polynomials belonging to the family (2) arbitrarily intersect the vertical segments $(x_i, Y_i), i = 1, \ldots, n$, and the intersection points are not interrelated. It is important to study families which intersect (some of the) vertical segments in certain interdependent way. An interesting practical situation happens when we know that (some of) the vertical segments are traced by the family (2) monotonically in certain direction. Such a knowledge can be taken into account when estimating (outer and inner) bounds for the envelope of the family.

Example 1. As an example, let us fix an arbitrary set (x, Y) of vertical segments and denote by $p_t(x, Y)$ the family of polynomials

$$\left\{p_t(x, Y; \xi) = \sum_{i=1}^{n} l_i(\xi)(\mu(Y_i) + t\varepsilon_i\rho(Y_i)) \mid t \in [-1, 1]\right\}, \quad \xi \in R \tag{10}$$

where $\mu(Y_i)$ and $\rho(Y_i) > 0$ are the center and the radius, respectively, of the interval Y_i for $i = 1, \ldots, n$ and $\varepsilon_i = 1$ for $i = 1, \ldots, n$. For any fixed $\xi = x_i, i = 1, \ldots, n$, the functions $p_t(x, Y; \xi)$ are monotone increasing on t. Geometrically, the intersection points of the graphs of the polynomials $p_t(x, Y)$ with the vertical segments (x, Y) are tracing the latter in positive direction (from left to right) whenever the parameter t traces the interval $[-1, 1]$ from left to right. The envelope of this family can be substantially narrower than the envelope of a family that randomly intersects the segments, as is the case with family (2).

Example 2. For this example, we substitute in (10) different values for the ε_i's. Fix k (that is the subinterval $[x_k, x_{k+1}]$), and set $\varepsilon_k = 1, \varepsilon_{k-1} = -1, \varepsilon_{k-2} = 1, \varepsilon_{k-3} = -1, \ldots$, respectively, $\varepsilon_{k+1} = 1, \varepsilon_{k+2} = -1, \varepsilon_{k+3} = 1, \varepsilon_{k+4} = -1, \ldots$, etc. Denote the family (10) with this choice of the ε_i's by $q_t^{(k)}(x, Y; \xi)$. The intersection points of the family $q_t^{(k)}$ with the vertical segments (x, Y) trace the latter monotonically whenever t varies from -1 to 1. In particular, the two vertical segments (x_k, Y_k), (x_{k+1}, Y_{k+1}) are traced in positive direction, the next two neighboring (towards outside) segments (x_{k-1}, Y_{k-1}), (x_{k+2}, Y_{k+2}) are traced in negative direction, and so on in an alternating manner. Recalling the polynomials $p_{n,k}^+$ and $p_{n,k}^-$ from (8), we see that $q_t^{(k)}(x, Y; \xi) \mid_{t=1} = p_{n,k}^+$ and $q_t^{(k)}(x, Y; \xi) \mid_{t=-1} = p_{n,k}^-$ for $\xi \in R$. This suggests that the family $q_t^{(k)}$ can be presented in the following way:

$$\left\{q_t^{(k)}(x, Y; \xi) = \frac{1-t}{2} \cdot p_{n,k}^-(\xi) + \frac{1+t}{2} \cdot p_{n,k}^+(\xi) \mid t \in [0, 1]\right\}, \quad \xi \in R. \tag{11}$$

The envelope of the family $q_t^{(k)}$ is equal to $p_n(x, Y)$ in the interval $[x_k, x_{k+1}]$. This is not true outside this interval in general.

Example 2 points out one special case of a one-parameter family whose envelope reaches (at least in certain intervals) the interval interpolation polynomial (7) generated by this family. However, this family presents an "extreme case". Normally, the interval interpolation polynomial presents a too rough tool for enclosing parametric families. In order to obtain tighter inclusions, we have to impose additional assumptions and to look for more sophisticated tools.

It is easy to give numerical examples showing that for one set (x, Y) of vertical segments, the envelopes of the families (10), (11), and (2) substantially differ. On the other hand, in practice we may have information about monotonicity of functions (as result of experiments over observed quantities) with respect to certain parameters that can be varied in the course of the experiments. Such information can be used to avoid overestimation of the envelope of the corresponding parametric family.

As another motivation for the present study, let us mention that our approach allows us to generalize the classical setting when interpolation is related not just to discrete numerical values but to a function f belonging to a given class. Recall that for a sufficiently smooth f, the distance between $y(t) = f(t; \cdot)$ and the corresponding interpolation polynomial $p_n(x, y(t))$ can be estimated by (see e.g. [24, 2])

$$|f(t;\xi) - p_n(x, y(t); \xi)| = \frac{1}{n!} \cdot \left| \frac{\partial^n f(t;\xi^0)}{\partial \xi^n} \right| \cdot \prod_{i=1}^{n} |\xi - x_i| \tag{12}$$

where ξ^0 belongs to the interval comprising the points $\xi, x_1, x_2, \ldots, x_n$. Symbolically, $\xi^0 \in [\xi \vee x_1 \vee x_2 \vee \ldots \vee x_n]$. We are interested in corresponding estimates in the situation when intervals are known for the values $f(x_i)$ (see [13] and [2] for a similar setting). In what follows, we consider functions f depending on a real-valued parameter.

