Department of Mathematics
The University of Southwestern Louisiana
Ph.D. Qualifying Examination: Numerical Analysis
January, 1991

Instructions: This portion of the exam is open-book, open notes; it is recommended that you have the Golub / van Loan and Ames texts, as well as the 1979 R. E. Moore monograph with you. Make sure to organize and check your work.

1. Consider the nonlinear boundary value problem
$$u_{xx} = -u^2 \qquad u(0) = -1, \quad u(1) = -1.$$
   (a) Using central differences on $[0, 1]$ with 2 equally spaced internal mesh points, write down a nonlinear system of equations which approximates this nonlinear boundary value problem. Do an iteration of Newton's method, starting with $u_1 = u_2 = -.5$, on this system, and compute the resulting residual.
   (b) Show that the discretized problem has a unique solution in the box $[-1, 0] \times [-1, 0]$ by using an interval Newton method or the Krawczyk method.

   Hint: You may assume that $u_1 = u_2$; showing that this is so on your paper is a bonus.
2. Use an interval technique to show that the original differential equation in problem 1 has a unique solution whose values lie in the interval $[-1.1, -.7]$.
   Hint: (5.28) on p. 70 of [Moore, 1979] remains true if altered to read
$$\max_t\{P(X)(t)\} - \min_t\{P(X)(t)\} \le c\left(\max_t\{X(t)\} - \min_t\{X(t)\}\right).$$
3. Suppose
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$
can be written $A = U\Sigma V'$, where U and V are orthogonal,
$$U \approx \begin{pmatrix} 0.2093 & 0.9644 & 0.1617 \\ 0.5038 & 0.0353 & -0.8631 \\ 0.8380 & -0.2621 & 0.4785 \end{pmatrix},$$
$$\Sigma \approx \begin{pmatrix} 17.4125 & 0 & 0 \\ 0 & 0.8752 & 0 \\ 0 & 0 & 0.1969 \end{pmatrix}, \quad \text{and}$$
$$V \approx \begin{pmatrix} 0.4647 & -0.8333 & 0.2995 \\ 0.5538 & 0.0095 & -0.8326 \\ 0.6910 & 0.5528 & 0.4659 \end{pmatrix}.$$
Suppose we want to solve the system $AX = B$, where
$$B = [1, -1, 1]^T,$$
but that, due to noise in the data, we do not wish to deal with any system of equations with condition number equal to 25 or greater. Use the above singular value decomposition to write down the solution of minimum norm to the rank-two system of equations nearest to $AX = B$, such that the computations proceed with a matrix whose condition number is less than 25. Explain what we mean by "nearest" here.