INTERVAL ANALYSIS: NONDIFFERENTIABLE PROBLEMS

Introduction.

Nondifferentiable problems arise in various places in global optimization. One example is in l_1 and l_∞ optimization. That is,

$$\min_x \phi(x) = \min \|F\|_1 = \min_x \sum_{i=1}^m |f_i(x)| \qquad (1)$$

and

$$\min_x \phi(x) = \min \|F\|_\infty = \min_x \left\{ \max_{1\le i\le m} |f_i(x)| \right\} \qquad (2)$$

where x is an n-vector, arise in data fitting, etc., and ϕ has a discontinuous gradient. In other problems, piecewise linear or piecewise quadratic approximations are used, and the gradient or the Hessian matrix are discontinuous. In fact, in some problems, even the objective function can be discontinuous.

Much thought has been given to non-differentiability in algorithms to find local optima, and various techniques have been developed for local optimization. Some of these techniques can be used directly in interval global optimization algorithms. However, the power of interval arithmetic to bound the range of a point-valued function, even if that function is discontinuous, can be used to design effective algorithms for non-differentiable or discontinuous problems whose structure is virtually identical to that of algorithms for differentiable or continuous problems.

Posing as continuous problems.

Several techniques are available for re-posing problems as differentiable problems, in particular for Problem 1 and Problem 2. One such technique, suggested in [4, p. 74] and elsewhere, involves rewriting the forms $|e|$, $\max\{e_1, e_2\}$, and $\min\{e_1, e_2\}$ occurring in variable expressions in the objective and constraints in terms of additional constraints, as follows:

- Replace an expression $|e|$ by a new variable x_{n+1} and the two constraints $x_{n+1} \ge 0$ and $x_{n+1}^2 = e^2$.
- Replace $\max\{e_1, e_2\}$ by
$$(e_1 + e_2 + |e_1 - e_2|)/2$$
.
- Replace $\min\{e_1, e_2\}$ by
$$(e_1 + e_2 - |e_1 - e_2|)/2$$
.

Alternately, as explained in [1] and elsewhere, the entire Problem 1 and Problem 2 can be replaced by constrained problems. In particular, Problem 1 can be replaced by

$$\begin{array}{l} \text{minimize } \sum_{i=1}^m v_i \\ \quad \text{subject to } v_i \ \ge \ f_i(x), \ i = 1, \ldots, m, \\ \qquad\qquad\quad\ v_i \ \ge \ -f_i(x), \ i = 1, \ldots, m, \\ \text{where the } v_i \text{ are new variables.} \end{array} \qquad (3)$$

Likewise, Problem 2 can be replaced by

$$\begin{array}{l} \text{minimize } \quad v \\ \quad \text{subject to } v \ \ge \ f_i(x), \ i = 1, \ldots, m, \\ \qquad\qquad\quad\ v \ \ge \ -f_i(x), \ i = 1, \ldots, m, \\ \text{where } v \text{ is a new variable.} \end{array} \qquad (4)$$

A special method for minimax problems.

In [3], a special interval algorithm for Problem 2 is presented.

Treating as continuous problems.

Due to inclusion properties of interval arithmetic, interval algorithms based on a particular degree of smoothness can be effective, essentially unchanged when less smoothness is present. In particular,

- If the objective function is discontinuous, algorithms designed for continuous objective functions can be used effectively.
- If the function is non-smooth (that is, if the gradient has discontinuities), then algorithms based on second-order information can often be used effectively.

For a brief discussion and further references for these general algorithms, see **Interval analysis: Unconstrained and constrained optimization**. For a more in-depth discussion of how continuous algorithms can be used for discontinuous problems, see [2, Ch. 6]. The main ideas are highlighted below.

Minima of $\phi:\ \mathbf{R}^n \to \mathbf{R}^1$ can still be located when the objective ϕ is discontinuous because bounds on the range of ϕ are all that is necessary to do a branch and bound search. For a simple example, suppose

$$\phi(x) = \left\{ \begin{array}{rl} x^2 & \text{if } x \le 1, \\ 1+x & \text{if } x > 1, \end{array} \right\} \tag{5}$$

and suppose the interval $[-2, 2]$ is to be searched for global minima. For illustration purposes, suppose $\phi(0.25) = 0.125$ has been evaluated, so that 0.125 is an upper bound on the global optimum, and suppose the subinterval $\boldsymbol{x} = [0.5, 1.5]$ is to be analyzed. To obtain an interval enclosure for the range of ϕ over $\boldsymbol{x}$, we take

$$\begin{array}{rcl} \phi(x) & \in & [0.5, 1.0]^2 \underline{\cup} (1 + [1.0, 1.5]) \\ & = & [0.25, 1.0] \underline{\cup} [2.0, 2.5] = [0.25, 2.5], \end{array}$$

where $\boldsymbol{a} \underline{\cup} \boldsymbol{b}$ is the smallest interval that contains both $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, since $0.125 < [0.25, 2.5]$, a minimum of ϕ cannot possibly occur within the interval $[0.5, 1.5]$.