Besides the practical motivations for the present study, there is also an element of "purely academic interest". We have seen that the envelope of the family (2) can be expressed by means of interval arithmetic in a very simple form. The family (11) cannot be expressed by standard interval arithmetic, but it can be presented by the directed interval arithmetic, which will be discussed in Section 3. It will be shown that a large class of parametric families can be successfully treated by directed interval arithmetic.

2 Interpolation of families of functions depending on a parameter

Let $f(t;\xi)$ be a real function defined on $T^* \bigotimes X$, $T^* \in IR$, $X \subseteq R$. Assume that $f(t;\xi)$ is continuous on $t \in T^*$ for every $\xi \in X$. For every fixed $t \in T^*$, $f(t;\cdot)$ is a function defined on X, which we shall sometimes denote by $y(t) = f(t;\cdot)$. Denote the family of all $y(t)$ for $t \in T = [t^-, t^+] \subseteq T^*$ by

$$y(T) = f(T;\cdot) = \{f(t;\cdot) \mid t \in T\}. \tag{13}$$

For every $\xi \in X$, we have $f(T;\xi) \in IR$ so that the (envelope of the) family (13) is an interval-valued function on X.

Fix $T \in IR$, $T \subseteq T^*$, and denote by $X_{f,T}$ the set of all $\xi \in X$, such that $f(t;\xi)$ is monotone in t on T. If f is differentiable with respect to t, then for any fixed $\xi \in X_{f,T}$, the value of $\partial f(t;\xi)/\partial t$ does not change sign whenever t traces T. However, this sign may be different for two $\xi_1, \xi_2 \in X_{f,T}$, $\xi_1 \neq \xi_2$.

Let $y(t) \in y(T)$, and let $p_n(x, y(t);\xi)$ be the interpolation polynomial to $y(t)$ of degree $n-1$ along the mesh $x = \{x_i\}_{i=1}^n \in X,\ x_1 < x_2 < \cdots < x_n$. Let $y_i(t) = f(t;x_i),\ i = 1, \ldots, n$. Using (1) we have

$$p_n\bigl(x, y(t);\xi\bigr) = \sum_{i=1}^n l_i(\xi) y_i(t), \qquad l_i(\xi) = \prod_{j=1,\, j\neq i}^n \bigl((\xi - x_j)/(x_i - x_j)\bigr). \tag{14}$$

Denote the range of $f(t;x_i)$ over T by $y_i(T) = f(T;x_i) = \{f(t;x_i) \mid t \in T\}$. Assume that the interval polynomial (7) has been generated by the values $y_i(T)$ of the interval-valued function (13) at the mesh points x_i, and consider the distance between both interval-valued functions (13) and (7) at points different from the mesh points x. Note that $p_n(x, Y;\cdot)$ with $Y = \{y_i(T)\}_{i=1}^n$, as defined by (3) or (7), may include polynomials $p_n(x, y;\xi) = \sum_{i=1}^n l_i(\xi) y_i,\ y_i \in y_i(T),\ i = 1, \ldots, n$, which do not interpolate any individual function $f(t;\cdot)$ from the family $\{f(t;\cdot) \mid t \in T\}$ unless all $y_i(T)$ are degenerate intervals since we have $y \neq y(t) = f(t;x)$, in general. Therefore for the distance between the intervals $p_n(x, Y;\xi)$ and $f(T;\xi)$ at $\xi \neq x_i$, we cannot use estimates in terms of smoothness of f analogous to (12) that are valid for the degenerate case $T = t \in T^*$. The following theorem deals with an example of a family $y(T)$ of the type (13), which can be approximated in a certain interval by the interval-valued polynomial (7). The distance between the family $y(T)$ and the interval polynomial can be estimated in terms of the smoothness of y. As a measure for the distance between two intervals $A, B \in IR$, we take $r(A,B) = r\big([a^-,a^+],[b^-,b^+]\big) = \max\big\{|a^- - b^-|, |a^+ - b^+|\big\}$. We also use $|A| = \max\big\{|a^-|, |a^+|\big\}$.

