Department of Mathematics
The University of Southwestern Louisiana
Ph.D. Qualifying Examination: Numerical Analysis
August, 1992

Instructions: This portion of the exam is open-book, open notes; it is recommended that you have the Kincaid / Cheney and Gill / Murray / Wright texts, as well as your notes with you. Make sure to organize and check your work.

1. Compute the condition number of $f(x) = \sin(x)$ near $x = 10^6\pi + \pi/4$.

2. Show that the dual of

$$\min x_1 - 2x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 3, \\ 0 \le x_1 &\le 1, \\ x_2 &= 1, \\ x_3 &\quad \text{arbitrary} \end{aligned}$$

is

$$\max 3y_1 - y_2 + y_3$$

subject to

$$\left\{ \begin{array}{lllll} y_1 - y_2 & & + r_1 & = & 1 \\ y_1 & + y_3 & & = & -2 \\ y_1 & & & = & 1 \end{array} \right\} \qquad \left\{ \begin{array}{l} r_1 \ge 0 \\ y_2 \ge 0 \end{array} \right\}.$$

(Hint: You must show that the Lagrange multipliers of any solution of the primal problem form a solution to the dual problem. This involves showing that these Lagrange multipliers are both feasible for the dual and make the dual objective function optimal. Let y_1, y_2, y_3, and r_1 be the Lagrange multipliers corresponding to the constraints $x_1 + x_2 + x_3 = 3$, $-x_1 \ge -1$, $x_2 = 1$, and $x_1 \ge 0$, respectively. You can then follow the general argument on pp. 75–76 of Gill / Murray / Wright.)

3. If A is an m by n matrix, with $m \gg n$, then we may wish to find a least squares solution to the (usually) overdetermined system of equations $Ax = b$. A common technique is the following algorithm.
   (i) Form a QR factorization, i.e., write $A = QR$, where $Q'Q = I$ and R is right triangular.
   (ii) form $w = Q'b$,
   (iii) For $i = n$ to 1 by -1:
$$x_i \leftarrow \frac{w_i - \sum_{j=i+1}^{n} r_{i,j} x_j}{r_{i,i}}.$$

   Note that Q in the above algorithm is m by m; in certain instances, m may be so large that it is undesirable to store Q. An alternative is to factor A in the form $A = LQ$, where L is *left* or lower triangular, and Q is a (different) n by n orthogonal matrix (i.e. $Q'Q = I$). In such instances, only Q and $L'L$ are stored. Write an algorithm as above, assuming you are able to obtain Q and $L'L$. *(Hint: In this case, you may need to work with a more ill-conditioned system.)*

4. Discretize
$$u_{xx} = -u^2 \qquad u(0) = u(1) = 0$$
   with a finite element method with basis functions
$$\phi_1(x) = \begin{cases} 4x & \text{if } 0 \le x \le .25 \\ 1 - 4(x - .25) & \text{if } .25 \le x \le .5 \\ 0 & \text{otherwise,} \end{cases}$$
$$\phi_2(x) = \begin{cases} 4(x - .25) & \text{if } .25 \le x \le .5 \\ 1 - 4(x - .5) & \text{if } .5 \le x \le .75 \\ 0 & \text{otherwise,} \end{cases}$$
$$\phi_3(x) = \begin{cases} 4(x - .5) & \text{if } .5 \le x \le .75 \\ 1 - 4(x - .75) & \text{if } .75 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Contrast the system that you get with that you get when you discretize a linear PDE. How might you approximate the solution(s) to the discretized system?