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ON SOME OPTIMAL INCLUSION APPROXIMATIONS BY DISKS*

Ljiljana D. Petković and Miroslav Trajković

In this paper we construct the best possible circular including approximations for the complex-valued sets obtained under the transformations $z \mapsto z^{1/m}$ and $z \mapsto \ln z$. The diameters of these disks are equal to the diameters of the mapped regions.

О НЕКОТОРЫХ ОПТИМАЛЬНЫХ ВКЛЮЧАЮЩИХ АППРОКСИМАЦИЯХ С ПОМОЩЬЮ КРУГОВ

Л.Петкович, М.Трайкович

В работе мы строим наилучшие возможные круговые включающие аппроксимации для комплекснозначных множеств, полученных при трансформациях $z \mapsto z^{1/m}$ и $z \mapsto \ln z$. Диаметры этих кругов равны диаметрам отображенных областей.

1. Introduction

Let $Z = \{z : |z - \zeta| \leq r\}$ be a disk in the complex plane with the center $\zeta = mid\, Z \in \mathbb{C}$ and the radius $r = rad\, Z > 0$, denoted shorter by $Z = \{\zeta; r\}$. The boundary of the disk Z will be denoted by Γ and the set of all disks by $K(\mathbb{C})$. Furthermore, let f be a complex function, defined on the union of all disks from $H \subseteq K(\mathbb{C})$, such that the complex-valued

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set $f(Z) = \{f(z) : z \in Z\}$ is closed for each $Z \in H_1 \subseteq H$. In general, the range $f(Z)$ is not a disk, which makes difficulties in calculations. For this reason it is convenient to introduce a circular covering approximation, denoted by $I(f(Z))$, which completely includes the range $f(Z)$ for each $Z \in H_1$, that is, $I(f(Z)) \supseteq f(Z)$. The disk $I(f(Z))$ is called a **circular including approximation**, or shorter, **I-approximation.** The practical point is to find an I-approximation for given f and Z as good as possible.

Evidently, the best I-approximation to the closed range $f(Z)$ would be a disk with the diameter equal to the diameter

$$d = diam\{f(Z)\} = \max_{z_1, z_2 \in Z} |f(z_1) - f(z_2)| \tag{1}$$

of the range $f(Z)$ under the condition that this disk contains completely $f(Z)$. As it was proved in [8] if such disk exists than it is unique and its center w (say) is the mean of the diametrical segment lines. This disk is called the **diametrical including approximation** or **D-form** for $f(Z)$, denoted by $I_d(f(Z)) = \{w; d/2\}$. The enclosing condition is given by the inequality

$$|f(z) - w| \leq \frac{d}{2} \quad (z \in Z). \tag{2}$$

Complex circular functions in D-form can be of great importance as it was illustrated in [5] and [10]. Although the use of non-optimal disks for the range of values can give better results for some classes of problems because of some convenient properties (inclusion isotony, intersection), we stress some good features of optimal disks, especially in various calculations with complex variables. For example, the so-called *inclusion calculus of residues*, presented in Chapter X (written by M.S. Petković) of the book [4], just requires as small as possible disks. Furthermore, we note an outstanding problem in optimization theory which consists of finding $\max_{z \in Z} |f(z)|$, $Z \in K(\mathbb{C})$. Namely, since the absolute value of a disk Z is defined as

$$|Z| := |mid\, Z| + rad\, Z,$$

then obviously

$$|f(z)|_{z \in Z} \leq \max_{z \in Z} |f(z)| \leq \big|mid\, I\big(f(Z)\big)\big| + rad\, I\big(f(Z)\big). \tag{3}$$

Therefore, the upper bound will be determined more precisely if the radius $rad\, I(f(Z))$ of the covering disk $I(f(Z))$ is as small as possible (in the best case, equal to $\frac{1}{2} diam\{f(Z)\}$).

The study of D-form of elementary (library) functions has special importance. Some diametrical circular approximations for elementary functions were given by Börsken [2]. The further investigations were presented for the range $Z^{1/k} = \{z^{1/k} : z \in Z\}$ in [9] and for the range $e^Z = \{e^z : z \in Z\}$ in [8]. The subject of this paper is to construct I_d−approximation for the range $\ln g(Z)$, $0 \notin g(Z) = \{g(z) : z \in Z\}$ in an important special case when g defines a linear-fractional transformation, that is, $g(z) = (az+b)/(cz+d)$, $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$. We also construct the best possible including approximation for $Z^{1/m}$, $0 \notin Z$, which completes the investigation started in [9].

