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THE EXTENSION OF THE FRECHET DERIVATIVE CONCEPT IN THE INTERVAL-SEGMENT ANALYSIS

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In this paper we generalize the concept of Frechet derivative for the segment mappings and functions of one variable determined on the set of segments with non-negative and negative width. In [1], [4] only case of segments with non-negative width is discussed.

РАСШИРЕНИЕ ПОНЯТИЯ ПРОИЗВОДНОЙ ФРЕШЕ В ИНТЕРВАЛЬНО-СЕГМЕНТНОМ АНАЛИЗЕ

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Обобщается понятие производной Фреше для сегментных отображений и сегментных функций от одной переменной, действующих на множестве сегментов с неотрицательной и отрицательной шириной. В [1],[4] рассмотрен только случай сегментов неотрицательной ширины.

The interval analysis allows, for example, to obtain a more accurate upper and lower estimations for an analytical expression, that is to find the estimating intervals. The intervals may be open, half-open from the left or from the right and closed. All this considered, the closed intervals will be called below the segments, and the open intervals will be refered simply as intervals.

In this paper we will keep to the following notations:

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R is the set of all real numbers;

$x := [\underline{x}, \overline{x}] := \{\xi \mid \underline{x} \leq \xi \leq \overline{x}$ or $\overline{x} \leq \xi \leq \underline{x},\ \underline{x}, \overline{x} \in R\}$ is a segment;

$S(R)$ is the set of all segments.

The same reasoning as below for the set $S(R)$ may be carried easily out in the case of intervals and semi-intervals as well.

1. Arithmetic operations in $S(R)$

To distinguish from arithmetic operations over numbers, we will denote arithmetic operations over segments as $\dot{+}, \dot{-}, \dot{\times}, \dot{/}$.

Following the works [2,3], we introduce the notations

$\sigma :=$ if $|\underline{x}| \leq |\overline{x}|$ then $|\overline{x}|$ else $|\underline{x}|$;

$\kappa :=$ if $|\underline{x}| \leq |\overline{x}|$ then $|\underline{x}|$ else $|\overline{x}|$;

$\chi :=$ if $\sigma \neq 0$ then κ/σ else it is not defined;

$\alpha \vee \beta := [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}],\ \alpha \wedge \beta := [\max\{\alpha, \beta\}, \min\{\alpha, \beta\}]$;

dual $[\underline{x}, \overline{x}] := [\overline{x}, \underline{x}]$.

According to [2, 3], arithmetic operations are defined in the following way:

$$x + y := [\underline{x} + \underline{y}, \overline{x} + \overline{y}]; \qquad x - y := [\underline{x} - \overline{y}, \overline{x} - \underline{y}];$$

Let $\omega \in \{\vee, \wedge\}, \chi_x \leq \chi_y$, then

$x \times y :=$ if $\omega_x = \omega_y,\ 0 \leq \chi_x \leq 1$ then $\sigma_x \sigma_y (1\ \omega_x\ \chi_x \chi_y)$;

if $\omega_x = \omega_y,\ -1 \leq \chi_x \leq 0$ then $\sigma_x \sigma_y (1\ \omega_x\ \chi_x)$;

if $\omega_x \neq \omega_y,\ 0 \leq \chi_x \leq 1$ then $\sigma_x \kappa_y (1\ \omega_y\ \chi_x \chi_y^{-1})$;

if $\omega_x \neq \omega_y,\ -1 \leq \chi_x \leq 0,\ 0 < \chi_y < 1$ then $\sigma_x \kappa_y (1\ \omega_x\ \chi_x)$,

if $\omega_x \neq \omega_y,\ -1 \leq \chi_x \leq \chi_y < 0$ then it is not defined.

Remark. In the latter case in [2], [3] the result is supposed to be equal to [0,0]. But there exist the examples, which disprove this. The division is defined in the following way:

if $\chi_y > 0$ then $x \dot{/} y := x \dot{\times} (1 \dot{/} y) := x \times (\kappa_y^{-1} (1 \omega_y \chi_y))$ else it is not defined.

2. Introduction of norm in the set $S(R)$

If we introduce the addition of segments from $S(R)$ as we did it in the previous section, i.e. $x + y := [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$, and the multiplication of a real number by a segment according to the rule $\lambda \cdot x := [\lambda \circ \underline{x}, \lambda \circ \overline{x}]$, then it is easy to see that the set $S(R)$ with operations introduced will be a linear space like the two-dimensional vector space R^2. If R^2 is the Banach space with some given norm, it is evident that $S(R)$ will be also the Banachs space with the same norm.

For example, $S(R)$ with any of the norm

$$\| x \| := \max\{| \underline{x} |, | \overline{x} |\}, \quad \| x \| := (| \underline{x} |^2 + | \overline{x} |^2)^{1/2}, \quad \| x \| := | \underline{x} | + | \overline{x} |$$

is a complete linear normed space, i.e. a Banach space.

3. Frechet derivative

Let $S(D) := \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in D \subseteq R\}$. Then the mapping $F(x)$ from $S(D)$ to $S(R)$ will be named the segment mapping. The mapping $F(\xi)$ from $D \subseteq R$ to $S(R)$ will be called the segment function.

Let $S_1(R)$ and $S_2(R)$ be a linear normed spaces. A segment mapping $F(x) : S(D) \subseteq S_1(R) \to S_2(R)$ is called differentiable on a given segment $x \in S(D)$, if there exists such segment $a_x \in S(R)$, that for any $\epsilon > 0$ we can find $\delta > 0$ such that the inequality $\| h \| < \delta$ implies the inequality

$$\| F(x + h) - F(x) - a_x \dot{\times} h \| \leq \epsilon \| h \|, \tag{1}$$

where the sign $'-'$ means the subtraction in R^2, i.e. $x - y := [\underline{x} - \underline{y}, \overline{x} - \overline{y}]$.