Theorem 1. *Let $f(t;\xi)$ be monotone increasing (decreasing) with respect to $t \in T$ at the mesh points $x_{k-2j}, x_{k+2j+1}, j = 0,\ 1,\ 2,\ \ldots$ and monotone decreasing (increasing) at $x_{k-2j-1}, x_{k+2j},\ j = 0,\ 1,\ 2,\ \ldots$ (all mesh points are involved in alternating order starting from the points x_k, x_{k+1} towards the outside). The following properties hold for the interpolation polynomial $p_n(x, y(t); \xi)$, where $y(t) = f(t;\cdot)$:*

i) $p_n(x, y(t); \xi)$ is monotone increasing (decreasing) in t for $\xi \in [x_k, x_{k+1}]$;

ii) $\big\{p_n(x, y(t); \xi) \mid t \in T\big\} = p_n\big(x, y(T); \xi\big) = \sum_{i=1}^n l_i(\xi) f(T; x_i)$ for $\xi \in [x_k, x_{k+1}]$;

iii) if f is n times differentiable with respect to ξ, then for $\xi \in [x_k, x_{k+1}]$, we have

$$r\Big(p_n\big(x, y(T); \xi\big), f(T;\xi)\Big) \leq \frac{1}{n!} \cdot \left|\frac{\partial^n f(T;X)}{\partial \xi^n}\right| \cdot \prod_{i=1}^n |\xi - x_i|$$

where $X = [\xi \vee x_1 \vee x_2 \vee \ldots \vee x_n]$. That is, X is the smallest interval comprising the mesh points $\{x_i\}_{i=1}^n$ and ξ.

Proof. From $p(\xi) = p_n(x, y(t); \xi) = \sum_{i=1}^n l_i(\xi) f(t; x_i)$, we have

$$\frac{dp_n(\xi)}{dt} = \sum_{i=1}^n l_i(\xi) \frac{\partial f(t; x_i)}{\partial t}.$$

Let $x_k \leq \xi \leq x_{k+1}$. Then the polynomials of degree $(n-1)$ $l_{k-2j}(\xi)$, $l_{k+2j+1}(\xi)$, $j = 0,\ 1,\ 2,\ \ldots$, are positive for $\xi \in [x_k, x_{k+1}]$, whereas the polynomials $l_{k-2j-1}(\xi)$, $l_{k+2j}(\xi)$, $j = 0,\ 1,\ 2,\ \ldots$, are negative in $[x_k, x_{k+1}]$. The assumption of the theorem says that in $[x_k, x_{k+1}]$, we have sign $\left(\frac{\partial f(t;x_i)}{\partial t}\right) = \mathrm{sign}(l_i(\xi)) (= -\mathrm{sign}(l_i(\xi))$, resp.), $i = 1,\ 2,\ \ldots$. Hence $dp_n(\xi)/dt > 0$ (< 0, resp.) for $x_k \leq \xi \leq x_{k+1}$, and case i) is proved.

To show ii), we observe that for $\xi \in [x_k, x_{k+1}]$, the boundary functions of the interval polynomial $p_n(x, y(Y); \xi)$ and of the set $\{p_n(x, y(t); \xi) \mid t \in T\}$ are polynomials of degree $(n-1)$ that have same values at the mesh points $x = \{x_i\}_{i=1}^n$ and therefore coincide. However, the boundary functions of these sets may not coincide outside the interval $[x_k, x_{k+1}]$ where they are pieces of other polynomials. Actually, the boundary functions of the family $\{p_n(x, y(t); \xi) \mid t \in T\}$ are $p_{n,k}^-(\xi), p_{n,k}^+(\xi)$ for all ξ (see (8) and Example 2).

To demonstrate iii), note that for $t = t^-, t^+$, we have

$$\begin{aligned} |f(t;\xi) - p_n(x, y(t); \xi)| &= \frac{1}{n!} \cdot \left| \frac{\partial^n f(t; \xi^0)}{\partial \xi^n} \right| \cdot \prod_{i=1}^n |\xi - x_i| \\ &\le \frac{1}{n!} \cdot \left| \frac{\partial^n f(t; X)}{\partial \xi^n} \right| \cdot \prod_{i=1}^n |\xi - x_i| \end{aligned}$$

where $\xi^0 \in X = [\xi \vee x_1 \vee x_2 \vee \ldots \vee x_n]$. Variation of this inequality for $t \in T$ and using the monotonicity of f and p_n implies the validity of iii) in $[x_k, x_{k+1}]$.

For the particular family $f(T;\xi)$ described in the theorem, case iii) of Theorem 1 gives an estimate for the distance between $f(T, \xi)$ and the corresponding family of interpolating polynomial functions $\{p_n(x, f(t, \cdot)) \mid t \in T\}$ in $[x_k, x_{k+1}]$.