We presume that the readers are familiar with the basic properties of circular complex arithmetic. For the details see the book [1] by Alefeld and Herzberger.

2. Some general results

Solving the mentioned problem we will first prove the following more general result.

Lemma 1. *Let D be an arbitrary disk having the boundary Γ_D and the diameter d, and let G be a closed complex-valued range bounded by the smooth curve Γ_G. If P and Q are two adjacent points of intersection or points of tangency of the boundaries Γ_D and Γ_G, then there exists at least one point on the arc $\widehat{PQ} \subset \Gamma_G$ (excluding the points P and Q) whose the normal passes through the center of the disk D.*

Proof. Let $\rho = \rho(\theta) = d/2 + g(\theta)$ define the distance from the pole C_D to a point on the boundary Γ_G. Here $\theta \in (0, \pi)$ is the angle in reference to the positive part of the real axis. Since $\rho(\theta_Q) = \rho(\theta_P) = d/2$, we have $g(\theta_P) = g(\theta_Q) = 0$. Considering the derivative $g'(\theta)$ along the arc $\widehat{QP} \subset \Gamma_G$ between the points Q and P we conclude that this derivative must change its sign on this arc (not necessarily for $\theta \in (\theta_Q, \theta_P)$; for instance, if $g(\theta)$ is a multiple-valued function on this interval). Hence, it follows that there must exist at least one point, say T_G, on the arc $\widehat{QP}$

where the tangent is parallel to the tangent on the circular arc belonging to Γ_D in the corresponding point T_D. But the normal $\overline{C_D T_D}$ evidently passes trough the point T_G which means that $\overline{C_D T_G}$ is the normal of the curve Γ_G in the point T_G. This proves the lemma. □

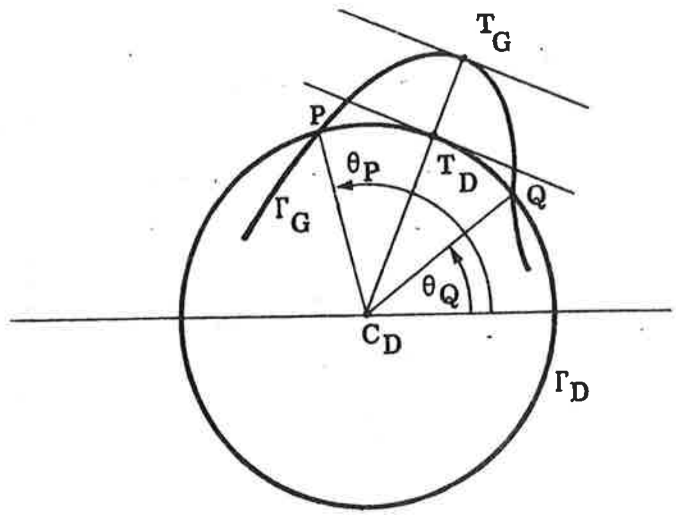

Fig. 1

According to Lemma 1 we can easily state the following assertion:

Lemma 2. *If the disk D does not completely include the range G with the smooth boundary Γ_G then there exists at least one point on Γ_G outside D such that its normal passes trough the center of the disk D.*

Taking a complementary assertion from Lemma 2 we obtain

Lemma 3. *The disk D will completely contain the region G with the smooth boundary Γ_G if and only if all points belonging to the contour Γ_G whose normals pass trough the center of D lie in the interior of Γ_G.*

In connection with the assertion of Lemma 3 it should be noted, in the case when the boundary Γ_G completely lies inside the disk D, there exists at least one point on Γ_G whose normal passes through the center C_D. To prove this it is sufficient to construct a circle centered at C_D which intersects the boundary Γ_G and then directly apply Lemma 1.

3. Diametrical disks for the m-th root

In this section we continue our study on diametrical including approximations presented in [9]. We are concerned with the problem of finding the diametrical disks for the range $Z^{1/m} = \{z : z^m \in Z,\ 0 \notin Z,\ m \in \mathbb{N}\}$ which has been partly considered in [9]. Our aim is to give an answer to the open problem proposed by L. Petković in 1986 [7]. This result completes the study from the paper [9] concerning the diametrical disk for the range $Z^{1/m}$, $0 \notin Z$. We note that the solution presented uses a different approach than McCoy's [3] and is somewhat simpler than McCoy's answer [3].

In the determination of the diametrical including approximation for the range $Z^{1/m}$ we need the following lemmas (the proof of the first one is omitted because of simplicity).