Inequality (1) may be written in the abbreviated form

$$F(x + h) - F(x) - a_x \dot{\times} h = o(h), \tag{2}$$

where $\| o(h) \| / \| h \| \to 0$ for $\| h \| \to 0$.

Notice that the segment a_x may have a positive $(\overline{a}_x - \underline{a}_x \geq 0)$, zero $(\overline{a}_x - \underline{a}_x = 0)$, and negative width $(\overline{a}_x - \underline{a}_x < 0)$.

The expression $a_x \dot{\times} h$ (obviously, for any $h \in S_x(x)$ it is an element of the space $S_y(R)$) is called the Frechet differential (or the strong differential) of a segment mapping $F(x)$.The segment a_x itself is called the

Frechet derivative (strong derivative) of a segment mapping $F(x)$ wich will be denoted by $F'(x)$.

We adduce, without proofs, some properties following from definition of Frechet derivative.

1) If the segment mapping F is differentiable on a segment x, then it is continuous on this segment.
2) If the segment mapping is differentiable on a segment x, then the corresponding Frechet derivative is uniquely defined.
3) If $F(x) := c =$ const, where $c \in S(R)$, then $F'(x) = [0,0]$.
4) The derivative of a "linear" mapping $F(x) := a \dot{\times} x + b$, where $a, b \in S(R)$, is a, i.e. $F'(x) = a$.
5) (Analogue of the differentiation of composite function).

Let $S_X(R), S_Y(R), S_Z(R)$ be linear normed spaces, let $U(x_0)$ be neighbourhood of the segment $x_0 \in S_X(x)$, let F be a segment mapping of this neighbourhood to $S_Y(R), y_0 := F(x_0)$, $V(y_0)$ be a neighbourhood $y_0 \in S_Y(R)$ and let G be its segment mapping to $S_Z(R)$. Then if the mapping F is differentiable on the segment x_0 and G is differentiable on the segment y_0, then the mapping $H := GF$ which is defined in some neighbourhood of the segment x_0 is differentiable on the segment x_0, and

$$H'(x_0) = G'(y_0) \dot{\times} F'(x_0). \tag{3}$$

6) Let F and G be two continuous segment mappings, operating from $S_X(D)$ to $S_Y(R)$. If F and G are differentiable on the segment x_0, then the mapping $F \dot{+} G$ and $\alpha \dot{\times} F$ (α is a number) is also differentiable on this segment, and, besides,

$$(F \dot{+} G)'(x_0) = F'(x_0) \dot{+} G'(x_0), \tag{4}$$

$$(\alpha \dot{\times} F)'(x_0) = \alpha \dot{\times} F'(x_0). \tag{5}$$

4. Segment function derivative

From definition of segment function we can see that it is the special case of segment mapping, which has a point segment, i.e. a real number as argument. Then the Frechet derivative for a segment function will be defined in the following way.

A segment function $F(\xi) : D \to S(R)$, where D is the real domain of the function $F(\xi)$ and the range of values of this function belongs to a linear normed space $S_Y(R)$, is called differentiable at given point $\xi \in D$, if there exists an element $a(\xi) \in S(R)$, such that for any $\epsilon > 0$ we can find $\delta > 0$ such that inequality $\mid \Delta\xi \mid < \delta$ implies

$$\| F(\xi + \Delta\xi) - F(\xi) - a(\xi) \dot{\times} \Delta\xi \| < \epsilon \mid \Delta\xi \mid, \tag{6}$$

or

$$\lim_{\Delta\xi \to 0} (F(\xi + \Delta\xi) - F(\xi)) \dot{/} \Delta\xi = a(\xi). \tag{7}$$

A function $F'(\xi) := a(\xi)$ will be called Frechet derivative at point ξ.

Evidently, the properties of Frechet derivative of segment mapping listed above will be also true for a Frechet derivative of segment function. The following statement is valid.

Theorem. *Let $F(\xi) := [\underline{f}(\xi), \overline{f}(\xi)]$ be a segment function defined for $D \subseteq R$. Then $F(\xi)$ has Frechet derivative at every point $\xi \in D$ if and only if $\underline{f}(\xi)$ and $\overline{f}(\xi)$ are differentiable at point $\xi \in D$ and $F'(\xi) = [\underline{f}'(\xi), \overline{f}'(\xi)]$.*

A similar statement for the case of the set of segments with non-negative width is supplied in [1], [4].

An interesting property of the Frechet derivative should be noted.

If for a segment fuction $F(\xi)$ the width of segment $\overline{f}(\xi) - \underline{f}(\xi)$ does not decrease, then $\overline{f}'(\xi) - \underline{f}'(\xi) \geq 0$. If the width of the segment $\overline{f}(\xi) - \underline{f}(\xi)$ does not decrease, then $\overline{f}'(\xi) - \underline{f}'(\xi) < 0$. And vice versa.

Example. Consider the function $F(\xi) := [0.9, 1.0] \dot{\times} \xi$.

For do this, let $F'(\xi) = [0.9, 1.0]$ for $\xi > 0$ and $F'(\xi) = [1.0, 0.9]$ for $\xi < 0$.

For the considered function, if $\xi > 0$ the difference $\overline{f}(\xi) - \underline{f}(\xi)$ increases, and $F'(\xi) = [0.9, 1.0]$; if $\xi < 0$ the difference $\overline{f}(\xi) - \underline{f}(\xi)$ decreases, and $F'(\xi) = [1.0, 0.9]$.

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