Department of Mathematics
University of Louisiana at Lafayette

Numerical Analysis Comprehensive Examination

Thursday, August 11, 2011, 9:00AM to 1:00PM, MDD 208

This exam is closed book, but calculators are permitted (and will be useful). Please use your own paper, and put you name on each sheet.

1. Compute the condition number of
$$f(x) = \sqrt{x^2 - 1}$$
near $x = 1.001$.

   Hint: Recall that the condition number is the relative change in f divided by the relative change in x. You can derive the formula for the condition number of f by taking the limit of the ratio of these relative changes.

   If there are small errors in x do you think values of f computed with floating point arithmetic with 6 decimal digits will be meaningful, regardless of how f is computed?

2. Consider approximating
$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$
in the interval $x \in [-0.1, 0.1]$ by a degree-2 polynomial.

   (a) Write down the degree-2 Taylor polynomial, and give a bound on the error in $[-0.1, 0.1]$.

   (b) Write down the degree 2 interpolating polynomial at the points $x_0 = -0.1$, $x_1 = 0$, and $x_2 = 0.1$, and give a bound for the error. Write the polynomial in power form so it can be compared to part (a).

   (c) Write down the least-squares approximation to f by a polynomial of degree 2, with respect to the dot product
$$(f, g) = \int_{-0.1}^{0.1} f(x)g(x)dx.$$
   Hint: Assume you know
$$\int_{-0.1}^{0.1} \frac{e^x - 1}{x} \approx 0.200111.$$

   (d) Compare the coefficients in (a), (b), and (c).

   (e) For values of x very near zero, if you use floating point arithmetic, do you think it would be better to evaluate f using the system-provided exponential or one of the quadratic approximants? Why?

3. For $x = (x_1, x_2)^T$, define
$$G(x) = \begin{pmatrix} \dfrac{x_1}{2} \cdot \dfrac{x_1^2 + x_2^2 + 1}{x_1^2 + x_2^2} \\[2ex] \dfrac{x_2}{2} \cdot \dfrac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2} \end{pmatrix}.$$

- (a) Show that G has a unique fixed point in the box

$$\boldsymbol{x} = \begin{pmatrix} [0.9, 1.1] \\ [-0.1, 0.1] \end{pmatrix}.$$

- (b) Do three iterations of fixed-point iteration on G, starting with $x^{(0)} = (0.9, 0.1)^T$.
- (c) Based on part (b), what do you think the quality and rate of convergence are?

4. Suppose $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}.$$

Compute the solution x to $Ax = b$, where $b = (0, 1/3, 2/3, 1)^T$. (You may use floating point approximations.)

5. Let $A = \begin{bmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 1.5 \end{bmatrix}$. Determine α such that the condition number $\kappa(A)$ is minimized. Use the maximum norm.

6. Give an estimate for the rate of convergence of the power method for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Derive your convergence rate with appropriate and well-organized calculations.

7. Find the region of absolute stability in the complex plane of the implicit method for initial value problems $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ given by

$$y_{j+1} = y_j + h\left[-\frac{1}{2}f(t_j, y_j) + \frac{3}{2}f(t_{j+1}, y_{j+1})\right].$$