Lemma 4. *The function $h(\varphi) = \sin m\varphi / \sin\varphi$ is monotonically decreasing on the interval $(0, \frac{1}{m}\arcsin p)$, where $0 < p < 1$ and $m \geq 1$.*

Lemma 5. *Let $m \geq 2$ and $p \in [0, 1)$. Then*

$$(1+p)^{1/m} + (1-p)^{1/m} \leq 2 \tag{4}$$

and

$$(1+p)^{1/m} - (1-p)^{1/m} \geq 2\sin\Big(\frac{\arcsin p}{m}\Big). \tag{5}$$

Proof. The proof of (4) was given in [6]. To prove (5) we first use the general binomial formula and find

$$(1+p)^{1/m} - (1-p)^{1/m} = \frac{2}{m}p + 2\sum_{\lambda=2}^{\infty} a_\lambda p^\lambda, \tag{6}$$

where

$$a_{2k} = 0,\ a_{2k+1} = \frac{1}{(2k+1)!m^{2k+1}} \prod_{s=1}^{k} [(2s-1)m-1][2sm-1]\ (k = 1, 2, \ldots). \tag{7}$$

Let us introduce the function $f(p) = \sin(\frac{\arcsin p}{m})$ and note that it satisfies the following differential equation

$$(1-p^2)f''(p) - pf'(p) + \frac{1}{m}f(p) = 0 \tag{8}$$

with the initial conditions $f(0) = 0$, $f'(0) = \frac{1}{m}$, $f''(0) = 0$. Setting $f(p) = \sum_{\lambda=0}^{\infty} b_\lambda p^\lambda$, we solve the differential equation (8) by the well-known method using the series. In this way, taking into account the initial conditions, we obtain

$$b_{2k} = 0,\ b_1 = \frac{1}{m},\ b_{2k+1} = \frac{1}{(2k+1)!m^{2k+1}} \prod_{s=1}^{k} [(2s-1)^2 m^2 - 1] \quad (k = 1, 2, \ldots). \tag{9}$$

Thus

$$f(p) = \sin\left(\frac{\arcsin p}{m}\right) = \frac{1}{m}p + \sum_{\lambda=2}^{\infty} b_\lambda p^\lambda, \tag{10}$$

where the coefficients b_λ are given by (9).

Using the expansions (6) and (10) we get

$$(1+p)^{1/m} - (1-p)^{1/m} - 2\sin\left(\frac{\arcsin p}{m}\right) = 2\sum_{k=1}^{\infty}(a_{2k+1} - b_{2k+1})p^{2k+1}.$$

To prove (5) it suffices to show that $a_{2k+1} - b_{2k+1} \geq 0$ for all $m \geq 2$ and $k = 1, 2, \ldots$. But it is trivial since from (7) and (9) there follows

$$a_{2k+1} - b_{2k+1} = \frac{1}{(2k+1)!m^{2k+1}} \prod_{s=1}^{k} (m-2)[(2s-1)m - 1] \geq 0. \qquad \square$$

From (4) and (5) we immediately obtain

Corollary. *Let $m \geq 2$ and $p \in [0, 1)$. Then*

$$(1+p)^{1/m} - (1-p)^{1/m} \geq [(1+p)^{1/m} + (1-p)^{1/m}] \sin\left(\frac{\arcsin p}{m}\right). \tag{11}$$

We note that the inequality (11) has also appeared in McCoy's papers [3] but was obtained by a quite different approach.

Let us return to our problem of finding the diametrical including approximation $I_d(Z^{1/m})$. Let Z be a circular complex interval given by a disk with the center $c = mid\, Z \in \mathbb{C}$ and the radius $r = rad\, Z > 0$, that is, $Z = \{z : |z - c| \leq r\}$, denoted shorter by $Z = \{c; r\}$. Let us assume that the disk Z does not contain the origin, that is, $p := r/|c| < 1$. As it was shown in [9], the construction of the diametrical disk for the range $Z^{1/m}$ reduces to the following problem (see Fig. 2):

Let $D = \{u_0; d/2\}$ be the disk with the center

$$u_0 := \frac{(1+p)^{1/m} + (1-p)^{1/m}}{2} \tag{12}$$

and the radius

$$r_0 := \frac{d}{2} = \frac{(1+p)^{1/m} - (1-p)^{1/m}}{2}, \tag{13}$$

and let G be the image (one of m) of the disk $\{1; p\}$, $p \in (0, 1)$ under $w = z^{1/m}$ with $\arg z^{1/m} \in [-\frac{1}{m} \arcsin p, \frac{1}{m} \arcsin p]$. The question arises whether G is completely contained inside the disk D.