Case ii) of Theorem 1 says that for the family $y(t) = f(t; \cdot)$, the corresponding family of interval polynomials for $\xi \in [x_k, x_{k+1}]$ equals the generated interval interpolation polynomial. That is,

$$\{p_n(x, f(t; x); \xi) \mid t \in T\} = \sum_{i=1}^n l_i(\xi) f(T; x_i), \qquad \xi \in [x^k, x^{k+1}]. \tag{15}$$

However, relation (15) is true only for $\xi \in [x^k, x^{k+1}]$ and for the very restrictive case considered in the theorem. In what follows, we look for similar interval-arithmetic estimates under the more general assumption that $f(t;\xi)$ is monotone on t at each mesh point $\xi = x_i$, without specifying the monotonicity of f at x_i as this was required in Theorem 1. To this end, we use extended interval arithmetic. The results can be equally well formulated either by using normal intervals and nonstandard operations [14–16, 18], or by using directed intervals and extended interval arithmetic [9, 11, 12, 19, 22]. We use the latter form here, so we next give some basic concepts of the extended interval arithmetic over directed intervals.

3 An interpolation polynomial involving directed intervals

A directed interval on R is a pair of reals $[a^-, a^+]$, $a^-, a^+ \in R$. The set of all directed intervals is denoted by D. Addition of directed intervals and multiplication by a real number $\alpha \in R$ are defined as extensions of (4), that is:

$$[a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+],\ [a^-, a^+], [b^-, b^+] \in D; \quad (16)$$

$$\alpha[a^-, a^+] = [\alpha a^{-\mathrm{sign}(\alpha)}, \alpha a^{\mathrm{sign}(\alpha)}],\ [a^-, a^+] \in D, \alpha \neq 0; \quad 0[a^-, a^+] = 0. \quad (17)$$

Whenever appropriate, we denote directed intervals by boldface letters. The basic operations (16) and (17) involve a variety of derived operations. We define *negation* by $-\mathbf{A} = (-1)\mathbf{A} = [-a^+, -a^-]$ and *subtraction* by $\mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B}) = [a^- - b^+, a^+ - b^-]$. To every $\mathbf{A}$, there exists an *additive inverse* directed interval $-_h\mathbf{A} = [-a^-, -a^+]$, generating the operation *hyperbolic subtraction* $\mathbf{A} -_h \mathbf{B} = \mathbf{A} + (-_h\mathbf{B}) = [a^- - b^-, a^+ - b^+]$. The *conjugated* (or dual) directed interval is defined by $\mathbf{A}_- = -(-_h\mathbf{A}) = -_h(-\mathbf{A}) = [a^+, a^-]$. Note that $\mathbf{A}-_h\mathbf{B} = \mathbf{A}-\mathbf{B}_-$, which is $\neq \mathbf{A}-\mathbf{B}$ in general. The *direction* of $\mathbf{A}$ is defined by $\tau(\mathbf{A}) = +$, if $a^- \leq a^+$, and $\tau(\mathbf{A}) = -$, otherwise. Directed intervals with positive direction are called positively directed (not to be confused with positive!) or *proper* intervals. Denote $\mathbf{A}_+ = \mathbf{A}$. Then the directed interval $\mathbf{A}_{\tau(\mathbf{A})}$ has a positive direction for every $\mathbf{A} \in D$ and is called the *proper part* (or the *prop*) of $\mathbf{A}$. Symbolically, $\mathrm{prop}(\mathbf{A}) = \mathbf{A}_{\tau(\mathbf{A})}$. The set of all proper intervals is equivalent to the set of normal intervals IR and will be denoted by IR. The set of negatively directed (improper) intervals is denoted by IR_-. The set D with the operations (16), (17) satisfies all basic relations of a linear space, except for the relation $(\alpha + \beta)\mathbf{C} = \alpha\mathbf{C} + \beta\mathbf{C}$. This relation is replaced in D by the simple conditionally distributive law $(\alpha + \beta)\mathbf{C}_{\mathrm{sign}(\alpha+\beta)} = \alpha\mathbf{C}_{\mathrm{sign}(\alpha)} + \beta\mathbf{C}_{\mathrm{sign}(\beta)}$ [18, 19].

