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INTERVAL DISCRETE MODELS AND MULTIOBJECTIVITY. COMPLEXITY ESTIMATES

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We consider known extremal problems on graphs (travelling salesman problem, spanning trees problem, chains problem between a pair of vertices, matching problem) with interval weights. The objective function being an interval weights sum generates the Pareto Set of solutions. The investigation of maximal cardinality of Pareto Set by methods of multiobjective optimization shows that the problems under consideration are intractable when they have criteria only of weight form. However, this problems besides the travelling salesman problem have the polynomial computation complexity in case of maximin criteria.

ИНТЕРВАЛЬНЫЕ ДИСКРЕТНЫЕ МОДЕЛИ И МНОГОКРИТЕРИАЛЬНОСТЬ. ОЦЕНКИ СЛОЖНОСТИ

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Рассматриваются известные экстремальные задачи на графах (о коммивояжере, о кратчайшей связующей сети, о кратчайшей цепи между парой вершин, о совершенных паросочетаниях), ребра которых взвешены интервальными весами. Целевая функция, представляющая собой сумму интервальных весов, порождает паретовское множество решений. Исследование (с помощью аппарата многокритериальности) максимальной мощности паретовского множества показало, что рассматриваемые интервальные задачи являются труднорешаемыми в случае весовых критериев. Однако при введении критериев максиминного вида вычислительная сложность указанных задач, кроме задачи о коммивояжере, оказывается полиномиальной.

In the field of mathematical simulation under conditions of uncertainty to obtain an adequate model of an object investigated, we construct an interval model ([1],[2]).

We consider the case where all initial data that are the objective function parameters are specified as estimates, that is approximately. By now, this case is studied as applied to the linear programming problems [3].

The problem under investigation is concerned with extremal interval problems on graphs. These settings arise, for example, in the simulation land tenure problems [4] where harvest forecasting can't be specified in the form of uniquely defined parameters. There is a similar situation in the electronic devices design, and transport networks [6] and al.

Generally, the problems in question can be described in the following way.

There is a n-vertex graph $G = (V, E)$ in which an interval weight $w(e) = [w_1(e), w_2(e)]$ is assigned to every edge $e \in E$.

A feasible solution of the problem examined is defined in the form of a subgraph $x = (V_x, E_x)$, $V_x \subseteq V$, $E_x \subseteq E$. We denote the set of all feasible solutions by $X = \{x\}$. On X, the objective function (OF) is defined

$$F(x) = \sum_{e \in E_x} w(e). \tag{1}$$

We must find an element $x_0 \in X$ on which the values of OF (1) attain an extremum required, for example, the maximum.

The case of non-interval weights $w(e)$, $e \in E$, is well studied in extremal problems on graphs such that the problems of travelling salesman (x is a Hamiltonian cycle), of shortest chain between a pair of vertices (x is a path between a pair vertices), of minimal spanning tree (x is a spanning tree), of matching (x is matching). For these problems, the problem of optimum determining does not arise, algorithms of finding an optimum being known for each of them.

In the case where parameters of OF (1) are specified intervally, a fundamental question arises of defining the concept of an optimum $x_0 \in X$. All the more, the question of algorithms for finding the best solution remains open.

In the present paper, a constructive solution for above problems is proposed by means of reducing an interval mathematical programming problem to a discrete multiobjective (vector) problem.

First of all, using well-known definitions of binary relations on intervals [1], [7], we refer to the meaning of the best or ideal solution concept.

Definition. A solution $x_0 \in X$ is said to be ideal if on x_0, values of magnitudes of the lower boundary

$$w_1(x) = \sum_{e \in E_x} w_1(e), \tag{2}$$

and of the upper boundary

$$w_2(x) = \sum_{e \in E_x} w_2(e) \tag{3}$$

attain the minimum on the interval $w(x) = [w_1(x), w_2(x)]$ obtained as the result of interval summing of weights $w(e)$ over all $e \in E_{x_0}$, as well as the maximum width

$$d(e) = w_2(e) - w_1(e) \tag{4}$$

of intervals of weights for edges $e \in E_{x_0}$.

A widespread desire of problem-setters to minimize a variance value over the set of possible outcomes stipulated taking account of index (4). As a particular example where taking account of the minimizing index of the form (4) is necessary, one can designate the land tenure problem [4].