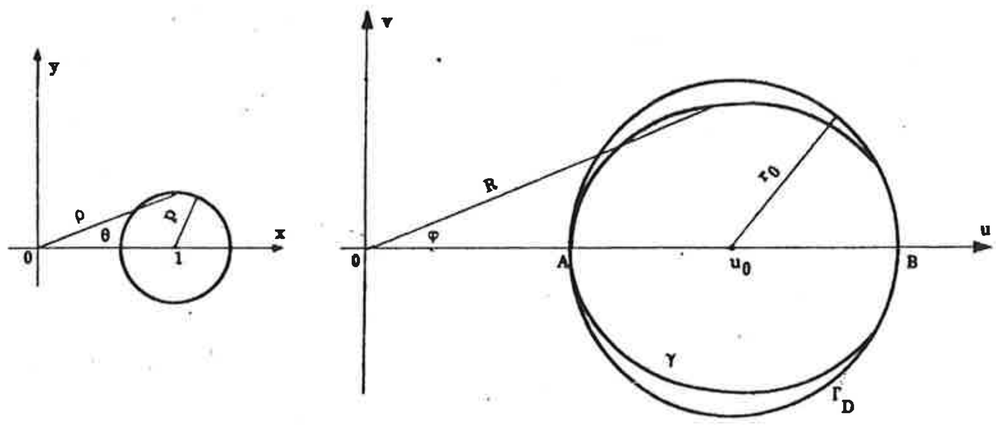

Fig. 2

The boundary $\Gamma_D = \{z : |z - 1| = p,\ p < 1\}$ of the disk $Z_0 = \{1; p\}$ shown in Fig. 2a can be represented in the polar coordinate system (ρ, θ) as

$$\rho^2 - 2\rho \cos\theta + 1 - p^2 = 0. \tag{14}$$

Under the transformation $w = z^{1/m}$ the disk Z_0 maps to the m closed regions $G_0, G_1, ..., G_{m-1}$ of the same form with the axes passing through the origin and rotated through the angle $2\pi/m$. For our purpose it is sufficient to consider only one of these regions since all diametrical disks enclosuring the images are of the same form (and rotated through the angle $2\pi/m$). For simplicity we choose the region G_0 whose axis coincides with the real axis and $|\arg z^{1/m}| < \frac{1}{m}\arcsin p$, $z \in Z_0$ (see Fig. 2b). This region will be called the *principal-value* range, or shorter, *p.v.* range. Under $w = z^{1/m} = R\exp(i\varphi) = u + iv$ the contour (11) maps to m contours $\Gamma_{G_0}, \Gamma_{G_1}, ..., \Gamma_{G_{m-1}}$. The contour $\gamma := \Gamma_{G_0}$ of the considered *p.v.* region G_0, called *p.v. branch*, intersects the real axis u in the points A and B. The equation of the contour γ in the polar coordinate system (R, φ) is given by

$$R^{2m} - 2R^m \cos m\varphi + 1 - p^2 = 0, \quad |\varphi| < \frac{1}{m}\arcsin p. \tag{15}$$

Using the rectangular coordinate system (u, v) in the w-complex plane, from (15) we find

$$\frac{dv}{du} = -\frac{R^m \cos\varphi - \cos(m-1)\varphi}{R^m \sin\varphi + \sin(m-1)\varphi} \tag{16}$$

in an arbitrary point on the boundary γ. Besides, solving the biquadratic equation (15) we obtain

$$R^m = \cos m\varphi \pm \sqrt{p^2 - \sin^2 m\varphi}. \tag{17}$$

It is easy to see that the intersecting points A and B of the contour γ and the real axis u have the coordinates $A\big((1-p)^{1/m}, 0\big)$ and $B\big((1+p)^{1/m}, 0\big)$. Let us consider the disk D with the center $u_0 = (u_A + u_B)/2 = \big((1-p)^{1/m} + (1+p)^{1/m}\big)/2$ and the diameter $d = 2r_0 = u_B - u_A = (1+p)^{1/m} - (1-p)^{1/m}$. The boundary $\Gamma_D = \{w : |w - u_0| = r_0\}$ of this disk passes through the points A and B. Since the boundary γ of the region G_0 also traverses the points A and B, we conclude that the diameter of the region G_0 is equal to the diameter $d = \overline{AB}$ of the disk D if $G_0 \subset D$. Therefore, if we prove that the disk D completely includes the closed region G_0 then the disk D is the **diametrical disk** for the *p.v.* range G_0, that is, $D = I_d(G_0)$. The answer is given in the following theorem.