Since D is an extension of IR, we assume that all relations in IR hold also in the set of proper intervals. For a proper interval $\mathbf{A} \in D$, we write $A = \mathbf{A} = \mathrm{prop}(\mathbf{A}) \in IR$. In particular, inclusion between proper intervals is well defined in the usual manner. We define inclusion between directed intervals via inclusion between their corresponding props by setting $\mathbf{A} \subseteq \mathbf{B} \Longleftrightarrow \mathrm{prop}(\mathbf{A}) \subseteq \mathrm{prop}(\mathbf{B})$ for any two $\mathbf{A}$, $\mathbf{B} \in D$ such that $\mathbf{B} \neq \mathbf{A}_-$. For two dual intervals $\mathbf{A}$, $\mathbf{B} = \mathbf{A}_-$, we may postulate that the negatively directed interval is included in the positively directed one. If an expression involves both directed and normal intervals, we shall consider the normal intervals as proper directed intervals. If an interval expression involves at least one directed interval or at least one (purely) directed operation or relation (such as conjugation), then this expression will be considered as an expression between directed intervals. In accordance with these stipulations, inclusion between normal and directed intervals also make sense, namely, $A \subseteq \mathbf{B} \Longleftrightarrow A \subseteq \text{prop}(\mathbf{B})$, respectively, $\mathbf{A} \subseteq B \Longleftrightarrow \text{prop}(\mathbf{A}) \subseteq B$. For $a \in R$, $\mathbf{A} \in D$, the inclusion $a \in \mathbf{A}$ is equivalent to $a \in A$ or $a \in \text{prop}(\mathbf{A})$. We note that this concept of inclusion slightly differs from the one considered by E. Kaucher [12], and each one can be expressed by the other. The distance between two directed intervals is defined as $r(\mathbf{A}, \mathbf{B}) = |\mathbf{A} -_h \mathbf{B}|$, where $|\mathbf{A}| = \max\{|a^-|, |a^+|\}$. The width is defined by $\omega(\mathbf{A}) = \omega(\text{prop}(\mathbf{A})) = |a^+ - a^-|$.

We next give two propositions involving expressions for the sum of directed intervals in terms of the set-theoretic operations for joint (connected union) $\bigcup$ and intersection $\bigcap$. The joint and the intersection of two equally directed intervals are directed intervals having the direction of the arguments involved and their props are defined by $\text{prop}(\mathbf{A} \bigcup \mathbf{B}) = \text{prop}(\mathbf{A}) \bigcup \text{prop}(\mathbf{B})$. $\text{prop}(\mathbf{A} \bigcap \mathbf{B}) = \text{prop}(\mathbf{A}) \bigcap \text{prop}(\mathbf{B})$, respectively.

Proposition 1. *i) For* $\mathbf{A},\ \mathbf{B} \in D$ *such that* $\tau(\mathbf{A}) \neq \tau(\mathbf{B})$, *we have*

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{cases} \bigcap_{a \in A} (a + \mathbf{B}), & \text{if } \omega(\mathbf{A}) \leq \omega(\mathbf{B}), \\ \bigcap_{b \in B} (\mathbf{A} + b), & \text{if } \omega(\mathbf{A}) \geq \omega(\mathbf{B}); \end{cases} \\ &= \left(\bigcap_{a \in A} (a + \mathbf{B}) \right) \bigcup \left(\bigcap_{b \in B} (\mathbf{A} + b) \right). \end{aligned} \tag{18}$$

ii) For $\mathbf{A},\ \mathbf{B} \in D$ *such that* $\tau(\mathbf{A}) = \tau(\mathbf{B})$,

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \bigcup_{a \in A} (a + \mathbf{B}) = \bigcup_{b \in B} (\mathbf{A} + b) \\ &= \left(\bigcup_{a \in A} (a + \mathbf{B}) \right) \bigcap \left(\bigcup_{b \in B} (\mathbf{A} + b) \right). \end{aligned} \tag{19}$$

We note that the joint (18) involves an empty interval; the intersection (19) involves two equal directed intervals. Formulae (18) and (19) express the duality between the expressions for $\mathbf{A} + \mathbf{B}$ in both cases, which is equivalent to considering both expressions $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + \mathbf{B}_-$. To clarify this, we present Proposition 1 i) in the following equivalent form

Proposition 2. *For* $\mathbf{A},\ \mathbf{B} \in IR$, *we have*

$$\mathbf{A} + \mathbf{B}_- = \begin{cases} \bigcap_{b\in B}(\mathbf{A} + b), & \textit{if } \mathbf{A} + \mathbf{B}_- \in IR, \\ \bigcap_{a\in A}(a + \mathbf{B}), & \textit{if } \mathbf{A}_- + \mathbf{B} \in IR_-. \end{cases}$$

The next proposition deals with the sum of n directed intervals and is a generalization of Proposition 1. We first introduce some notation. Let $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in D^n$ be a vector of directed intervals. If all components $\mathbf{A}_i$, $i = 1, \ldots, n$, of $\mathbf{A}$ have same direction $\tau(\mathbf{A}_i)$, then the direction of the vector $\mathbf{A}$ is defined by $\tau(\mathbf{A}) = \tau(\mathbf{A}_i)$. For a real vector $a = (a_1, a_2, \ldots, a_n) \in R^n$, the inclusion $a \in \mathbf{A}$ means that $a_i \in \mathbf{A}_i$, $i = 1, \ldots, n$. Also let $\Sigma(\mathbf{A}) = \sum_{i=1}^n \mathbf{A}_i$. In particular, $\Sigma(a) = \sum_{i=1}^n a_i$.

