Math. 556-01
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Numerical Analysis Comprehensive Examination

Tuesday, August 11, 1998, 1:00PM–5:00PM, MDD 206

Instructions: This exam is closed-book. Present your answers clearly and completely, and check your answers carefully.

1. Compute the condition number of $f(x) = \sin(x)$ near $x = 10^6\pi + \pi/4$.

2. Use the reverse mode of automatic differentiation to compute the gradient of
$$f(x, y) = x^2 \sin(y) + y^2$$
at $(x, y) = (1, \frac{\pi}{4})$. Write down the matrix and the intermediate steps, as if you were giving the example to a class.

3. Given that
$$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{135} f^{(4)}(\theta),$$
for some $\theta \in [-1, 1]$, find an interval enclosure to $\arctan(0.1)$. Use the above integration formula, rather than other knowledge.
Hint: If $f(x) = 1/(1 + x^2)$*, then*
$$f^{(4)}(x) = \frac{384x^4}{(1 + x^2)^5} - \frac{288x^2}{(1 + x^2)^4} + \frac{24}{(1 + x^2)^3}$$
Also, be careful when you make the change of variables.

4. Describe two methods solving the least squares problem
$$Az = b$$
where $A \in C^{m\times n}$, $z \in C^{n\times 1}$, $b \in C^{m\times 1}$, and apply one of them to find the least squares solution of
$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

5. Use the interval Gauss–Seidel method with inverse midpoint preconditioner to prove that there exists a unique solution to

$$\begin{aligned} x_1^2 - 4x_2 &= 0 \\ x_2^2 - 2x_1 + 4x_2 &= 0 \end{aligned}$$

within the box $\boldsymbol{X} = ([-0.1, 0.1], [-0.1, 0.1])^T$.

6. Discretize

$$u_{xx} = -u^2, \qquad u(0) = u(1) = 0$$

with a finite element method with basis functions

$$\phi_1(x) = \begin{cases} 4x & \text{if } 0 \le x \le .25 \\ 1 - 4(x - .25) & \text{if } .25 \le x \le .5 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_2(x) = \begin{cases} 4(x - .25) & \text{if } .25 \le x \le .5 \\ 1 - 4(x - .5) & \text{if } .5 \le x \le .75 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_3(x) = \begin{cases} 4(x - .5) & \text{if } .5 \le x \le .75 \\ 1 - 4(x - .75) & \text{if } .75 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Contrast the system that you get with that you get when you discretize a linear PDE. How might you approximate the solution(s) to the discretized system?

7. Set up an implicit finite difference scheme with three internal mesh points in space for the following parabolic initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + 2u, \ 0 < x < 1, \ 0 < t < T, \\ u(0, t) = f(t), \ u(1, t) = g(t), \ 0 < t < T, \\ u(x, 0) = h(t), \quad 0 \le x \le 1, \end{cases}$$

and carry out a stability analysis.

Hint: You may either use the Fourier method or compute eigenvalues, etc.