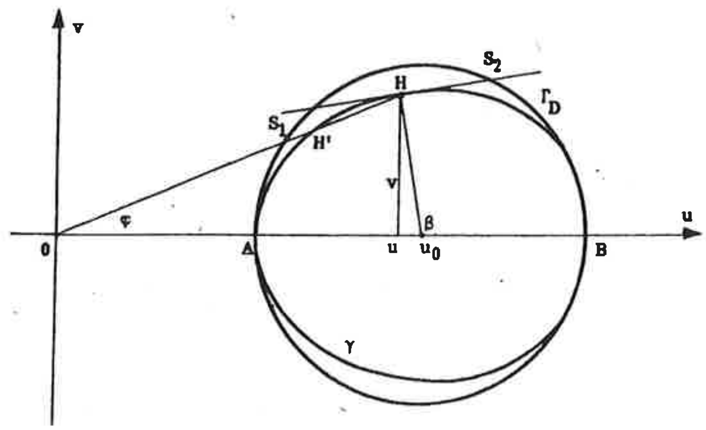

Fig. 3

Theorem 1. *The disk $D = \{u_0; r_0\}$, where u_0 and r_0 are given by (12) and (13), completely contains the p.v. region G_0 if and only if the inequality (11) holds.*

Proof. Let $H(u, v)$ be an arbitrary point on the contour γ whose normal passes through the center u_0 of the disk D. Since

$$\left.\frac{dv}{du}\right|_{w=w_H} = -\frac{1}{\tan\beta},$$

from (16) and Fig. 3 we obtain

$$u_0 = u - \frac{v}{\tan\beta} = u + v\frac{dv}{du} = R\cos\varphi - R\sin\varphi\frac{R^m\cos\varphi - \cos(m-1)\varphi}{R^m\sin\varphi + \sin(m-1)\varphi},$$

wherefrom

$$u_0 = \frac{R\sin m\varphi}{R^m\sin\varphi + \sin(m-1)\varphi}. \tag{18}$$

Substituting R^m (given by (17)) in (18) and solving the linear equation we find the modulus R of the radius vectors $\overrightarrow{OH'}$ and $\overrightarrow{OH}$,

$$R_{H,H'} = u_0\cos\varphi \pm u_0\frac{\sin\varphi}{\sin m\varphi}\sqrt{p^2 - \sin^2 m\varphi}. \tag{19}$$

Using the polar coordinate system (R, φ) the contour Γ_D of the disk D can be written as

$$R^2 - 2Ru_0\cos\varphi + (1-p^2)^{1/m} = 0. \tag{20}$$

Let S_1 and S_2 be the points of intersection of the circle Γ_D and the straight line $\overline{OH}$. Then from (20) we find the distances R_{S_1} and R_{S_2} of the intersection points S_1 and S_2 from the origin,

$$R_{S_2,S_1} = u_0 \cos\varphi \pm \sqrt{u_0^2 \cos^2\varphi - (1-p^2)^{1/m}}. \tag{21}$$

To prove that the disk D contains the region G_0, according to Lemma 3 it is necessary and sufficient that the point H belong to the interior of Γ_G, denoted by $int\,\Gamma_D$. From Fig. 3 we see that $H \in int\,\Gamma_D$ if and only if the inequalities $R_{S_1} \leq R_{H'}$ and $R_H \leq R_{S_2}$ hold. Taking into account (19) and (21), the two last inequalities reduce to the inequality

$$u_0 \frac{\sin\varphi}{\sin m\varphi}\sqrt{p^2 - \sin^2 m\varphi} \leq \sqrt{u_0^2\cos^2\varphi - (1-p^2)^{1/m}}.$$

After a short rearrangement this inequality becomes

$$r_0 \sin m\varphi \geq u_0 p \sin\varphi, \qquad 0 < \varphi < \frac{\arcsin p}{m}.$$

or

$$\frac{\sin m\varphi}{\sin\varphi} \geq \frac{u_0 p}{r_0}.$$

Since the function $h(\varphi) = \sin m\varphi / \sin\varphi$ is monotonically nonincreasing on the interval $(0, \frac{1}{m}\arcsin p)$ (Lemma 4), it is sufficient to prove the inequality

$$\frac{\sin(m \cdot \frac{1}{m}\arcsin p)}{\sin(\frac{1}{m}\arcsin p)} \geq \frac{u_o p}{r_0},$$

that is

$$(1+p)^{1/m} - (1-p)^{1/m} \geq [(1+p)^{1/m} + (1-p)^{1/m}]\sin\Big(\frac{\arcsin p}{m}\Big).$$

This is the inequality (11) proved above and, therefore, the proof of Theorem 1 is completed. □