Let $\mathbf{A}' = (\mathbf{A}_{i_1}, \mathbf{A}_{i_2}, \ldots, \mathbf{A}_{i_k}) \in D^k$ and $\mathbf{A}'' = (\mathbf{A}_{i_{k+1}}, \mathbf{A}_{i_{k+2}}, \ldots, \mathbf{A}_{i_n}) \in D^{n-k}$ be two subsets of the interval vector $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$, such that $\mathbf{A} = \{\mathbf{A}', \mathbf{A}''\}$, $1 \le k \le n$. The couple $(\mathbf{A}', \mathbf{A}'')$ will be called a partition of $\mathbf{A}$.

Proposition 3. *Let* $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) \in D^n$, *and let* $(\mathbf{A}', \mathbf{A}'')$ *be a partition of* $\mathbf{A}$.

i) If all components of $\mathbf{A}$ *are of same direction and* $(\mathbf{A}', \mathbf{A}'')$ *is an arbitrary partition of* $\mathbf{A}$, *then*

$$\begin{aligned} \sum_{i=1}^n \mathbf{A}_i &= \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n = \Sigma(\mathbf{A}') + \Sigma(\mathbf{A}'') \\ &= \bigcup_{a'\in A'} \Sigma(a') + \Sigma(\mathbf{A}'') = \bigcup_{a''\in A''} \Sigma(\mathbf{A}') + \Sigma(a''). \end{aligned}$$

ii) If the components of $\mathbf{A}$ *are of different directions and* $(\mathbf{A}', \mathbf{A}'')$ *is a partition of* $\mathbf{A}$ *such that both* $\mathbf{A}'$ *and* $\mathbf{A}''$ *comprise intervals of same direction (and, hence,* $(\mathbf{A}'$ *and* $\mathbf{A}'')$ *are of opposite direction), then*

$$\begin{aligned} \sum_{i=1}^n \mathbf{A}_i &= \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n = \Sigma(\mathbf{A}') + \Sigma(\mathbf{A}'') \\ &= \begin{cases} \bigcap_{\Sigma(a')\in\Sigma(A')} \Sigma(a') + \Sigma(\mathbf{A}''), & \textit{if } \omega(\Sigma(\mathbf{A}')) \le \omega(\Sigma(\mathbf{A}'')) \\ \bigcap_{\Sigma(a'')\in\Sigma(A'')} \Sigma(\mathbf{A}') + \Sigma(a''), & \textit{if } \omega(\Sigma(\mathbf{A}')) \ge \omega(\Sigma(\mathbf{A}'')) \end{cases} \\ &= \begin{cases} \bigcap_{a'\in A'} \Sigma(a') + \Sigma(\mathbf{A}''), & \textit{if } \omega(\Sigma(\mathbf{A}')) \le \omega(\Sigma(\mathbf{A}'')) \\ \bigcap_{a''\in A''} \Sigma(\mathbf{A}') + \Sigma(a''), & \textit{if } \omega(\Sigma(\mathbf{A}')) \ge \omega(\Sigma(\mathbf{A}'')) \end{cases} \end{aligned}$$

$$= \bigcap_{a' \in A'} \left(\Sigma(a') + \Sigma(\mathbf{A}'')\right) \bigcup \bigcap_{a'' \in A''} \left(\Sigma(\mathbf{A}') + \Sigma(a'')\right).$$

Let us consider now an interpolation polynomial of the form (7) involving directed intervals. Let $x = \{x_i\}_{i=1}^n \in X$ be a mesh. Let $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) \in D^n$ be a vector of directed intervals. Consider the expression

$$p_n(x, \mathbf{Y}; \xi) = \sum_{i=1}^{n} l_i(\xi)\mathbf{Y}_i. \tag{20}$$

For every fixed ξ, the value of (20) is a directed interval that can be computed by means of (16) and (17). At the mesh points, the values of (20) are the given directed intervals $\mathbf{Y}_i$.

We next give a set-theoretic interpretation for the value of (20) at the arbitrary point ξ. If all $\mathbf{Y}_i$, $i = 1, \ldots, n$ have the same direction, then according to Proposition 3 i), the directed interval $p_n(x, \mathbf{Y}; \xi)$ has same direction and proper part $\mathrm{prop}(p_n(x, \mathbf{Y}; \xi)) = p_n(x, Y; \xi)$. If $\mathbf{Y}_i$ are of different directions, let us consider a partition $(\mathbf{Y}', \mathbf{Y}'')$ of $\mathbf{Y}$ such that $\mathbf{Y}'$ and $\mathbf{Y}''$ consist of equally directed intervals and $\tau(\mathbf{Y}') \neq \tau(\mathbf{Y}'')$. To be more specific, we assume that $\tau(\mathbf{Y}') = -$, $\tau(\mathbf{Y}'') = +$ and that $\mathbf{Y}' = (\mathbf{Y}_{i_1}, \mathbf{Y}_{i_2}, \ldots, \mathbf{Y}_{i_k}) \in D^k$ and $\mathbf{Y}'' = (\mathbf{Y}_{i_{k+1}}, \mathbf{Y}_{i_{k+2}}, \ldots, \mathbf{Y}_{i_n}) \in D^{n-k}$ for some k, $1 \le k \le n$. We shall further denote by $y' \in R^k$ and $y'' \in R^{n-k}$ two real vectors such that $y' \in \mathbf{Y}'$ and $y'' \in \mathbf{Y}''$ (or equivalently, $y' \in Y'$ and $y'' \in Y''$). Using Proposition 3 ii), we obtain

