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INTERVAL METHODS BASED ON A POSTERIORI ESTIMATES

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In this paper some methods based on a posteriori estimates are presented.

ИНТЕРВАЛЬНЫЕ МЕТОДЫ, ОСНОВАННЫЕ НА АПОСТЕРИОРНЫХ ОЦЕНКАХ

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Представлены методы, основанные на апостериорных оценках. Для построения интервального решения предварительно решается несколько вспомогательных, в общем случае неинтервальных задач.

Let R^n be a space of n-dimensional vectors. In what follows, we denote interval numbers $\mathbf{a} = [\underline{a}, \overline{a}]$ with bold font: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}$. Similarily, $\mathbf{R}^n$ is the space of n-dimensional interval vectors, where $wid(\mathbf{a}) = \underline{a} - \overline{a}, mid(\mathbf{a}) = (\underline{a} + \overline{a})/2$.

By the example of an operator equation we explain the essence of our algorithm. Consider the operator equation

$$L(u, k) = 0, \tag{1}$$

where k is a vector of parameters, $k \in \mathbf{k}$. Denote by $\mathbf{U}$ the set of solutions of the problem

$$\mathbf{U} = \{u \mid (\exists k \in \mathbf{k}), (L(u, k) = 0)\}. \tag{2}$$

In general, the method is based on the reduction of problem (1) to some sequence of non-interval problems

$$L_i(u_i, k_i) = 0, \ i = 1, 2, \dots . \tag{3}$$

Solving numerically these problems we obtain solutions $u_i, i = 1, 2, \dots$. Combining these solutions we can construct an interval solution $\mathbf{u}$ of problem (1) so that $\mathbf{u} \supseteq \mathbf{U}$.

Consider this approach on the following examples.

Interval extension

Let f be a function in the special form [5]

$$f(x) = \sum_{i+j=0}^{2} x_i x_j. \tag{4}$$

To construct an interval extension, we a posteriori calculate a partial derivative $\partial f / \partial x_i$. In certain cases, using information about this derivative we can construct an optimal interval extension.

Estimation of the minimum for a strongly convex function

Let U be a convex finite region in R^n, f be a strongly convex function, $f \in C^1(U)$ and $\{e_i\}_1^n$ is an orthonormal basis in R^n. Denote by x_* the minimal point of

$$f(x_*) = \min f(x)$$

and let x_0 be a point liying in the neighbourhood of x_*. We may obtain x_0 by some numerical minimization method.

Our goal is the construction of a parallelepiped $P \ni x_*$ [7]. At first we construct the plane $l0_i \ni x_0$ ortohonal to e_i. Further, we seek the point x_1 and the plane $l1_i$ such that the point x_* lies beetween the planes $l0_i$ and $l1_i$. Hence we have region $U_i \ni x_*$.
Then

$$P = \bigcap_{i=1}^{n} U_i.$$

Systems of nonlinear equations

Let us consider systems of nonlinear equations in the form [4]

$$x_i = f_i(x, k), \;\; i = 1, 2, ...n, \tag{5}$$

where
$x \in R^n$ is a vector of variables;
$k \in R^n$ is a vector of parameters, $k \in \mathbf{k}$.

We transform system (5) to the form

$$\begin{aligned} \overline{x} &= F^1(\underline{x}, \overline{x}), \\ \underline{x} &= F^2(\underline{x}, \overline{x}), \end{aligned} \tag{6}$$

or

$$\mathbf{x} = \mathbf{F}(\mathbf{x}). \tag{7}$$

Let x^0 be a numerical solution of system (5) with $k \in \mathbf{k}$, let $\mathbf{X}$ be the set of solutions of (5). Then x^0 is an approximation the exact solutions and $x^0 \in \mathbf{X}$. We can solve the system (7) by the simple iteration method

$$\mathbf{x}^{j+1} = \mathbf{F}(\mathbf{x}^j), \;\; j = 0, 1, ... \,. \tag{8}$$

where $\mathbf{x}^0 \supseteq \mathbf{X}$. For the construction of $\mathbf{x}^0$ we use a numerical solution of system (6). We solve system (6) by a simple iteration method similarly to the one used for non-interval systems in R^{2n} with initial vector x^0. Then

$$\mathbf{x}^0 = mid(\mathbf{x}) + d[-1, 1]wid(\mathbf{x}),$$

where d is a parameter.