According to Theorem 1 it follows that the disk $D = \{u_0; r_0\}$ is the **diametrical disk** for the region G_0. Hence, for the disk $Z = \{c; r\}$, $|c| > r$

we establish the diametrical including approximations for the region $Z^{1/m}$ as the union of m diametrical disks, that is

$$I_d(Z^{1/m}) = \bigcup_{k=0}^{m-1} \left\{u_0|c|^{1/m} exp\Big(i\frac{\arg c + 2k\pi}{m}\Big); r_0|c|^{1/m}\right\}.$$

4. Diametrical disks for the logarithmic function

In this section we are concerned with the construction of the diametrical disk for the range of the function $f(z) = \ln z$. Since $\ln z$ is *infinitely-many-valued* function we will consider in the sequel only the *principal-value* of $\ln z$ assuming that $\ln z = \ln|z| + i \arg z$, $\arg z \in [0, 2\pi)$. Therefore, speaking about the diametrical disk for $\ln Z$ we assume only one set $\ln Z = \{\ln|z| + i \arg z : z \in Z,\ 0 \le \arg z < 2\pi\}$ called *principal-value* range, or shorter, *p.v.* range.

The diametrical disk $I_d(\ln Z)$ has been established by Börsken [2]. Here we give a new derivation which uses partly Börsken's results but with a new simple approach in the estimation procedure based on the use of circular arithmetic and Theorem 1. First, according to (1), we will find the diameter of the range $\ln Z$, $0 \notin Z = \{z; \rho\}$. We introduce $p = \rho/|z| (< 1)$ and obtain

$$\begin{aligned}|\ln z_1 - \ln z_2|_{z_1,z_2\in Z} &= |\ln(1+pe^{i\alpha}) - \ln(1+pe^{i\beta})|_{\alpha,\beta\in[0,2\pi)}\\ &= \left|\int_0^p \frac{e^{i\alpha}}{1+te^{i\alpha}}\,dt - \int_0^p \frac{e^{i\beta}}{1+te^{i\beta}}\,dt\right| \le \int_0^p \left|\frac{e^{i\alpha}}{1+te^{i\alpha}} - \frac{e^{i\beta}}{1+te^{i\beta}}\right| dt.\end{aligned}$$

Using circular arithmetic operations and (3) we find

$$\begin{aligned}\left|\frac{e^{i\alpha}}{1+te^{i\alpha}} - \frac{e^{i\beta}}{1+te^{i\beta}}\right| &= \frac{1}{t}\left|\frac{1}{1+te^{i\beta}} - \frac{1}{1+te^{i\alpha}}\right|\\ &\le \frac{1}{t}\left|\frac{1}{1+t\{0;1\}} - \frac{1}{1+t\{0;1\}}\right| = \frac{1}{t}\left|\left\{0; \frac{2t}{1-t^2}\right\}\right| = \frac{1}{1+t} + \frac{1}{1-t}.\end{aligned}$$

According to this we estimate

$$\int_0^p \left|\frac{e^{i\alpha}}{1+te^{i\alpha}} - \frac{e^{i\beta}}{1+te^{i\beta}}\right| dt \le \int_0^p \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \ln\frac{1+p}{1-p},$$

that is

$$|\ln z_1 - \ln z_2| \leq \ln \frac{1+p}{1-p} \tag{22}$$

for all $z_1, z_2 \in Z$. Therefore, the diameter of the range $\ln Z$ is given by

$$d = diam\,\{\ln Z\} = \max_{z_1, z_2 \in Z} |\ln z_1 - \ln z_2| = \ln \frac{1+p}{1-p} = \ln \frac{|z|+\rho}{|z|-\rho}.$$

It is easy to show that the boundary of the range $\ln Z$ is centrally symmetrical in reference to the two mutually perpendicular axes which are parallel to the real and imaginary axes (see Börsken [2]). According to this we can conclude that the equality in (22) appears for $z_1^* = z + \rho e^{i \arg z}$ and $z_2^* = z - \rho e^{i \arg z}$, which means that these points lie on the diametrical segment line. As mentioned previously, if a diametrical disk exists (that is, if the enclosing condition (2) is valid), then its center is the mean of diametrical segment lines. For this reason the possible center of the diametrical disk for $\ln Z$ must be the point determined by

$$A = \frac{\ln z_1^* + \ln z_2^*}{2} = \ln \sqrt{|z|^2 - \rho^2} + i \arg z.$$

Thus, the disk $\{A; d/2\}$ will be the diametrical disk for the range $\ln Z$ if the enclosing condition

$$|\ln \zeta - A| \leq \frac{d}{2} = \frac{1}{2} \ln \frac{1+p}{1-p} \quad (p = \rho/|z|) \tag{23}$$

is fulfilled for all $\zeta \in Z$. The check of this inequality can be avoided if we use Lemma 3.