$$\begin{aligned} p_n(x, \mathbf{Y}; \xi) &= \sum_{i=1}^{n} l_i(\xi)\mathbf{Y}_i = \sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j} \\ &= \bigcap_{y' \in Y'} \left(\sum_{j=1}^{k} l_{i_j}(\xi) y_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j} \right) \\ &\quad \bigcup \bigcap_{y'' \in Y''} \left(\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi) y_{i_j} \right). \end{aligned} \tag{21}$$

The first intersection $\bigcap_{y' \in Y'} \left(\sum_{j=1}^{k} l_{i_j}(\xi) y_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j}\right)$ in expression (21) involves only positively directed (proper) intervals, whereas the second intersection $\bigcap_{y''\in Y''}\left(\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j}+\sum_{j=k+1}^{n} l_{i_j}(\xi)y_{i_j}\right)$ involves only negatively directed (improper) intervals. One of the intersections is always empty, unless both intersections produce the same (real) values. Namely, if the inequality $\omega\left(\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j}\right)<\omega\left(\sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j}\right)$ holds, then the second intersection is empty. If the opposite inequality holds, then the first intersection is empty. If an equality takes place, then both intersections have equal real values. This shows that the boundary functions of (20) are piecewise polynomial functions.

Polynomials of the form (20) find application in the computation of L_k-compatible systems of interval segments as introduced in [17].

4 Interpolation of parametric families using directed ranges

Let $f(t)$ be a continuous and monotone function on $T=[t^-,t^+]\in IR$, denoted by $f\in CM(T)$. The directed interval $[f(t^-),f(t^+)]$ is called the *directed range* of f and is denoted by $f[T]$ or $\mathbf{f}[T]$. Clearly, the directed range $\mathbf{f}[T]$ includes information about: i) the range $f(t)$ and ii) the kind of monotonicity of f on T (nondecreasing/nonincreasing). We give some simple rules for computing with ranges and directed ranges of monotone functions (see also [25]). We denote for brevity $\tau(f[T])=\tau_f$.

Rule 1. If $f,g\in CM(T)$, then $\mathbf{f}[T]+\mathbf{g}[T]\subseteq(f+g)(T)\subseteq f(T)+g(T)$. For the width of $f(T)+g(T)$, we have $\omega_1=\omega(\mathbf{f}[T]+\mathbf{g}[T])\le\omega((f+g)(T))\le\omega(f(T)+g(T))=\omega_2$. The upper bound of $\omega((f+g)(T))$ can be improved by $\omega((f+g)(T))\le(\omega_1+\omega_2)/2$.

Rule 2 [18]**.** If in addition to the assumption of Rule 1 $f,g\in CM(T)$, we assume $h=f+g\in CM(T)$, then $\mathbf{h}[T]=(f+g)[T]=\mathbf{f}[T]+\mathbf{g}[T]$.

Rule 3 [18]**.** If f is monotone on $T\in IR$ and $\alpha\in R$, then for $h=\alpha f$ we have $\mathbf{h}[T]=\alpha\mathbf{f}[T]_{\mathrm{sign}(\alpha)}$.

Rules 1–3 imply

Rule 4. If i) $f_i \in CM(T)$, $i = 1, \ldots, n$, and ii) $\alpha_i \in R$, $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} \alpha_i \mathbf{f}_i[T]_{\mathrm{sign}(\alpha_i)} \subseteq \left(\sum_{i=1}^{n} \alpha_i f_i\right)(T) \subseteq \sum_{i=1}^{n} \alpha_i f_i(T). \tag{22}$$

Rule 5. If in Rule 4 in addition to i) – ii), we assume iii) $h = \sum_{i=1}^{n} \alpha_i f_i \in CM(T)$, then

$$\mathbf{h}[T] = \left(\sum_{i=1}^{n} \alpha_i f_i\right)[T] = \sum_{i=1}^{n} \alpha_i \mathbf{f}_i[T]_{\mathrm{sign}(\alpha_i)}. \tag{23}$$

Remark 1. Note that Rule 1 does not presume monotonicity of $f+g$ and that $\mathbf{f}[T] + \mathbf{g}[T]$ gives substantially inner bounds for $(f+g)(T)$ if f and g are of differently monotonicities. For equally monotone functions f and g, the sum is also monotone, and we can apply Rule 2. However, Rule 1 is valid also for equally monotone functions. In this case, $\mathrm{prop}(\mathbf{f}[T] + \mathbf{g}[T]) = f(T) + g(T)$. Rule 1 can be also expressed in the following way: If $f, g \in CM(T)$, then $(f+g)(T)$ lies between (w. r. t. $\subseteq$) $\mathbf{f}[T] + \mathbf{g}[T]$ and $\mathbf{f}[T] + \mathbf{g}[T]_{\tau_f \tau_g}$.

