Department of Mathematics
University of Louisiana at Lafayette

Numerical Analysis Comprehensive Examination

Tuesday, August 7, 2012, 9:00AM to 1:00PM, MDD 208

This exam is closed book, but calculators are permitted (and will be useful). Please use your own paper, and put you name on each sheet.

1. Consider
$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}. \tag{1}$$
   (a) Compute the condition number with respect to $\|\cdot\|_\infty$ of A.
   (b) If one were computing the solution of $Ax = b$, with A as in (1), and one were using floating point arithmetic with 5 significant digits, how many digits would you expect would be correct in the computed solution for x? (A formal proof is not needed — use the common rule of thumb.)

2. Assume that $f \in \boldsymbol{C}^2[a,b]$. Let $M = \max_{x\in[a,b]} |f''(x)|$.
   (a) Prove that
$$\left| \int_a^b f(x)dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \le (b-a)^3 \frac{M}{24}.$$
   (b) Prove that
$$\left| \int_a^b f(x)dx - \sum_{j=0}^{N-1} \frac{b-a}{N} f\left(\frac{a_j + a_{j+1}}{2}\right) \right| \le \frac{(b-a)^3 M}{24N^2},$$
   where $a_j = a + (j-1)h$ with $h = (b-a)/N$.

3. Consider the iteration
$$x^{(k+1)} = Gx^{(k)} + q, \quad \text{where } G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and } q = \begin{pmatrix} \frac{1}{6} f_0 \\ \frac{1}{4} f_1 \\ \frac{1}{4} f_2 \\ \frac{1}{6} f_3 \end{pmatrix}. \tag{2}$$
   (a) Prove that the iteration (2) has a fixed point x^*.
   (b) Find the smallest constant $C > 0$ you can such that $\|x^{(k+1)} - x^*\|_1 \le C \|x^{(k)} - x^*\|_1$.
   (c) Prove that
$$\|x^{(k)} - x^*\| \le \|G\|^k \|x^{(0)} - x^*\| \quad \text{and} \quad \|x^{(k)} - x^*\| \le \frac{\|G\|^k}{(1 - \|G\|)} \|x^{(1)} - x^{(0)}\|$$

4. (Related to problem 3) Now consider the system

$$Ax = f, \qquad \text{where } A = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix} \quad \text{and } f = \begin{pmatrix} 0 \\ \sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}. \tag{3}$$

   (a) Explain the relationship between (3) and the iteration (2).

   (b) Solve the system (3) by hand. (You may do it numerically, carrying three significant figures.)

   (c) Do two iterations of (2), starting with $x^{(0)} = q$. Compare what you get to the value C you obtained in 3b and to the approximate solution you obtained in the step previous to this one.

5. Consider approximating $f(x) = \cos(x) - 1$ in the interval $x \in [-0.1, 0.1]$ by a degree-2 polynomial.

   (a) Write down the degree-2 Taylor polynomial, and give a bound on the error in $[-0.1, 0.1]$.

   (b) Write down the degree 2 interpolating polynomial at the points $x_0 = -0.1$, $x_1 = 0$, and $x_2 = 0.1$, and give a bound for the error. Write the polynomial in power form so it can be compared to part (a).

   (c) Write down the least-squares approximation to f by a polynomial of degree 2, with respect to the dot product

$$(f, g) = \int_{-0.1}^{0.1} f(x)g(x)dx.$$

   (d) Compare the coefficients in (a), (b), and (c).

   (e) For values of x very near zero, if you use floating point arithmetic, do you think it would be better to evaluate f using the system-provided cosine function or one of the quadratic approximants? Why?

6. Suppose $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{6} & 1 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and } U = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & \frac{15}{4} & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

   Compute the solution x to $Ax = b$, where $b = (0, 0.866, 0.866, 0)^T$, without first forming A. (You may use floating point approximations with 3 significant decimal digits, if you so desire.)

7. Consider the initial value problem

$$y'(t) = f(t, y), \quad y(t_0) = y_0, \tag{4}$$

   and consider the implicit solution method

$$t_{k+1} = t_k + h, \quad y^{(k+1)} = y^{(k)} + \frac{h}{2}\left(f^{(k)} + f^{(k+1)}\right), \tag{5}$$

   where $f^{(k)} = f(t_k, y_k)$ and y_k is the approximation given by the method to $y(t_k)$.

(a) What is the order of this method? Why?

(b) Compute the region of stability in the complex plane of this method.

8. Do three steps of Newton's method,

   (a) on the equation $f(z) = z^2 + 1$, $z^{(0)} = 0.2 + 0.8i$, and

   (b) on the system of equations
   $$F(x, y) = \begin{pmatrix} x^2 - y^2 + 1 \\ 2xy \end{pmatrix},$$
   with starting vector $x^{(0)} = (0.2, 0.8)^T$.

   (c) Compare the results of your last two computations. What do you find? Why?

9. Using Taylor's formula, find an expression for the discretization error in approximating $f'(x_0)$ by the formula
$$f'(x_0) \approx \frac{1}{2h} \{-3f(x_0) + f(x_0 + h) - f(x_0 + 2h)\}.$$