Solution of ODEs

Consider ODEs of the form [3,6,7]:

$$x_i' = f_i(t, x, k), \quad i = 1, ..., n, t \in (0, l), \tag{9}$$

$$x(0) = x_0, \tag{10}$$

where $x \in R^n$ is a vector of variables,
$x_0 \in R^n$ is an initial values vector, $x_0 \in \mathbf{x}_0$,

$k \in R^n$ is a vector of parameters, $k \in \mathbf{k}$.
In order to construct an interval solution we first solve approximately the problem (9),(10) for some specific values of k and x_0 using for example, the Runger-Kutta method. As a result, we obtain an approximate solution x^h. Using these values, we construct the Hermite cubic splines S.

Later we shall use the next deviations

$$\phi(t, x, k, S) = f(t, S, k) - S', \quad t \in (0, l). \tag{11}$$

For the construction of an interval solution we use solutions of two additional linear ODE systems

$$v_1' = Wv_1 + w, \quad t \in (0, l), v_1(0) = 0$$

and

$$v_2' = Wv_2, \quad t \in (0, l), v_1(0) = z_0,$$

where $W = \{W_{ij}\}$ is a special matrix, the vector w has components $w_i = 1$, the vector $z_i = wid(\mathbf{x}_0)/2$. Then we solve the additional problem as in solving problem (9),(10) and construct corresponding splines S^1, S^2. We will seek an interval solution of the form

$$\mathbf{x} = S + [-1, 1]S^1 + \alpha S^2.$$

Let us consider the width of the interval solution

$$\rho(t) = wid\,\mathbf{x}(t).$$

Theorem 1. *Let S, S^1, S^2 be the Hermit cubic splines. Then*

$$\rho(t) \le Ch^3 \|x\|_{W^4_\infty[0,l]} S^1(t).$$

Partial differential equation

For example let us consider the model elliptic boundary value problem

$$Lu = f(x, u), \;\; x \; in \; \Omega, \tag{12}$$

$$u(x) = 0, \;\; x \; on \; \partial\overline{\Omega}, \tag{13}$$

where Ω is a bounded open convex domain in R^2, with the piecewise smooth boundary $\partial\overline{\Omega}$,

$$Lu = \sum_{i+j}^{2} \partial^2 u / \partial x_i \partial x_j .$$

Assume that

$$\partial f(x,u)/\partial u \;\geq\; q(x) \geq 0$$

and that positive constant K exists such that

$$|f(x,\eta)| \leq K(1+|\eta|), \quad \forall x \in \Omega, \forall \eta \in [\min_{\Omega} u, \max_{\Omega} u].$$

We solve problem (12),(13) by the finite element methods (FEM) on the triangulation Ω_h. We obtain the numerical solution u^h and construct a special spline $S \in W_2^1(\Omega)$. Then we can use the deviation

$$\phi(x,S) = LS - f(x,S), \; x \in \Omega. \tag{14}$$

We solve numerically by FEM the additional problem

$$L_1 u_1 = 1, \; x \; in \; \Omega, \tag{15}$$

$$u_1(x) = 0, \; x \; on \; \partial\overline{\Omega}, \tag{16}$$

where $L_1 = Lu - qu$ and we also construct the special spline S_1. Then the interval solution is of the form

$$\mathbf{u} = S + \alpha S_1 + \beta,$$

where

$$\overline{\alpha} = \max_{\overline{\Omega}}(\phi/L_1 S_1, 0), \; \underline{\alpha} = \min_{\overline{\Omega}}(\phi/L_1 S_1, 0),$$

$$\overline{\beta} = \max_{\partial\Omega}(-\overline{\alpha} S_1 - S, 0), \; \underline{\beta} = \min_{\partial\Omega}(-\underline{\alpha} S_1 - S, 0).$$

Theorem 2. *Let $u \in W_\infty^6(\Omega), \rho(x) = wid\,\mathbf{u}(x), \; x \in \Omega$ then*

$$\rho(x) \leq Kh^2,$$

where h is the size of mesh, K is a constant independent of h.

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