Similar considerations apply if the gradient $\nabla\phi$ is discontinuous. In such cases, the gradient test (see **Interval analysis: Unconstrained and constrained optimization**) will keep boxes that either contain zeros of the gradient or critical points corresponding to gradient discontinuities where the gradient changes sign.

When the gradient is discontinuous, interval Newton methods can still be used for iteration, as well as to verify existence. (See [2, (6.4) and (6,5),p. 217] for a formula, and see **Interval analysis: Interval Newton methods** for an introduction to interval Newton methods, and see **Interval analysis: Interval fixed point theory** for an explanation of interval fixed point theory.). Application to problems with discontinuous gradients is based on extended interval arithmetic (with infinities) and astute computation of slope bounds; see **Interval analysis: The slope interval Newton method** for an explanation of slopes, and see [2, pp. 214–215] for details.

An example.

Consider

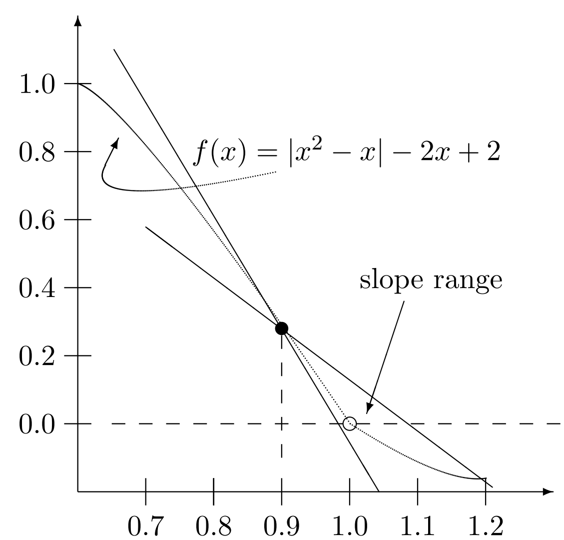
$$f(x) = |x^2 - x| - 2x + 2 = 0. \tag{6}$$

This function has both a root and a cusp at $x = 1$, with a left derivative of -3 and a right derivative of -1 at $x = 1$. If $1 \in \boldsymbol{x}$, then a slope enclosure is given by $\mathbf{S}(f, \boldsymbol{x}, x) = [-1, 1](\boldsymbol{x}+x-1)-2$.

Consider using the interval Newton method

$$\begin{array}{rcl} \tilde{\boldsymbol{x}} & \leftarrow & \check{x}^{(k)} - f(\check{x}^{(k)})/\mathbf{S}(f, \boldsymbol{x}^{(k)}, \check{x}^{(k)}) \\ \boldsymbol{x}^{(k+1)} & \leftarrow & \boldsymbol{x}^{(k)} \cap \tilde{\boldsymbol{x}}, \end{array}$$

with $\check{x}^{(k)}$ equal to the midpoint $\check{x} = 0.9$ of $\boldsymbol{x}^{(k)}$, and $\boldsymbol{x}^{(0)} = [0.7, 1.1]$, where $\mathbf{S}(f, \boldsymbol{x}^{(k)}, \check{x}^{(k)})$ is a bound on the slope enclosure of f at $\check{x}$. (See the figure for the concept of slope range.)



The concept of a slope range for a non-differentiable function.

An initial slope enclosure is then $\mathbf{S}(f, [0.7, 1.1], 0.9) = [-3, -1]$,

$$\tilde{\boldsymbol{x}} = .9 - .29/[-3, -1] = [.99\overline{6}, 1.19],$$

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and $\boldsymbol{x}^{(1)} = [0.99\overline{6}, 1.1]$. If this interval Newton method is iterated, then on iteration 3, existence of a root within $\boldsymbol{x}^{(3)}$ was proven, since $\boldsymbol{x}^{(3)} \subset \mathrm{int}(\boldsymbol{x}^{(2)})$, where $\mathrm{int}(\boldsymbol{x}^{(2)})$ is the interior of $\boldsymbol{x}^{(2)}$. For details, see [2, pp. 224–225].

References

[1] Gill, P. E., Murray, W., and Wright, M.: *Practical Optimization*, Academic Press, New York, 1981.

[2] Kearfott, R. B.: *Rigorous Global Search: Continuous Problems*, Kluwer, Dordrecht, Netherlands, 1996.

[3] Shen, Z., Neumaier, A., and Eiermann, M. C.: 'Solving Minimax Problems by Interval Methods', *BIT* **30** (1990), 742–751.

[4] Van Hentenryck, P., Michel, L., and Deville, Y.: *Numerica: A Modeling Language for Global Optimization*, MIT Press, Cambridge, MA, 1997.

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