Remark 2. In Rule 3 the lower index $\mathrm{sign}(\alpha_\mathrm{i})$ changes the direction of the directed range $\mathbf{f}[\mathrm{T}]$ according to the sign of α_i. Note that the multiplication by real number does not change the direction of the directed interval.

Remark 3. Rules 1 and 3 have a simple form for linear functions f and g (or f_i), since then the sum $f+g$ (or $\sum f_i$) is also linear and therefore monotone.

Remark 4. Rules 1 – 3 can be successfully incorporated into an algorithm which automatically finds ranges of functions and their derivatives, such as the one reported in [3]. An extended interval differentiation arithmetic can be developed in the sense of [23].

We next apply the arithmetic for directed intervals to interval-valued functions corresponding to parametric families of functions. Interval-valued functions generated by parametric families have been considered in an early paper on interval arithmetic [26]. Moreover, in this paper interval functions are defined as envelopes of parametric families.

Assume that $f(t;\xi)$ is continuous on $T^* \bigotimes X$ and that for every ξ belonging to some nonempty set $X_{f,T} \subseteq X$, $f(t;\xi) \in CM(T)$, $T = [t^-, t^+] \in IR$, $T \subseteq T^*$. In addition to the interval-valued function $f(T;\cdot) = \{f(t;\cdot) \mid t \in T\}$ defined on X, we can consider a mapping $\mathbf{f}[T;\cdot] : X_{f,T} \longrightarrow D$

defined for $\xi \in X_{f,T}$ by $\mathbf{f}[T;\xi] = \left[f(t^-;\xi), f(t^+;\xi)\right]$, which is the directed range of $f(t;\xi)$ over T.

Let $x_i \in X_{f,T}$, $i = 1, \ldots, n$, $x_1 < x_2 < \cdots < x_n$. That is, let the functions $f(t;x_i)$, $i = 1, \ldots, n$, be monotone on T. Then the directed ranges $\mathbf{f}[T;x_i]$ are defined by $\mathbf{f}[T;x_i] = [f(t^-;x_i), f(t^+;x_i)]$. Each $f(t;\cdot)$ generates an interpolation polynomial p_n passing through the points $(x, f(t;x)) = (x_i, f(t;x_i))_{i=1}^n$:

$$p_n\big(x, f(t;x);\xi\big) = \sum_{i=1}^{n} l_i(\xi) f(t;x_i). \tag{24}$$

Theorem 2. *Assume that the function $f(t;\xi)$ is continuous on $T^* \bigotimes X$ and that the functions $f(t;x_i)$, $i = 1, \ldots, n$, are monotone on $T \in T^*$. Then*
i) for every $\xi \in R$,

$$\sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\operatorname{sign}(l_i(\xi))} \subseteq p_n\big(x, f(T;x);\xi\big) \subseteq \sum_{i=1}^{n} l_i(\xi) f(T;x_i); \tag{25}$$

ii) if (24) is monotone on T at $\xi \in R$, then $p_n\big(x, f(T;x);\xi\big)$ reaches its lower bound in (25), i.e.

$$p_n\big(x, f(T;x);\xi\big) = \sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\operatorname{sign}(l_i(\xi))}. \tag{26}$$

If f is n times differentiable with respect to ξ, then

$$r\big(p_n(x, y(T);\xi), f(T;\xi)\big) \le \frac{1}{n!} \cdot \left|\frac{\partial^n f(T;X)}{\partial \xi^n}\right| \cdot \prod_{i=1}^{n} |\xi - x_i|$$

for $\xi \in [x_k, x_{k+1}]$, where X is an interval containing the mesh points $\{x_i\}_{i=1}^n$ and ξ.

Proof. The proof follows by fixing ξ and applying Rules 4 and 5. Equality (26) is obvious from the more detailed form

$$\begin{aligned} \mathbf{p}_n\big(x, \mathbf{f}[T;x];\xi\big) &= \big[p_n(x, f(t^-;x);\xi), p_n(x, f(t^+;x);\xi)\big] \\ &= \sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\operatorname{sign}(l_i(\xi))}. \end{aligned}$$

An open problem is to find estimates for the interpolation family in the situation when the family f is monotone at some of the knots x_i (and not at all of them).

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