From (2)–(4), we obtain the following criteria of feasible solution $x \in X$ quality estimates

$$F_1(x) = \sum_{e \in E_x} w_1(e) \to \min, \tag{5}$$

$$F_2(x) = \sum_{e \in E_x} w_2(e) \to \min, \tag{6}$$

$$F_3(x) = \max_{e \in E_x} d(e) \quad \to \min. \tag{7}$$

In the multiobjective situation, it is customary to assume that the solution quality is estimated by means of the vector objective function (VOF)

$$F(x) = (F_1(x), \ldots, F_N(x)), \tag{8}$$

whose criteria we suppose to be minimizable

$$F_\nu(x) \to \min, \qquad \nu = \overline{1, N}. \tag{9}$$

Moreover, if N-criterial problem is specified on a graph $G = (V, E)$ we suppose that to each edge $e \in E$ the weights $w_\nu(e)$, $\nu = \overline{1, N}$, are assigned. In this case, we are dealing with N-weighting of graph*.

VOF (8), (9) determines a Pareto set (PS) $\widetilde{X}$ [8] consisting of all Pareto optima. Element $\widetilde{x} \in X$ is called a Pareto optimum if there is no $x \in X$ such that $F_\nu(x) \le F_\nu(\widetilde{x})$, $\nu = \overline{1, N}$, here even one of inequalities being strict.

To find the best solution it is suffice to examine not all PS $\widetilde{X}$, but a subset of its representatives known as complete set of alternatives (CSA) $\hat{X}$.

A subset of minimal cardinality $\hat{X} \subseteq \widetilde{X}$ is called a complete set of alternatives if $F(\hat{X}) = F(\widetilde{X})$.

Therefore, determining the best solution of a given interval problem is reduced to the following stages. First, we structure VOF criteria (8), (9), after that a CSA $\hat{X}$ is obtained. At the final stage, using the choice and decision theory procedures [9], from $\hat{X}$, the desired "compromise" optimum x_0 is chosen.

The above transition from an interval problem to a multiobjective one provides certain algorithmic problems and we are coming now to their analysis.

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*Without imposing in the general case any constrains on the number of criteria N we suppose that the diversity of criteria does not restrict to expressions (5)–(7). For example, in the land tenure problems, decision-making persons apply a criterion of the following maximim form

$$F_\nu(x) = \min_{e \in E_x} \frac{w_1(e) + w_2(e)}{2} \to \max. \tag{10}$$

Computation complexity estimates

As is known [9], [10], in the case of classical (non-interval) setting of shortest spanning tree, of chain between a pair of vertices, and of matching problems, there are efficient algorithms of finding an optimum x_0. The algorithms mentioned are called efficient since their computation complexity is bounded from above by a polynomial of low power $O(n^c)$, where $c \leq 3$ ([10], [12]).

The question arises whether the above problems remain of polynomial computation complexity in the multiobjective (or interval) setting. If not, it is worth to find conditions under which these problems become intractable.

To state the result obtained let us agree to call the criteria of the form (5), (6) the criteria of weight form, and the criteria of the form (7)–(10) minimax criteria.

According to [12], we call a multiobjective problem intractable one if there is no algorithm that would guarantee the finding a CSA with polynomial complexity. One can consider the cardinality of the CSA $|\hat{X}|$ as a lower estimate of computation complexity of its finding. This implies that a multiobjective problem is intractable in the case if the maximal cardinality of the CSA $\mu(n)$ increases exponentially with the increase of the dimension of a problem n ($\mu(n) = \max|\hat{X}|$, where the maximum is taken over all n-vertex N-weighted graph).

In the sequel, using some results of [13]–[16] we state sufficient condition for the considered vector problems on graphs be intractable.

Theorem 1. *If the VOF of travelling salesman problem contains at least two criteria of weight form the maximal cardinality of the CSA is*

$$\mu(n) = \frac{1}{2}(n-1)!$$

The scheme of the proof of Theorem 1 is the following.

Consider a complete n-vertex graph whose edges are indexed by $s = 1, 2, \ldots, m$, $m = \binom{n}{2}$. To the edge e_s two weights $w_1(e_s) = 2^s$, $w_2(e_s) = 2^m - w_1(e_s) + 1$ are assigned.