Remark. Obviously, to prove that the disk $\{A; d/2\}$ is the diametrical disk for the *p.v.* range $\ln Z$, it is sufficient to prove that the disk $\{u_0; r_0\}$ is the diametrical disk for the *p.v.* range $\{\ln|z| + i \arg z \,:\, z \in \{1; p\}\}$, where

$$u_0 = \frac{1}{2} \ln\,(1-p^2), \qquad r_0 = \frac{d}{2} = \frac{1}{2} \ln \frac{1+p}{1-p}. \tag{24}$$

As in the previous section, in order to establish the diametrical disks for the range $\ln Z$ it suffices to consider the mapping of the disk $\{1; p\}$,

$0 \leq p < 1$ under the transformation $z \mapsto \ln z = u + i\,v$. Before stating the main result we give

Lemma 6. *Let* $y \in [0, \arcsin p]$, *where* $p = |z|/\rho < 1$. *Then*

$$\frac{1}{2p} \ln \frac{1+p}{1-p} \geq \frac{y}{\sin y}. \tag{25}$$

The proof of Lemma 6 is based on the inequality $\frac{1}{2} \ln \frac{1+p}{1-p} \geq \arcsin p$ and the fact that the maximum of the function $y \mapsto y/\sin y$ on the interval $[0, \arcsin p]$ is equal to $\arcsin p/p$. Now we are in the possibility to state

Theorem 2. *The disk* $D = \{u_0; r_0\}$, *where* u_0 *and* r_0 *are given by (24), completely includes the p.v. range* $G_0 := \{\ln|z| + i \arg z : z \in \{1; p\}\}$ *if and only if the inequality (25) holds.*

Proof. Regarding the boundary of the disk $\{1; p\}$ in the form (14) we obtain $\rho = \cos\theta \pm \sqrt{p^2 - sin^2\theta}$, $|\theta| \leq \arcsin p$. Hence the equation of the principal branch γ of the mapped region whose center u_0 lies on the positive part of the real u–axis in the $u0v$ coordinate system is given by

$$u = \ln\Big(\cos v \pm \sqrt{p^2 - \sin^2 v}\Big), \qquad |v| = |\theta| \leq \arcsin p. \tag{26}$$

The boundary (circumference) of the possible diametrical disk $\{u_0; r_0\}$ is

$$u = u_0 \pm \sqrt{r_0^2 - v^2}, \tag{27}$$

where u_0 and r_0 are given by (24).

In what follows we use a geometrical construction as in the proof of Theorem 1 omitting details. Let $H(u, v)$ be an arbitrary point on the boundary γ whose normal passes through the center u_0 of the disk D. For the normal segment line at this point we have

$$u_0 = u + v \frac{dv}{du}\bigg|_{H(u,v)}.$$

Since

$$\frac{du}{dv} = \pm\frac{\sin v}{\sqrt{p^2 - \sin^2 v}},$$

from the last two relations we obtain

$$u = u_0 \pm v\frac{\sqrt{p^2 - \sin^2 v}}{\sin v}. \tag{28}$$

Regarding v fixed, from (27) and (28) we conclude that the point $H(u, v)$ will belong to the interior of the disk $\{u_0; r_0\}$ if and only if the inequality

$$v\frac{\sqrt{p^2 - \sin^2 v}}{\sin v} \le \sqrt{r_0^2 - v^2} \tag{29}$$

holds. Because of the symmetry it is sufficient to take $v \ge 0$ and (29) reduces to the inequality (25) which is proved above. □

Therefore, the disk

$$\{A; d/2\} = I_d(\ln Z) = \left\{\ln\sqrt{|z|^2 - \rho^2} + i \arg z; \frac{1}{2}\ln\frac{|z| + \rho}{|z| - \rho}\right\} \tag{30}$$

is the **diametrical disk** for the range $\ln\{z; \rho\}$, $|z| > \rho$. We note that the condition (31) provides the validity of the inequality $|z| > \rho$ because the inversion of a disk not containing 0 produces a disk which also does not include 0.

Now, we will consider a more general case which involves a logarithmic function.