Using the binary representation of weights $w_\nu(e)$, $\nu = 1, 2$, it is easy to see that under above way of weighting each pair of feasible solution for a travelling salesman problem is a vector incongruent with respect to the value of VOF (8). Thus, for above way of weighting, the equalities $\hat{X} = \widetilde{X} = X$ hold. Taking into account that the cardinality of the set of Hamiltonian cycles in a complete n-vertex graph is equal to $\frac{1}{2}(n-1)!$, we complete the proof of the theorem.

The proofs of Theorems 2,3,4 are based on the scheme used in proving Theorem 1. The main difference for Theorem 2 consists in constructing a graph such that the cardinality of the set of feasible solutions on it would be maximal.

Theorem 2. *If a VOF of a chain between a pair of vertices problem contains at least two criteria of weight form, the maximal cardinality of CSA is bounded from below by the exponential function*

$$\mu(n) \geq 2^m, \qquad m = \lfloor n/2 \rfloor,$$

where $\lfloor a \rfloor$ is equal to the greatest integer not exceeding a.

Theorem 3. *For ever n, if VOF of a matching problem contains at least two criteria in the weight form, the maximal cardinality of CSA is equal*

$$\mu(n) = \frac{(2m)!}{m!2^m}, \qquad n = 2m.$$

By the spanning tree problem we call a mathematical setting such that the set of feasible solutions X consists of all spanning trees of a given connected N-weighted graph.

Theorem 4. *If a VOF of a spanning tree problem contains at least two criteria of the weigh form the maximal cardinality of CSA is equal to*

$$\mu(n) = n^{n-2}.$$

From the standpoint of applications, of particular interest is the justification of conditions determining polynomial resolvable classes of vector problems. Using the results of the paper [17] we succeeded to establish such classes for two-criterial settings.

We denote by N_1 the number of criteria of weight form contained in VOF (8)–(9), remaining $N - N_1$ criteria are of minimax form (7) or (10).

Theorem 5. *If $N = 2, N_1 \leq 1$ the computation complexity of finding CSA of a chain between a pair of vertices problem and of a spanning tree problem does not exceed $O(n^4)$.*

The proof of the theorem is based on constructing an algorithm α of complexity $\tau(\alpha) = O(n^4)$. The computation plan of the algorithm consists of the following steps: α_1 is a ranking (classification) of edges $e \in E$ of a given 2-weighted graph $G = (V, E)$ in order of decrease of the second weight $w_2(e)$; The result of the procedure α_1 is the set of values of the above second weights $\rho = \{\rho_1, \ldots, \rho_s, \ldots, \rho_l\}$, $\rho_s > \rho_{s+1}$, $s = 1, \ldots, l-1$; we denote by $G_s = (V, E_s)$ an spanning subgraph of a graph G consisting of edges with second weight $w_2(e) \leq \rho_s$, $e \in E_s$;

α_2 — is the procedure in which
1) for edges $e \in E_s$ the convolution of weights is computed:

$$w^\lambda(e) = \lambda_1 w_1(e) + \lambda_2 w_2(e), \qquad \lambda_1, \lambda_2 > 0,$$

2) on a graph G_s weighted by the weights $w^\lambda(e)$, an optimal solution is obtained (a shortest chain between a pair of vertices, the Dijkstra algorithm [10], for a shortest chain between a pair of vertices problem, the Prim and Kruskal algorithms [9]–[11], for a spanning tree problem) using algorithms with complexity equal to $O(n^2)$.

As the result of applying the procedure α_2 l times we obtain the set X_α of feasible solutions that contains the CSA desired. The proof of the latter statement can be found in [17]. The complexity of selecting the desired CSA from X_α does not exceed $O(n^2)$. By the inequality $l \leq |E| \leq O(n^2)$ this implies the final statement of Theorem 5.

Theorem 6. *If $N = 2$, $N_1 \leq 1$ the computation complexity of finding CSA of a matching problem does not exceed $O(n^5)$.*

The scheme of the proof of Theorem 6 differs from corresponding verification in the proof of Theorem 5 only by that essentially, that at step α_2, one uses the Lawler algorithm [10] instead the above ones of finding optimal matching; the complexity of the Lawler algorithm $O(n^3)$.

Returning to interval setting, one can refer them as hard problems if the quality estimate for solutions obtained is computed accounting indices of the (5) and (6).

On the other hand, we obtain effectively resolvable settings in the case where for any of several reasons, it proves to be sufficient to estimate the solution quality by means of a pair of criteria satisfying conditions of Theorems 5 and 6. We observe that these conditions are satisfied in the case where VOF consists of criteria (7) and (10).

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