Let $\delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \ne 0$ be the determinant of the linear-fractional transformation $g(z) = (az + b)/(cz + d)$ and let $c \ne 0$. Then this transformation maps a disk, which does not contain the point $-d/c$, into a disk. Let $Z = \{\zeta; r\}$ be a disk such that

$$\left|\zeta + \frac{d}{c}\right| > r, \tag{31}$$

(i.e. $-\frac{d}{c} \notin Z$). Since

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{-\delta/c^2}{z + d/c},$$

using the circular arithmetic operations we find that the linear-fractional transformation $z \mapsto g(z)$ maps the disk $Z = \{\zeta; r\}$ onto the disk $W = \{w; \rho\}$, where

$$w = mid\ W = \frac{a}{c} + \frac{-\frac{\delta}{c^2}(\bar{\zeta} + \bar{d}/\bar{c})}{|\zeta + d/c|^2 - r^2}, \quad \rho = rad\ W = \frac{\left|\frac{\delta}{c^2}\right| r}{|\zeta + d/c|^2 - r^2}. \tag{32}$$

Since the inverse of a disk is an exact operation the image W (in the form of a disk) coincides with the exact range $g(Z)$, that is

$$W = g(Z) = \left\{\frac{az+b}{cz+d} : z \in Z = \{\zeta; r\}\right\}.$$

This fact is convenient in all cases when we consider circular complex functions of the form $f(g(z))$ because $g(Z)$ is an exact range – a disk. Thus, it remains to find as good as possible I-approximation of $f(W)$. In this section we are concerned with the function $f(z) = \ln z$ and the construction of the diametrical disk for the range

$$\ln W = \left\{\ln \frac{az+b}{cz+d} : c \neq 0, \delta = ad - bc \neq 0, z \in Z = \{\zeta; r\},\ |\zeta + d/c| > r\right\}.$$

In the following we will construct diametrical disks for some inverse trigonometric and hyperbolic functions. We recall the relations which connect the mentioned inverse functions and the natural logarithm:

$$\arctan z = \frac{1}{2i} \ln \frac{i-z}{i+z}, \qquad \operatorname{arccot} z = \frac{1}{2i} \ln \frac{z+i}{z-i},$$
$$\operatorname{artanh} z = \frac{1}{2} \ln \frac{z+1}{-z+1}, \qquad \operatorname{arcoth} z = \frac{1}{2} \ln \frac{z+1}{z-1}.$$

As above, we assume the *principal-value* of $\ln(\cdot)$. Since all functions under the logarithm define linear-fractional transformations, according to the previous argumentation it is possible to construct diametrical disks for all four inverse functions. As an illustration we will consider the inverse functions $z \mapsto \arctan z$ and $z \mapsto \operatorname{artanh} z$.

Special case (i): arctan z.

In this case we have the linear-fractional transformation $g_1(z) = (-z+i)/(z+i)$. The requirement (31) for the disk $Z = \{\zeta; r\}$ reduces to the inequality $|\zeta + i| > r$. By virtue of (32) we have

$$W_1 = \left\{\frac{-z+i}{z+i} : z \in Z = \{\zeta; r\}\right\} = \{w_1; \rho_1\},$$

where

$$w_1 = -1 + \frac{2i(\bar{\zeta} - i)}{|\zeta + i|^2 - r^2}, \quad \rho_1 = \frac{2r}{|\zeta + i|^2 - r^2}.$$

Then we have

$$I_d(\tan^{-1} Z) = \left\{\frac{1}{2i} mid\, I_d(\ln W_1); \frac{1}{2} rad\, I_d(\ln W_1)\right\},$$

where I_d- approximation of $\ln W_1$ is given by (30).

Special case (ii): artanh z.

The linear-fractional transformation is $g_2(z) = (z+1)/(-z+1)$ and the inequality (4) reduces to $|z - 1| > r$. In view of (32) we get

$$W_2 = \left\{\frac{z+1}{-z+1} : z \in Z = \{\zeta; r\}\right\} = \{w_2; \rho_2\},$$

where

$$w_2 = -1 + \frac{-2(\bar{\zeta} - 1)}{|\zeta - 1|^2 - r^2}, \quad \rho_2 = \frac{2r}{|\zeta - 1|^2 - r^2}.$$

Following (30) we find

$$I_d(\text{artanh}\, Z) = \left\{\frac{1}{2} mid\, I_d(\ln W_2); \frac{1}{2} rad\, I_d(\ln W_2)\right\}.$$

In the similar way we can construct diametrical disks for the remaining two functions.

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L.D. Petković
Faculty of Mechanical Engineering
University of Niš
18000 Niš, Yugoslavia

M. Trajković
Faculty of Electronic Engineering
University of Niš
18000 Niš, Yugoslavia